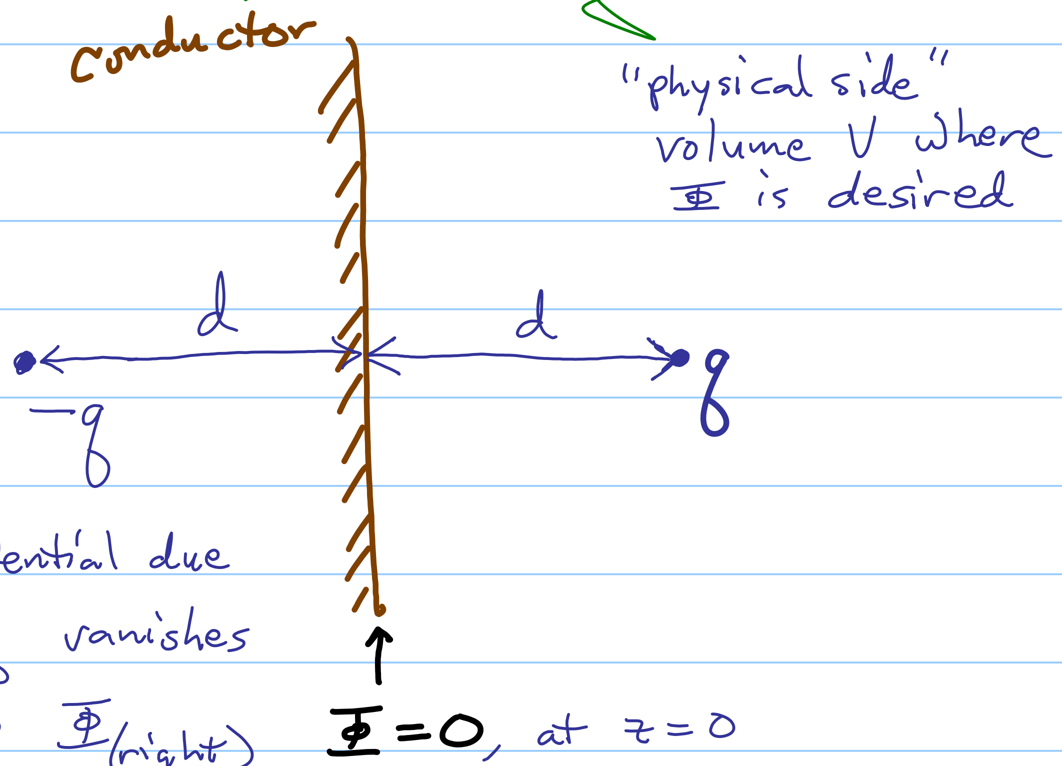
Chapter 2 Boundary value problems in electrostatics, part I

Sec. 2.1 For problems with charges near (a) boundary surface(s), we may use appropriately-placed "image charges" outside the volume V of interest, to simulate the boundary conditions.

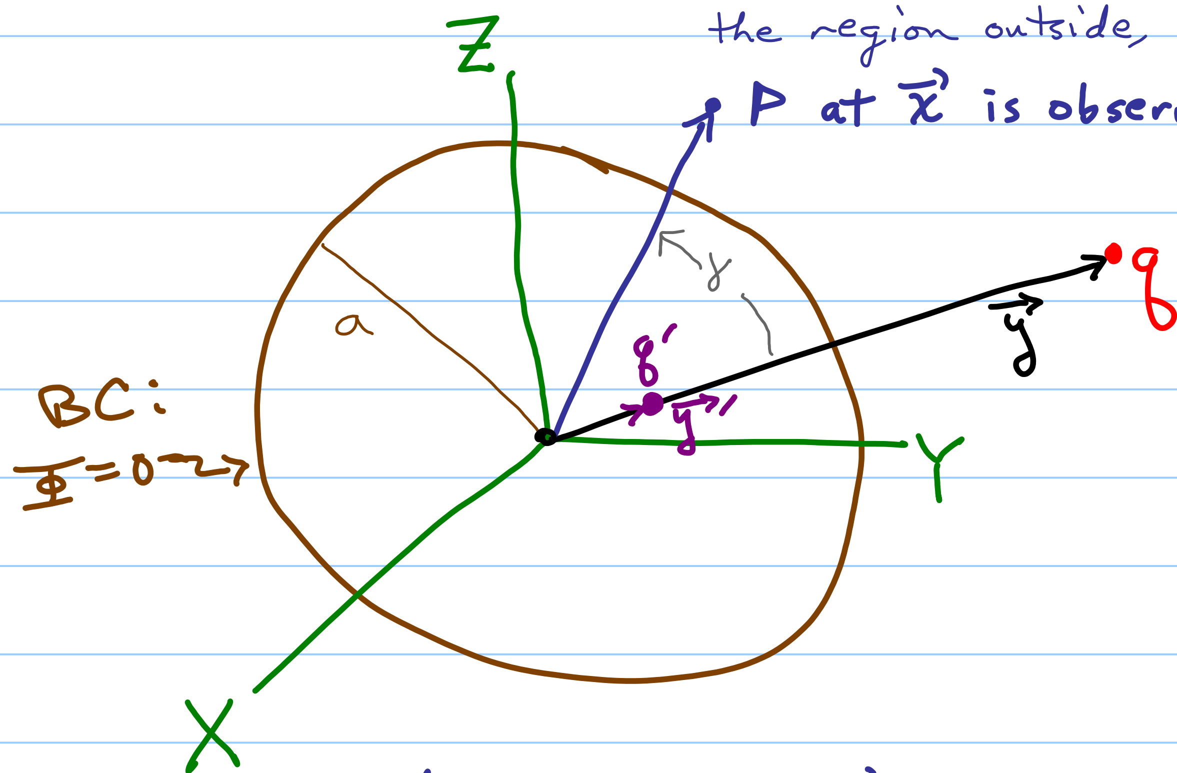
Note a charge placed outside of V will obey Laplace's equation inside V, since $\nabla^2 \frac{1}{|\vec{x}-\vec{x}_0|} = 0$ except near $\vec{x} \simeq \vec{x}_0$

e.g. the grounded, conducting plane



Clearly the potential due to q and $-q$ vanishes at $z=0$, so $\Phi_{(\text{right})}$ can be calculated using q and $(-q)$ as point charges, only!

Sec. 2.2 Consider a point charge q at position $\vec{y}$ outside a grounded, conducting sphere of radius a. The volume V of interest is the region outside, at $|\vec{y}| > a$.



Idea: place an image charge q' INSIDE the sphere at $\vec{y}'$, and choose $q', \vec{y}'$ such that $\Phi = 0$ on the sphere surface.

By symmetry, of course, $\vec{y}'$ must be along $\vec{y}$, i.e. $\vec{y}' \propto \vec{y}$, pointing towards q. So if we set $\vec{x} = x\hat{n}$, $\vec{y} = y\hat{n}'$, $\vec{y}' = y'\hat{n}'$, with $\hat{n}$ and $\hat{n}'$ unit vectors, the total potential in V is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right)$$

Now demand that

$$4\pi\epsilon_0 \Phi(x=a) = 0 = \frac{q}{|a\hat{n} - y\hat{n}'|} + \frac{q'}{|a\hat{n} - y'\hat{n}'|}$$

or

$$0 = \frac{q}{(a^2+y^2-2ay\cos\gamma)^{1/2}} + \frac{q'}{(a^2+y'^2-2ay'\cos\gamma)^{1/2}}$$

where $\hat{n}\cdot\hat{n}' \equiv \cos\gamma$

$$\Rightarrow \quad \frac{q'}{q} = -\frac{(a^2+y'^2-2ay'\cos\gamma)^{1/2}}{(a^2+y^2-2ay\cos\gamma)^{1/2}} \quad \Rightarrow \frac{q'}{q} < 0$$

square this and collect terms

$$\Rightarrow \left[\left(\frac{q'}{q}\right)^2(a^2+y^2) - (a^2+y'^2)\right] + \cos\gamma\left[-\left(\frac{q'}{q}\right)^2(2ay) + 2ay'\right] = 0$$

Since this equation holds for ALL γ, both terms in [ ] must separately vanish, giving 2 eqns and 2 unknowns (q'/q) and y':

SOLUTION:

$$\boxed{\frac{q'}{q} = -\frac{a}{y}, \quad y' = \frac{a^2}{y}}$$

Note that $y' < y$, so $\vec{y}'$ is outside of V, a necessary condition for an image charge.

We can now determine any desired observable, e.g.

$$\Phi(x,\gamma) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{(x^2+y^2-2xy\cos\gamma)^{1/2}} - \frac{qa/y}{\left(x^2+\left(\frac{a^2}{y}\right)^2-2x\frac{a^2}{y}\cos\gamma\right)^{1/2}}\right\}$$

which is valid at all $x > a$, $y > a$.

<u>Charge Density</u> One quantity often of interest is the charge density on any conducting surface.

For the sphere problem

we saw previously that
$$(\vec{E}_1 - \cancelto{0}{\vec{E}_2})\cdot\hat{n}_1 = \frac{\sigma}{\epsilon_0}$$

So here,
$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a^+}$$

or
$$\boxed{\sigma = -\frac{q}{4\pi a^2}\,\frac{a}{y}\,\frac{\left(1-a^2/y^2\right)}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma\right)^{3/2}}}$$

Consider the limit as $y \gg a$

$$\Rightarrow \sigma \xrightarrow[y \gg a]{} \frac{-q}{4\pi a^2}\,\frac{a}{y}\left(1+\left(-\tfrac{3}{2}\right)\left(-2\frac{a}{y}\right)\cos\gamma + O\left(\frac{a^2}{y^2}\right)\right)$$

$$\hookrightarrow \frac{-q}{4\pi a y} - \frac{3q}{4\pi y^2}\cos\gamma + O\left(\frac{a}{y^3}\right),$$

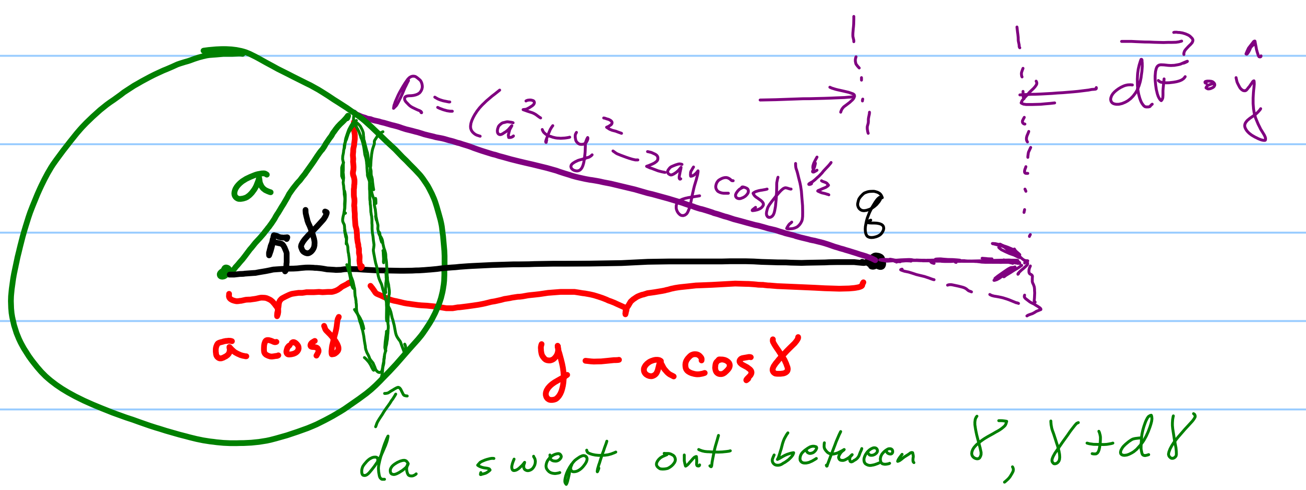
which looks approximately like a small net charge and a small electric dipole moment as $y \to \infty$.

Observe that $\oint \sigma da = q' = -q\frac{a}{y}$ at all y, which makes sense that the magnitude of charge on the surface equals the image charge magnitude!

Force on the charge q

Let's calculate directly the force on q, using the "real charges" we have just derived on the surface of the sphere. This is worth studying from different viewpoints, because with force and energy there can be factors of 2 or $\frac{1}{2}$ lurking.

method 1 integrate force from each $\sigma da'$:



So $$\vec{dF} \cdot \hat{y} = \left(\frac{y - a\cos\gamma}{R}\right) \frac{q}{4\pi\epsilon_0 R^2}\, \sigma(\gamma)\, 2\pi a^2 \sin\gamma \, d\gamma$$

$\equiv dF_y$

$$\Rightarrow dF_y = \frac{2\pi a^2 q}{4\pi\epsilon_0}\left\{\frac{-q}{4\pi a y}\left(1-\frac{a^2}{y^2}\right)\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{-3/2}\right\}$$

$\sigma(\gamma)$

$$\times \frac{\sin\gamma\, d\gamma}{(y^2+a^2-2ay\cos\gamma)^{3/2}}(y-a\cos\gamma)\cdot\frac{y^3}{(y^2)^{3/2}}$$

$$= \frac{-q^2}{8\pi\epsilon_0}\,\frac{a}{y}\left(1-\frac{a^2}{y^2}\right)y^3\,\frac{\sin\gamma\,(y-a\cos\gamma)}{(a^2+y^2-2ay\cos\gamma)^3}$$

Now integrate this from $\gamma=0$ to π, giving

$$\vec{F} = \frac{-q^2}{8\pi\epsilon_0}a(y^2-a^2)\left[\frac{2y}{(y^2-a^2)^3}\right]\hat{y}$$

Mathematica

or finally $\vec{F} = -\hat{y}\,\frac{q^2}{4\pi\epsilon_0}\,\frac{ay}{(y^2-a^2)^2}$

<u>method 2</u> interestingly, the above result agrees exactly with the trivial method of just computing the force on q due to the point charge q' at y'!

$\Rightarrow$ much simpler to compute that way!

<u>method 3</u> Use Newton's 3rd law, compute the force on the <u>sphere</u> due to q, then multiply by (-1) to get the force on q.

This gives the same answer also!

Section 2.3 Next treat a point charge q outside a charged, insulated, conducting sphere carrying total charge Q

Concept First solve for the potential in the case where q is at distance $y > a$ from a grounded sphere.

$\Rightarrow$ We saw previously that this induces a total charge $q' = -q\frac{a}{y}$ on the sphere, in order for the sphere to be an equipotential.

$\Rightarrow$ Start from that solution and add a small additional amount of charge

CLAIM: The additional charge must spread over the sphere surface symmetrically, because the sphere is an equipotential. If we keep adding charge this way until the total amount of charge is Q, this means we must add a total amount of extra charge equal to

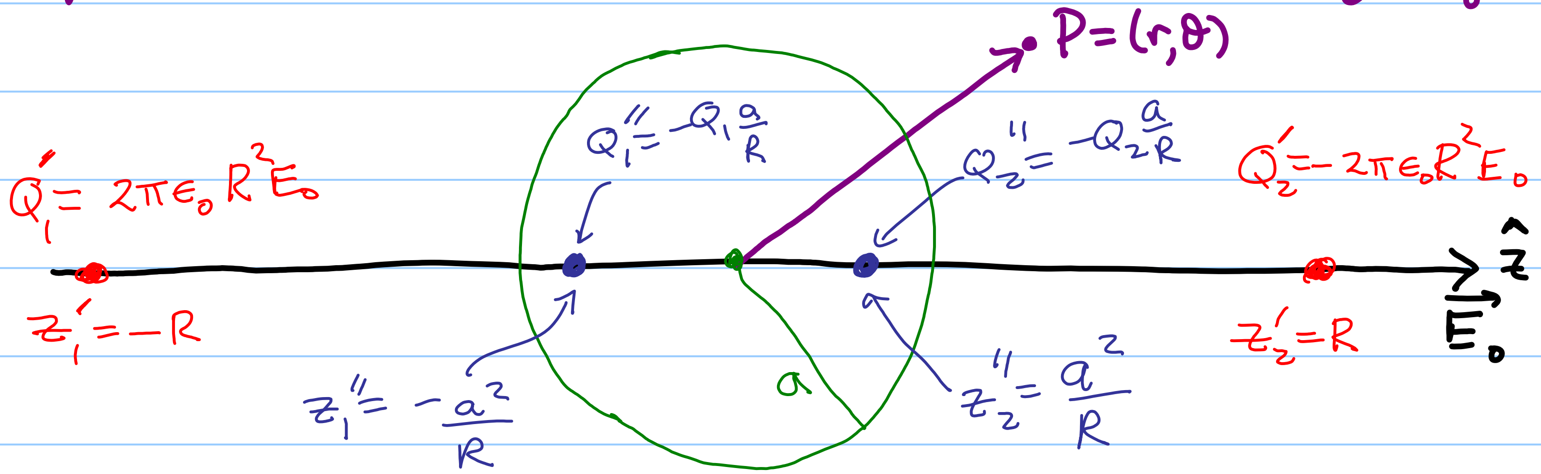
$$Q_{NEW} = Q - q' = Q + \frac{a}{y}q$$

We also know that the potential outside a spherically-symmetric distribution of charge Q_{NEW} is the same as if it were all at the center! 044

Therefore the total potential outside is equal to

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{y}\frac{1}{|\vec{x}-\frac{a^2}{y^2}\vec{y}|} + \frac{Q+\frac{a}{y}q}{|\vec{x}|} \right\}}$$

Section 2.5 Consider a grounded conducting sphere in a uniform external E-field, $\vec{E}_0 = E_0\hat{z}$



We solve this as a limiting case, with two charges far away $(R\to\infty)$ that set up the uniform electric field. The solution is again carried out using image charges.

$\Rightarrow$ The potential at all points outside the sphere is then

$$\Phi(r,\theta) = \lim_{R\to\infty} \frac{1}{4\pi\epsilon_0}\frac{Q_1'}{[R^2+r^2-2rR\cos(\pi-\theta)]^{1/2}}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{(-aQ_1'/R)}{[(a^2/R)^2+r^2-2r(a^2/R)\cos(\pi-\theta)]^{1/2}} + \frac{1}{4\pi\epsilon_0}\frac{Q_2'}{[R^2+r^2-2rR\cos\theta]^{1/2}}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{(-aQ_2'/R)}{[(a^2/R)^2+r^2-2r(a^2/R)\cos\theta]^{1/2}}$$

It is easiest to find the large $R \to \infty$ limit if we first make a binomial expansion

i.e. $(1+\epsilon)^\gamma = 1 + \gamma\epsilon + \frac{\gamma(\gamma-1)}{2!}\epsilon^2 + \ldots$

... skipping a few lines of algebra ...

$$\Rightarrow \Phi(r,\theta) = \lim_{R\to\infty} \frac{E_0 R}{2}\left[1 - \frac{r}{R}\cos\theta - 1 - \frac{r}{R}\cos\theta\right]$$

$$- \frac{E_0 R}{2}\frac{a}{r}\left[1 - \frac{a^2}{rR}\cos\theta - 1 - \frac{a^2}{rR}\cos\theta\right]$$

$+$ terms that vanish at $R \to \infty$

or $$\boxed{\Phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta}$$

Aside: observing that this Φ vanishes at $\theta = \frac{\pi}{2}$, we notice that we have solved another problem too, namely a grounded sphere + a grounded conducting plane at $z=0$.

Induced surface charge on the sphere:

$$\sigma(a,\theta) = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \epsilon_0 E_0\cos\theta + \epsilon_0 E_0 \frac{2a^3}{a}\cos\theta$$

or $$\boxed{\sigma(a,\theta) = 3\epsilon_0 E_0\cos\theta}$$

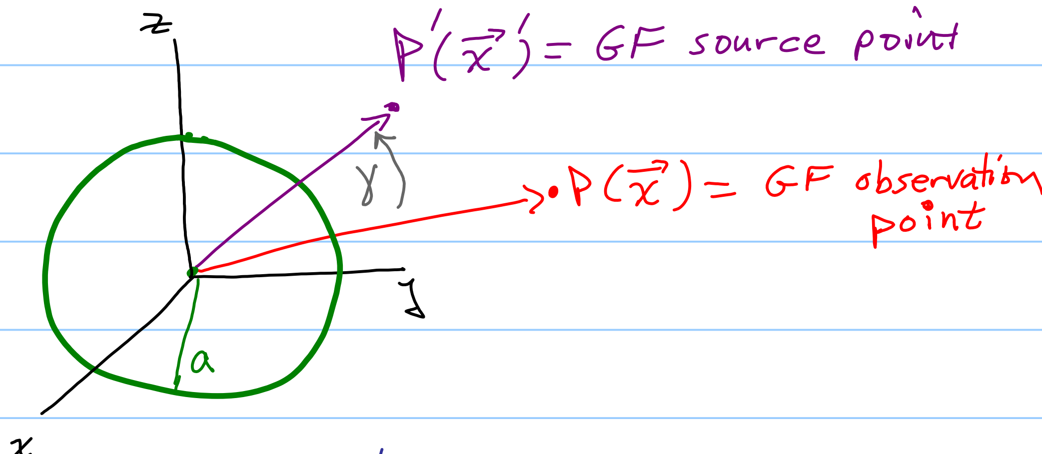
and since the integral vanishes, this means there is NO NET CHARGE on the sphere.

Dirichlet Green's Function for a sphere

Using the Green's function, we could solve many different problems. Recall that the Dirichlet GF for the exterior of a sphere must vanish at $r=a$, and it must obey $\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\delta(\vec{x}-\vec{x}\,')$ where $\vec{x}\,', \vec{x}$ are arbitrary positions outside the sphere, i.e. $G(|\vec{x}|=a, \vec{x}\,') = 0$.

This GF can be found using the method of images now, simply by inspection:

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} - \frac{a}{x'\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}\,'\right|}$$



Making this more explicit yields

$$\boxed{G(\vec{x},\vec{x}\,') = \left(x^2+x'^2-2xx'\cos\gamma\right)^{-1/2} - \left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{-1/2}}$$

To solve a general Dirichlet problem we will sometimes need the normal derivatives of G, i.e.

$$\boxed{\frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\Big|_{x'=a} = -\frac{\partial G(\vec{x},\vec{x}')}{\partial x'}\Big|_{x'=a} = \frac{a^2-x^2}{a(a^2+x^2-2ax\cos\gamma)^{1/2}}}$$

Next – A typical application: Find Φ outside a sphere whose two conducting hemispheres are held at equal & opposite potentials $\pm V$.

e.g. top is at $+V$ $(0 \le \theta \le \pi/2)$ (sphere radius a)

bottom (insulated from top) is at $-V$ $(\frac{\pi}{2} < \theta \le \pi)$

Solution Here $\rho = 0$ outside so there are only surface integrals in this Dirichlet BC problem.

$$\Rightarrow \Phi(r,\theta,\phi) = \frac{1}{4\pi}\underbrace{\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' a^2}_{\int da'} \frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\Big|_{x'=a} V(\theta',\phi')$$

$$= \frac{a^2 V}{4\pi}\int_0^{2\pi} d\phi' \left[\int_0^1 d(\cos\theta') - \int_{-1}^0 d(\cos\theta')\right] \times$$

$$\times \left\{\frac{a^2-x^2}{a(a^2+x^2-2ax\cos\gamma)^{3/2}}\right\}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

which has a complicated dependence on $\theta, \phi, \theta', \phi'$.

Jackson works this out for a special case, $\theta = 0 = \phi$.