Some mathematical preliminaries or review

⇒ Orthogonal Functions and expansions

– Expansions can provide an excellent method for electrostatic potential problems, and in other physics subfields, especially QM.

The simplest and usual case works with an orthonormal set of functions, say,

$$\{u_n(\xi)\}, \quad n = 1, 2, \ldots \infty$$

with the independent variable ξ having a finite range

$$a \le \xi \le b$$

with orthogonality relation written as:

$$\int_a^b u_n^*(\xi)\, u_m(\xi)\, d\xi = \delta_{nm}$$

Assume for now that the $u_n(\xi)$ are square-integrable, i.e.

$$N^2 \equiv \int_a^b |\bar{u}_n(\xi)|^2 d\xi < \infty$$

whereby one can construct normalized $u_n(\xi)$ as

$$u_n(\xi) = \frac{1}{N}\bar{u}_n(\xi)$$

Usually the set $\{u_n(\xi)\}$ is also <u>complete</u>, in which case any square-integrable function $f(\xi)$ over $a \le \xi \le b$ can be expanded as a linear combination, $f(\xi) = \sum_{n=1}^{\infty} a_n u_n(\xi)$

When $f(\xi)$ is known, the a_n can be determined by what Griffiths calls "Fourier's trick"

$$a_n = \int_a^b u_n^*(\xi) f(\xi)\, d\xi$$

$$= \int_a^b u_n^*(\xi') f(\xi')\, d\xi'$$

Plugging this last expression into the $f(\xi)$ expansion gives

$$f(\xi) = \int_a^b d\xi' \left[\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) \right] f(\xi')$$

This implies the completeness or closure relation,

$$\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) = \delta(\xi' - \xi)$$

Using notation slightly different from Jackson, here is one example of a Fourier series over a symmetric interval $-\frac{a}{2} \le x \le \frac{a}{2}$

$$U_m^e(x) = \sqrt{\frac{2-\delta_{m,0}}{a}} \cos\left(\frac{2\pi m x}{a}\right), \quad m = 0, 1, 2, \ldots$$

$$U_m^o(x) = \sqrt{\frac{2}{a}} \, , \sin\left(\frac{2\pi m x}{a}\right), \quad m = 1, 2, 3, \ldots$$

where the combination of both the even (e) and odd (o) sets gives a complete, orthonormal set, $\{ U_m^e(x), U_m^o(x) \}$, can write for any $f(x)$,

$$\Rightarrow \quad f(x) = \sum_{m=0}^{\infty} A_m U_m^e(x) + \sum_{m=1}^{\infty} B_m U_m^o(x)$$

where $$A_m = \left(\frac{2-\delta_{m,0}}{a}\right)^{1/2} \int_{-a/2}^{a/2} \cos\frac{2\pi m x}{a} f(x)\,dx$$

$$B_m = \left(\frac{2}{a}\right)^{1/2} \int_{-a/2}^{a/2} \sin\frac{2\pi m x}{a} f(x)\,dx$$

and these relations generalize to 2 or more dimensions in the natural way, e.g.

$$F(\xi,\eta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, u_m(\xi)\, v_n(\eta),$$

and, e.g. $$A_{mn} = \int_a^b d\xi \int_c^d d\eta\, u_m^*(\xi) v_n^*(\eta) F(\xi,\eta)$$

etc.

Generalization to an INFINITE INTERVAL:

We can rewrite the above sin, cos example in terms of exponentials, e.g.

let $$u_n(x) = a^{-1/2} e^{\frac{2i\pi n x}{a}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow \quad F(x) = \sum_{n=-\infty}^{\infty} A_n \frac{1}{\sqrt{a}} e^{2i\pi n x/a}$$

where $$A_n = \int_{-\frac{a}{2}}^{a/2} \frac{1}{\sqrt{a}} e^{-\frac{2i\pi n x}{a}} F(x)\,dx$$

We take the $a \longrightarrow \infty$ limit, defining:

$$k \equiv \frac{2\pi n}{a}$$

$$\sum_m \longrightarrow \int dm = \frac{a}{2\pi}\int dk$$

$$A_m \rightarrow \left(\frac{2\pi}{a}\right)^{1/2} A(k)$$

Then (a) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$

(b) $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x')\, dx'$

and

(c) $f(x) = \int_{-\infty}^{\infty} dx' f(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} \right]$

which again implies a completeness relation,

(d) $\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x')$

and we will need the "orthonormality relation",

(e) $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik'x}\, e^{ikx}\, dx = \delta(k-k')$

These are the basic equations of Fourier Analysis

Solution of partial differential equations by separation of variables

IDEA: when possible, there is great simplification if a PDE can be solved by the separable product form

e.g. $\Phi(x,y,z) = X(x)\, Y(y)\, Z(z)$

For any specific problem the solution may not have such a simple form, but it is still often useful to solve the PDE in this form as an initial step in the solution process.

Let's use this ansatz to solve

$$\nabla^2 \Phi(x,y,z) = 0 = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Plug in the above separable form, divide the entire equation by $X(x)\,Y(y)\,Z(z)$

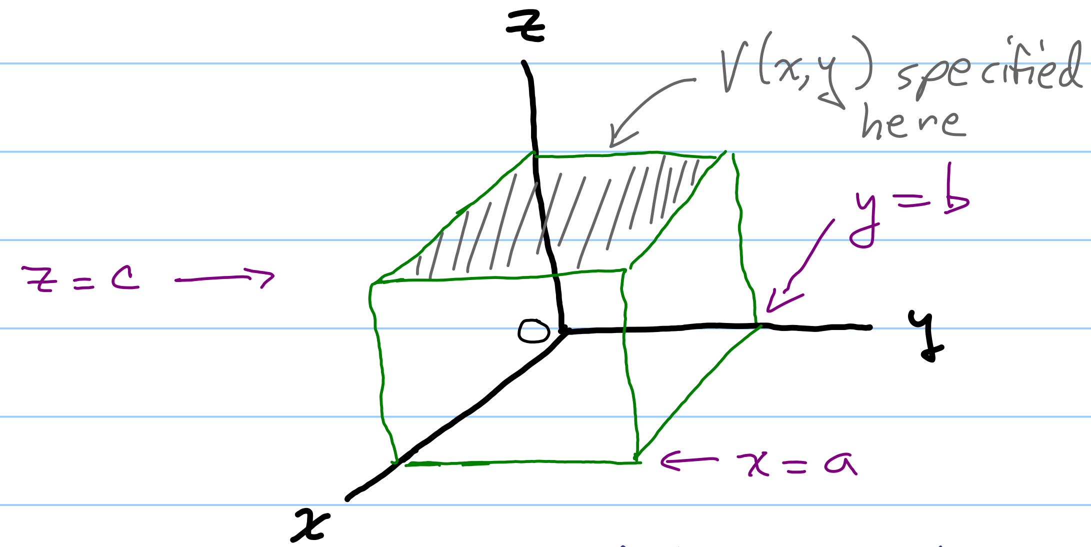
$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

and since this must hold for all x, y, z, each term must separately be a constant, i.e.

$$\frac{X''(x)}{X(x)} = -\alpha^2, \quad \frac{Y''(y)}{Y(y)} = -\beta^2$$

$$\frac{Z''(z)}{Z(z)} = \alpha^2 + \beta^2$$

separation constants



A useful rule of thumb for problems like this: Choose the sign of the separation constants to produce oscillatory solutions in any dimension having a

$\Phi = 0$ boundary condition at both limits.

e.g. suppose we are given the BCs to be

$$\Phi(0,y,z) = \Phi(a,y,z) = 0$$

$$\Phi(x,0,z) = \Phi(x,b,z) = 0$$

$$\Phi(x,y,0) = 0; \ \Phi(x,y,c) = V(x,y)$$

$\Rightarrow$ the above choice of signs of α, β works well for this PDE. Assume initially that α, β = real, but keep open the possibility that they could later be found to be imaginary or complex.

$\Rightarrow$ $X(x)$ = linear combinations of $e^{\pm i\alpha x}$

$Y(y)$ = " " " $e^{\pm i\beta y}$

$Z(z)$ = " " $e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$

By inspection the linear combinations needed in x, y are:

$$X_m(x) = \left(\frac{2}{a}\right)^{1/2} \sin\frac{m\pi x}{a}, \quad 0 \le x \le a,$$

$$\left(\text{i.e. } \alpha = \frac{m\pi}{a}, \ m = 1, 2, 3, \ldots\right)$$

and

$$Y_n(y) = \left(\frac{2}{b}\right)^{1/2} \sin\frac{n\pi y}{b}, \quad 0 \le y \le b$$

$$\left(\beta = \frac{n\pi}{b}, \ n = 1, 2, \ldots\right)$$

And the B.C. $Z(0)=0$ implies that the only solution in z must be proportional to

$$Z_{m,n}(z) = \sinh\left[\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right]$$

for $0 \le z \le c$

Hence an infinite set of product solutions that obey all BC's in the problem $X_m(x)\,Y_n(y)\,Z_{mn}(z)$ EXCEPT at $z=c$, where we must still find a way to satisfy the B.C.:

$$\Phi(x,y,c) = V(x,y)$$ ← specified + assumed to be known

Using the linearity of the PDE, we can superpose this set with coefficients A_{mn} still to be determined to match the BCs:

$$\Rightarrow \Phi(x,y,z) = \sum_{m,n=1}^{\infty} A_{mn} \left(\frac{2}{a}\right)^{1/2}\left(\frac{2}{b}\right)^{1/2} \sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{z}\,\sinh(\gamma_{mn} z)$$

where $\gamma_{mn} \equiv \sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\;\pi$

DEMAND

$$\xrightarrow[z\to c]{} V(x,y)$$

Use the orthonormality of the $\left(\frac{2}{a}\right)^{1/2}\sin\frac{m\pi x}{a}$ and

$\left[\text{i.e. } \int_0^a dx\,\frac{2}{a}\sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a} = \delta_{mm'}\right]$ $\left(\frac{2}{b}\right)^{1/2}\sin\frac{n\pi y}{b}$

Now apply "Fourier's Trick", mutiplying both sides by $\left(\frac{2}{a}\right)^{1/2}\sin\frac{m'\pi x}{a}\left(\frac{2}{b}\right)^{1/2}\sin\frac{n'\pi y}{b}$

and integrate, giving

$$\boxed{A_{mn} = \frac{\int_0^a dx\int_0^b dy \left(\frac{4}{ab}\right)^{1/2} V(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\sinh \gamma_{mn} c}, \quad \begin{array}{l} m = 1,2,\ldots\infty \\ n = 1,2,\ldots\infty \\ \gamma_{mn} = \sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\,\pi \end{array}}$$

Now when these constant coefficients are used in the infinite expansion above, then the above expression for $\Phi(x,y,z)$ will obey all the BCs and the PDE $\nabla^2\Phi = 0$.

Other little tricks:

- If $\Phi \neq 0$ is specified on any of the other sides, simply solve for one nonzero side at a time, but then use linearity and add them all up to make the final solution.

- If $\rho \neq 0$ inside as well, then we must use an appropriate Green's Function inside, as discussed in Chapter 3.