Now define $U \equiv \Phi_1 - \Phi_2$

$\Rightarrow \nabla^2 U = 0$ in V, and $U = 0$ on S

OR $\dfrac{\partial U}{\partial n} = 0$ on S.

Start from the 1st Green identity

with $\phi \to U$, $\psi \to U$

$$\Rightarrow \int_V \left[U \nabla^2 U + \nabla U \cdot \nabla U \right] d^3x = \oint_S U \frac{\partial U}{\partial n} da$$

($U\nabla^2 U \to 0$; $U \frac{\partial U}{\partial n} \to 0$ for either BC!)

$$\Rightarrow \int_V |\nabla U|^2 d^3x = 0 \quad \text{for either B.C.}$$

But since $|\nabla U|^2 \geq 0$ everywhere, the only possibility is that $\nabla U = 0$ everywhere in V.

$\Rightarrow U = \Phi_1 - \Phi_2 = $ constant in V

meaning that Φ_1 and Φ_2 are equivalent, as the physics is unchanged by $\Phi \to \Phi +$ constant.

Moreover, since either Dirichlet or Neumann BCs gives a unique solution, one sees that arbitrarily specifying BOTH Φ AND $\left.\frac{\partial \Phi}{\partial n}\right|_S$ will not be consistent.

To learn more about this issue, e.g. that Cauchy B.C.'s are problematic in this type of PDE, see Morse + Feshbach, volume I, or Arfken + Weber,

Green's Function solution of Poisson's equation

This method is especially useful and practical for solution of the following two types of problems:

(i) Potentials are specified on a boundary S

OR

(ii) The charge distribution ρ is given in a region that is inside of a conducting boundary

The Green's function can be viewed as the electrostatic potential at an observation point $\vec{x}$ due to a unit point charge at a source point $\vec{x}'$, subject to Dirichlet or Neumann boundary conditions on the relevant surfaces.

Mathematically, the Green's function $G(\vec{x}, \vec{x}')$ obeys

$$\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')$$

We already know a particular solution to be:

$$G(\text{particular}) = \frac{1}{|\vec{x} - \vec{x}'|}$$

since $$\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi \delta(\vec{x} - \vec{x}')$$

But this is not the most general solution.
More generally we should write

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

where F obeys Laplace's equation in V,
i.e. $$\nabla^2 F(\vec{x},\vec{x}') = 0$$

By choosing F appropriately, we can impose specific BCs on G, using tricks or more sophisticated math techniques.

One example of such a "trick" is the METHOD OF IMAGES, which determines F by placing "FAKE CHARGES" external to the volume V of interest.

Question How can we use $G(\vec{x},\vec{x}')$ to obtain the desired unique solution for $\Phi(\vec{x})$?

Solution Choose G (i.e. specify F) to eliminate one of the two surface integrals in the general formula for $\Phi(\vec{x})$.

e.g. return to Green's theorem, setting now

$$\phi \to \Phi(\vec{x}') \text{ and } \psi \to G(\vec{x},\vec{x}')$$

when we want to specify particular BCs on G, we use the abbreviations:

$G_D(\vec{x}, \vec{x}')$ = Dirichlet Green's Function (vanishes on surface(s) S)

$G_N(\vec{x}, \vec{x}')$ = Neumann BC Green's Function ($\frac{\partial G_N}{\partial n'}$ specified on S)

$$\Rightarrow \int_V \left[\Phi(\vec{x}') \underbrace{\nabla'^2 G(\vec{x}', \vec{x})}_{= -4\pi \delta(\vec{x}' - \vec{x})} - G(\vec{x}', \vec{x}) \underbrace{\nabla'^2 \Phi(\vec{x}')}_{= -\frac{\rho(\vec{x}')}{\epsilon_0}} \right] d^3x'$$

$$= \oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x}', \vec{x})}{\partial n'} - G(\vec{x}', \vec{x}) \frac{\partial \Phi(\vec{x}')}{\partial n'} \right] da'$$

$$= -4\pi \Phi(\vec{x}) + \int_V \frac{\rho(\vec{x}')}{\epsilon_0} G(\vec{x}', \vec{x}) d^3x'$$

Aside - We will see later that $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$

$$\Rightarrow \text{also } \frac{\partial G}{\partial n'}(\vec{x}', \vec{x}) = \frac{\partial}{\partial n'} G(\vec{x}, \vec{x}')$$

Thus we arrive at the key equation for Φ:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V G(\vec{x}, \vec{x}') \rho(\vec{x}') d^3x'$$

$$+ \frac{1}{4\pi} \oint_S \left[G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} - \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \Phi(\vec{x}') \right] da'$$

Apply this formula to the 2 standard cases:

Dirichlet BCs – $\Phi(\vec{x}')$ given on S

$\Rightarrow$ choose $G_D(\vec{x},\vec{x}') = 0$ for $\vec{x}$ on S or for $\vec{x}'$ on S

$$\Rightarrow \Phi(\vec{x}) = \int_V G_D(\vec{x},\vec{x}')\frac{\rho(\vec{x}')}{4\pi\epsilon_0}d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\Phi(\vec{x}')\,da'$$

For this Green's function choice, $\frac{\partial \Phi}{\partial n'}$ is irrelevant and not used!

Neumann BCs – $\frac{\partial \Phi}{\partial n'}$ is given on S

$\Rightarrow$ Specify $\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} = \frac{-4\pi}{S}$, a constant over S,

where $S = \oint da' =$ total area of surface S

Note: If we had chosen $\frac{\partial G}{\partial n'}\Big|_S = 0$, this would cause inconsistencies,

since $\oint_S \frac{\partial G_N(\vec{x},\vec{x}')}{\partial n'}da' = \int_V \nabla'^2 G_N(\vec{x},\vec{x}')\,d^3x'$

$= -4\pi$

$\Rightarrow$ our BC for G_N is consistent with this!

So in the Neumann BC problem,

$$\Phi(\vec{x}) = \int_V G_N(\vec{x},\vec{x}')\frac{\rho(\vec{x}')}{4\pi\epsilon_0}d^3x'$$

$$+\frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}')\frac{\partial \Phi}{\partial n'}(\vec{x}')\,da' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}(\vec{x},\vec{x}')da'$$

and the last term here is

$$-\frac{1}{4\pi}\left(\frac{4\pi}{S}\right)\oint_S \Phi(\vec{x}')\,da' = \langle \Phi \rangle_S$$

= mean value of Φ on S

= a constant that is usually dropped

whereby we obtain

$$\Phi(\vec{x}) = \int_V G_N(\vec{x},\vec{x}')\frac{\rho(\vec{x}')}{4\pi\epsilon_0}\,d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}')\frac{\partial \Phi}{\partial n'}(\vec{x}')\,da'$$

A third possible choice of Green's function, not considered by Jackson but sometimes useful, (instead of Dirichlet or Neumann BCs), is to simply apply the same BCs to G that Φ must obey on all boundaries S. For this choice, all surface integrals vanish and we obtain simply

$$\Phi(\vec{x}) = \int_V G(\vec{x},\vec{x}')\frac{\rho(\vec{x}')}{4\pi\epsilon_0}\,d^3x'$$

GF symmetry (1) can show that $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$ is automatically true, $\Rightarrow$ complete symmetry under interchange of source and observation points

(b) However, $G_N(\vec{x}, \vec{x}')$ need not automatically be equal to $G_N(\vec{x}', \vec{x})$, but one can demand this symmetry as a separate requirement, without loss of generality.

We will see examples of using these formulas in Chapters 2, 3.

Electrostatic Potential Energy

** Reading Assignment: Read through Jackson Sec. 2.6

The energy needed to take N charges,

$$q_1, q_2 \cdots q_N$$

from infinity, and assemble them at respective positions $\vec{x}_1, \vec{x}_2, \cdots \vec{x}_N$, is called the POTENTIAL ENERGY of the system, U.

$\Rightarrow$ This energy can be calculated by bringing in each charge, one-by-one:

e.g. $U_1 = 0$ $\qquad U_2 = q_2 \left(\dfrac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_2|} \right)$

$$U_3 = q_3 \left(\frac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_3|} + \frac{q_2}{4\pi\epsilon_0 |\vec{x}_2 - \vec{x}_3|} \right)$$

$$\vdots$$

$$U_N = q_N \left(\frac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_N|} + \cdots + \frac{q_{N-1}}{4\pi\epsilon_0 |\vec{x}_{N-1} - \vec{x}_N|} \right)$$

and

$$U = \sum_{i=1}^{N} U_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{x}_i - \vec{x}_j|}$$

This result can be rewritten as

$$U = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \qquad (1)$$

$i \neq j$ implies that "self-energy" terms are omitted w/ $i=j$

The most logical generalization of this expression (1) to a continuous distribution of charge $\rho(\vec{x})$ is evidently

$$U = \frac{1}{8\pi\epsilon_0} \int \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x\, d^3x'$$

or alternatively, using $\Phi(\vec{x}) = \int \frac{\rho(\vec{x}')\,d^3x'}{4\pi\epsilon_0|\vec{x}-\vec{x}'|}$

$$\Rightarrow U = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$$

Yet another form is $U = -\frac{\epsilon_0}{2}\int_{\text{all space}} \Phi(\vec{x})\nabla^2\Phi(\vec{x})\,d^3x$

and after applying Green's 1st identity with all surface terms at ∞ neglected,

$$\Rightarrow U = \frac{\epsilon_0}{2}\int_{\text{all space}} |\nabla U|^2 d^3x$$

or finally

$$U = \frac{\epsilon_0}{2}\int_{\text{all space}} |\vec{E}|^2 d^3x \qquad (2)$$

This result is usually interpreted as implying that the $\vec{E}$-field at any point $\vec{x}$ in space stores an energy density equal to $u(\vec{x}) = \frac{\epsilon_0}{2}|\vec{E}|^2$ (i.e. J/m^3)

Observe: We have an apparent inconsistency in our derivation!

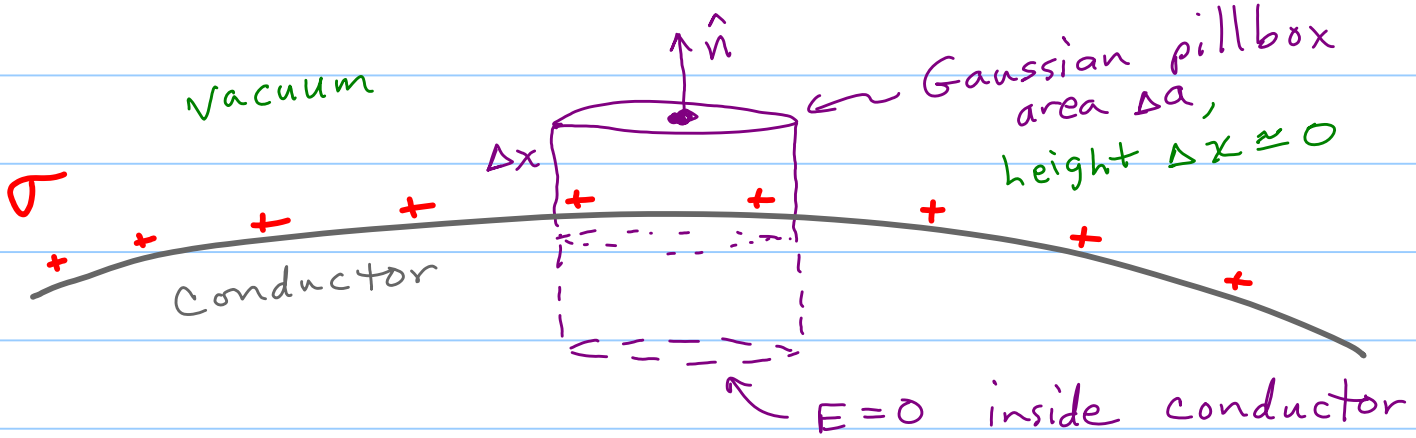
Namely, U in Eq. (1) can be positive OR negative, but U in Eq. (2) is positive definite!

Resolution This is because Eq. 2 includes the "self-energy" terms, i.e. when $\vec{x} = \vec{x}'$, whereas Eq(1) excludes them!

Electrostatic Pressure

Consider the situation where charge density σ is on the surface of a conductor.

$\Rightarrow$ Electrostatic forces cause an outward pressure, as we now explore in detail:



Gauss's Law

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} = E\,\Delta a = \frac{\sigma\,\Delta a}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} \Rightarrow$$ at the surface there is an energy density

$$u(\vec{x}) = \frac{\epsilon_0}{2}\frac{\sigma^2}{\epsilon_0^2} = \frac{\text{energy}}{\text{unit volume}}$$

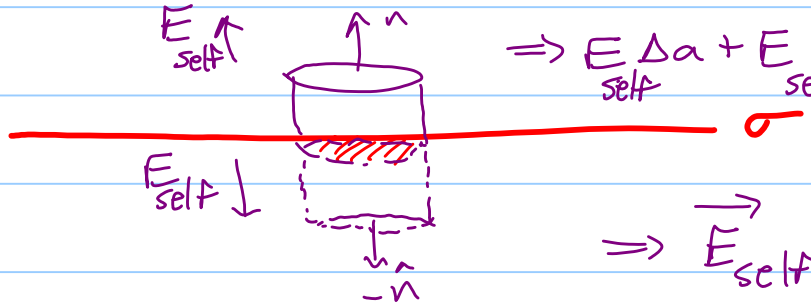
Suppose we now move a differential surface area element outward, in the direction $\hat{n}$, by a small distance Δx

$\Rightarrow$ the E-field does work

$$\Delta W = F_{other}\,\Delta x = (\sigma\,\Delta a)\,E_{other}\,\Delta x$$

In order to deduce what "E_{other}" is for this problem, consider what the electric field produced by this σ would be if this thin layer of charge were not on a conductor

$\Rightarrow$ Then



$$\Rightarrow E_{self}\,\Delta a + E_{self}\,\Delta a = \frac{\sigma}{\epsilon_0}\,\Delta a$$

$$\Rightarrow \vec{E}_{self} = \begin{cases} \frac{\sigma}{2\epsilon_0}\,\hat{n}, & \text{above} \\ -\frac{\sigma}{2\epsilon_0}\,\hat{n}, & \text{below} \end{cases}$$

The other charges in the universe must therefore produce an electric field $\vec{E}_{other} = \frac{\sigma}{2\epsilon_0}\,\hat{n}$, which cancels $\vec{E}_{self}$ below the surface (inside the conductor), and doubles the total field outside. We are interested only in the force due to those "other" charges, since charge cannot exert a net force on itself.

$\Rightarrow \vec{F}_{other} = (\sigma\,\Delta a)\,\frac{\sigma}{2\epsilon_0}\,\hat{n}$, or the $\frac{\text{force}}{\text{unit area}}$ = PRESSURE is

$$\boxed{\frac{\Delta \vec{F}}{\Delta a} = \frac{\sigma^2}{2\epsilon_0}\,\hat{n}}$$