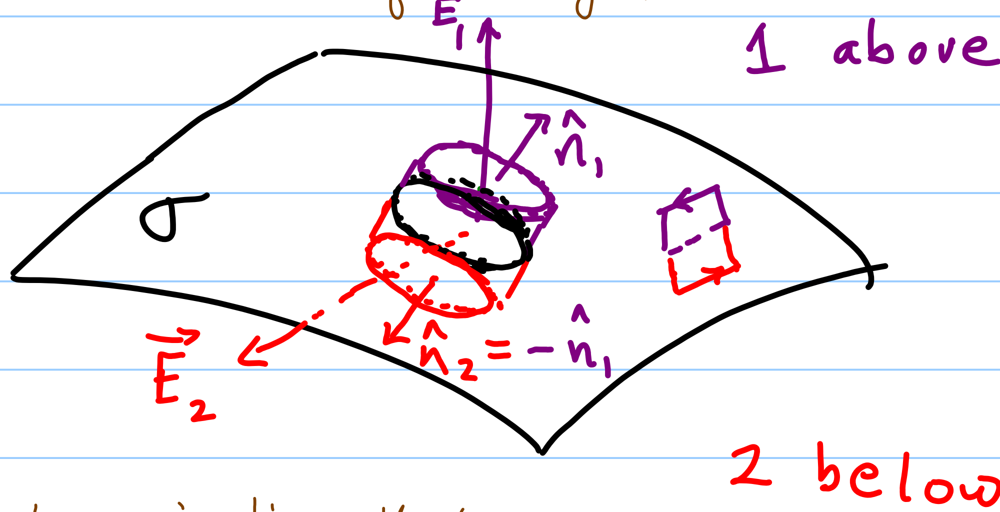
Implications of a surface charge distribution

Consider the electric field very close to a surface distribution of charge, σ



Now, Gauss's Law implies that

$$\oint \vec{E}\cdot\vec{da} = (\vec{E}_1\cdot\hat{n}_1 + \vec{E}_2\cdot\hat{n}_2)\,da = \frac{\sigma\,da}{\epsilon_0}$$

or $$\boxed{(\vec{E}_1 - \vec{E}_2)\cdot\hat{n}_1 = \frac{\sigma}{\epsilon_0}}$$

and since $\oint \vec{E}\cdot\vec{dl} = 0$, $\Rightarrow \boxed{\vec{E}_1\cdot\vec{l}_t = \vec{E}_2\cdot\vec{l}_t}$

where $\vec{l}_t$ is any vector lying in the plane tangent to the surface at that point.

Similarly, consider the effect of a dipole layer

i.e. Consider a layer of oriented dipole molecules on a surface, consisting of surface charge densities $+\sigma(\vec{x})$ and $-\sigma(\vec{x})$ separated by a tiny distance $a(\vec{x})$.

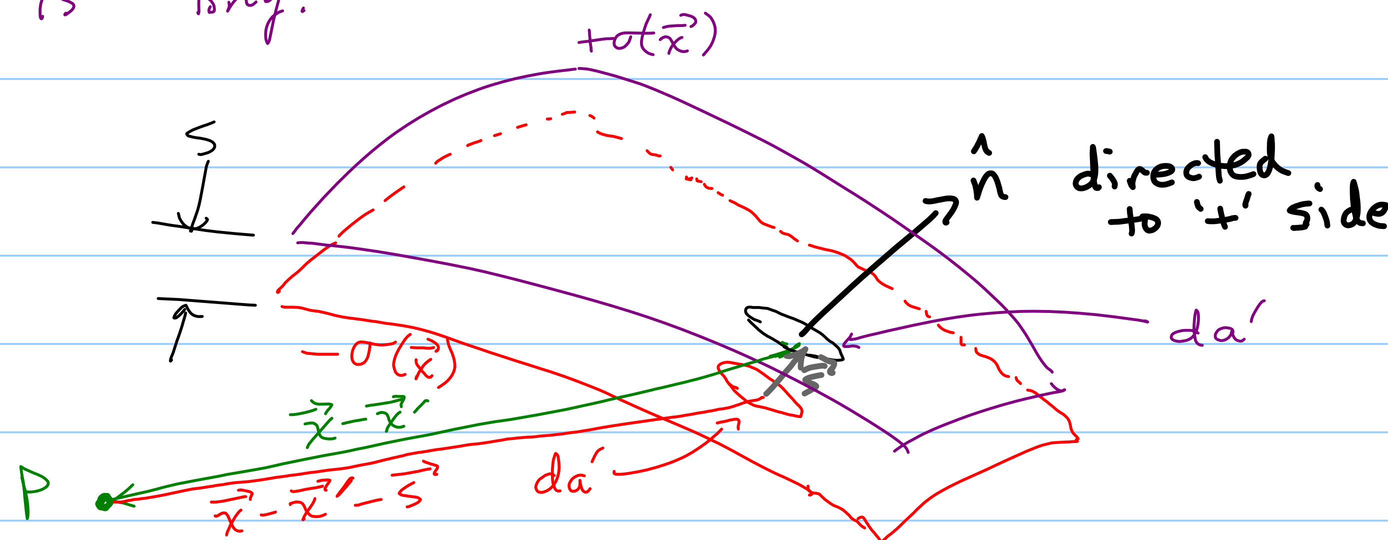
Then define $\mathbb{D}(\vec{x}) = \sigma(\vec{x})\, s(\vec{x})$

Alternatively one can view a dipole layer as some $\dfrac{\#\text{ dipole molecules}}{\text{unit area}} \equiv n(\vec{x})$

each molecule having dipole moment $\vec{d} = e\vec{s}$

$$\Rightarrow \quad \mathbb{D}(\vec{x}) = n(\vec{x})\,\vec{d} = n(\vec{x})\, e\vec{s}$$

Assume that the charge separation distance $\vec{s}$ is tiny.



Now, for a single layer of charge σ, the potential is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|}$$

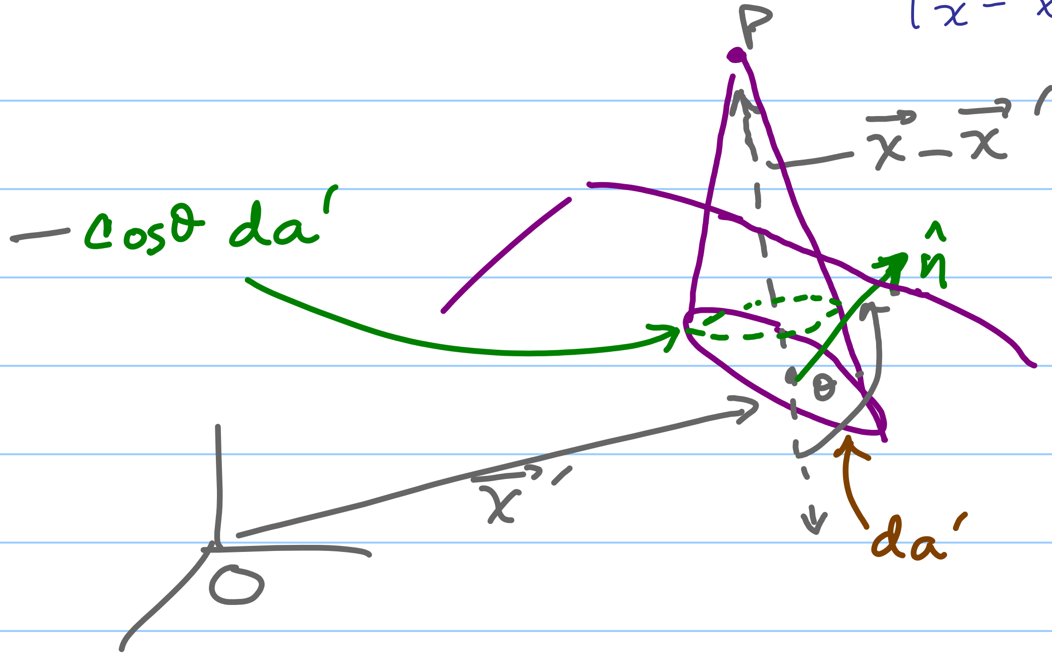
So for our dipole layer we have potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\int_{S'} \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|} - \int \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'+\vec{s}|}\right]$$

Now Taylor expand for small $\vec{s} = \hat{n}s$

$$\frac{1}{|\vec{x}-\vec{x}'+\vec{s}|} \simeq \frac{1}{|\vec{x}-\vec{x}'|} + s\,\hat{n}\cdot\nabla'\frac{1}{|\vec{x}-\vec{x}'|} + O(s^2)$$

$$\Rightarrow \Phi(\vec{x}) = \cancel{-}\frac{1}{4\pi\epsilon_0}\int \sigma(\vec{x}\,')s\,\hat{n}\circ\frac{(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}da'\,\cancel{(-1)}$$



This can be re-expressed as $d\Omega'$, the solid angle subtended by da' from P, i.e.

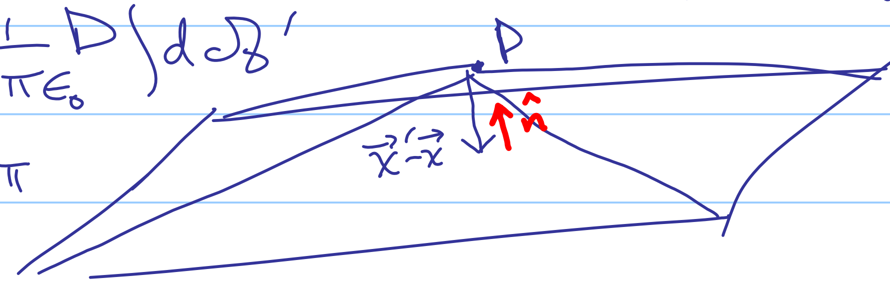
$$-d\Omega' = -\frac{(\vec{x}\,'-\vec{x})\circ\hat{n}\,da'}{|\vec{x}-\vec{x}\,'|^3} =$$ positive($-d\Omega'$) is for $\vec{x}$ <u>above</u> plane

giving

$$\boxed{\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int D(\vec{x}\,')\,d\Omega'}$$

e.g. for a point just <u>above</u> a large dipolar plane, constant $D(\vec{x}) = D$,

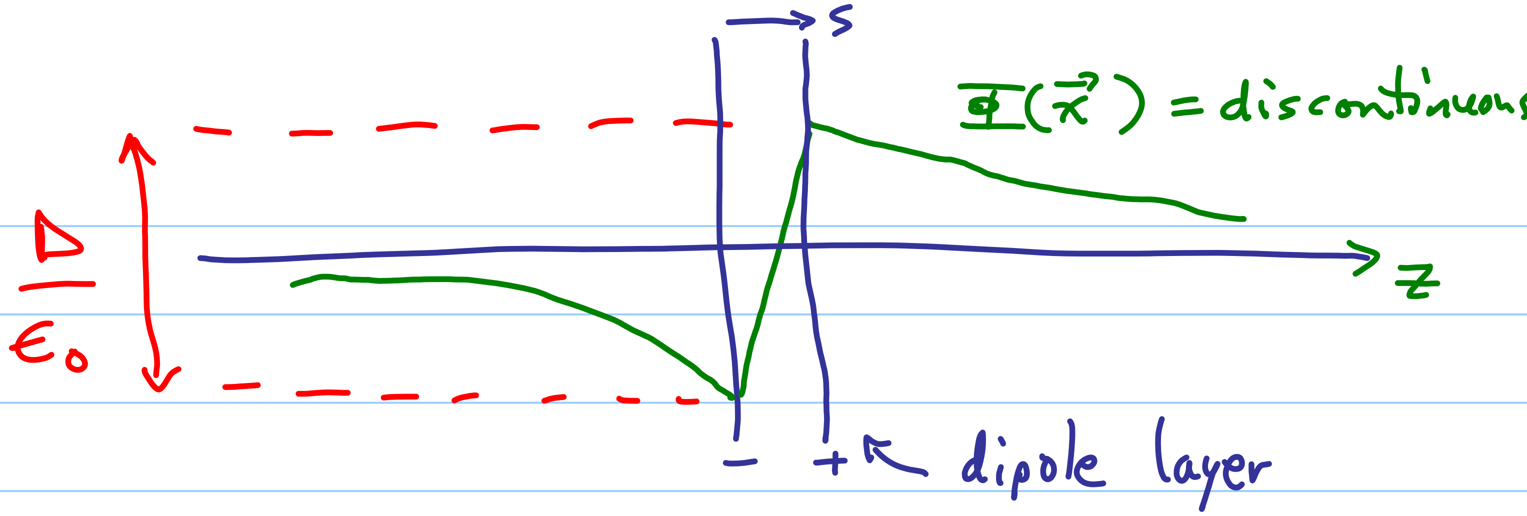
$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}D\int d\Omega'$$

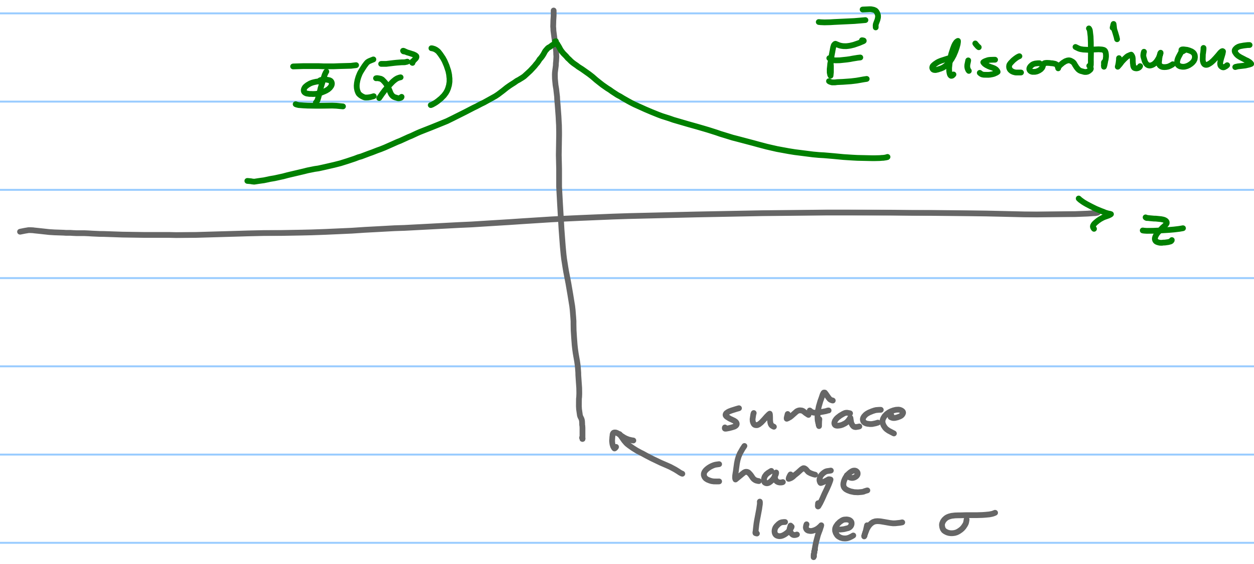


and $\int d\Omega' \rightarrow -2\pi$

so $\Phi^{\text{above}}(\vec{x}) \longrightarrow \dfrac{D}{2\epsilon_0}$, $\Phi^{\text{below}}(\vec{x}) \longrightarrow -\dfrac{D}{2\epsilon_0}$

and in general $\boxed{\Phi^{\text{above}}(\vec{x}) - \Phi^{\text{below}}(\vec{x}) = \dfrac{D}{\epsilon_0}}$





Poisson's and Laplace's equations

Again, a consequence of $\nabla \times \vec{E} = 0$ and $\nabla \cdot \vec{E} = \rho/\epsilon_0$ is that $\vec{E} = -\nabla \Phi$ where Φ obeys Poisson's eqn,

$$\boxed{\nabla^2 \Phi = -\rho/\epsilon_0}$$

A key special case is this equation in free space, with $\rho = 0$, where

$$\boxed{\nabla^2 \Phi = 0} = \text{Laplace's equation}$$

We will spend much time this year solving these kinds of equations.

An important check

Does $$\Phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \frac{d^3x'}{|\vec{x}-\vec{x}'|}$$

truly obey $\nabla^2 \Phi = -\rho/\epsilon_0$?

We can test this by taking derivatives, etc., everywhere except near the singularity at $\vec{x}=\vec{x}'$! Extra caution is always needed at singularities.

e.g., write $$\nabla^2 \Phi = \left\{ \int_{V_1} + \int_{V_2} \right\} \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} d^3x'$$

where V_2 = a tiny sphere of radius ϵ about $\vec{x}'=\vec{x}$

V_1 = everywhere else, outside of V_2, where $\vec{x}' \neq \vec{x}$

So within V_1 we just differentiate normally

$$\Rightarrow \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -\nabla \cdot \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right)$$

and with a variable change to $\vec{r} = \vec{x}-\vec{x}'$

$$\Rightarrow \nabla_x = \nabla_r \Rightarrow -\nabla_r \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3} \nabla_r \cdot \vec{r} + \vec{r} \cdot \nabla \frac{1}{r^3}$$

$$= \frac{3}{r^3} - \frac{3\,\vec{r}\cdot\vec{r}}{r^5} = 0 \quad \text{if } \vec{r} \neq 0 \text{, true in } V_1$$

hence $$\int_{V_1} d^3x' \rho(\vec{x}') \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = 0 \;!$$

Next, notice that $$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = \nabla'^2 \frac{1}{|\vec{x}-\vec{x}'|}$$ ← prove to yourself!

and moreover, V_2 is so small that

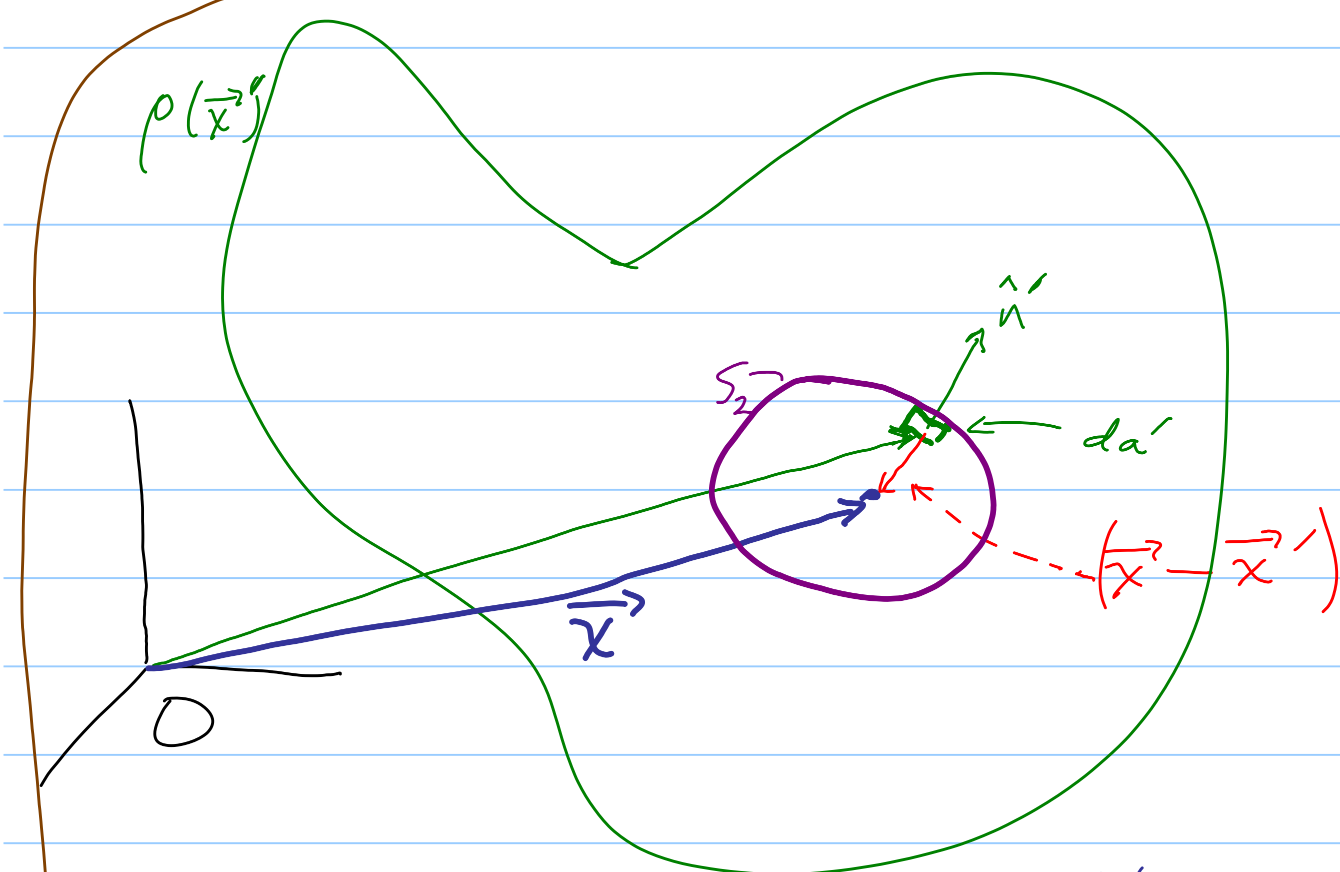
$\rho(\vec{x}') \simeq \rho(\vec{x})$ inside V_2,

i.e. approx. constant w.r.t. $\vec{x}'$

Thus $\displaystyle\int_{V_2} \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} d^3x'$

$$\longrightarrow \frac{\rho(\vec{x})}{4\pi\epsilon_0} \int_{V_2} \nabla' \cdot \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3x'$$

$$= \frac{\rho(\vec{x})}{4\pi\epsilon_0} \oint_{S_2} \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \cdot \hat{n}' da' \quad \begin{pmatrix}\text{divergence} \\ \text{theorem}\end{pmatrix}$$



Note from this diagram that $\hat{n}' \cdot (\vec{x}-\vec{x}') = -|\vec{x}-\vec{x}'|$

$$= -\frac{\rho(\vec{x})}{4\pi\epsilon_0} \oint_{S_2} d\Omega' = -\frac{\rho(\vec{x})}{\epsilon_0}$$

Conclusions: We have thus proved that

$$\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}$$ as was desired

Moreover, we have demonstrated that $$\nabla^2 \frac{1}{|\vec{x}-\vec{x}\,'|} = -4\pi\,\delta(\vec{x}-\vec{x}\,')$$

GREEN'S THEOREM

In some problems we don't know ρ or σ everywhere, but we can still determine Φ, $\vec{E}$.
e.g. a conductor has $\vec{E}=0$ inside, whereby
$\Phi(\vec{x}) = \text{constant}$ everywhere on + through it.
In such a problem, we might know Φ on the conductor, but not ρ.

Green's function methods are particularly useful for such problems.

I. 1^{st} Green Identity

Consider two scalar functions ϕ, ψ. They obey the following identity

$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$$

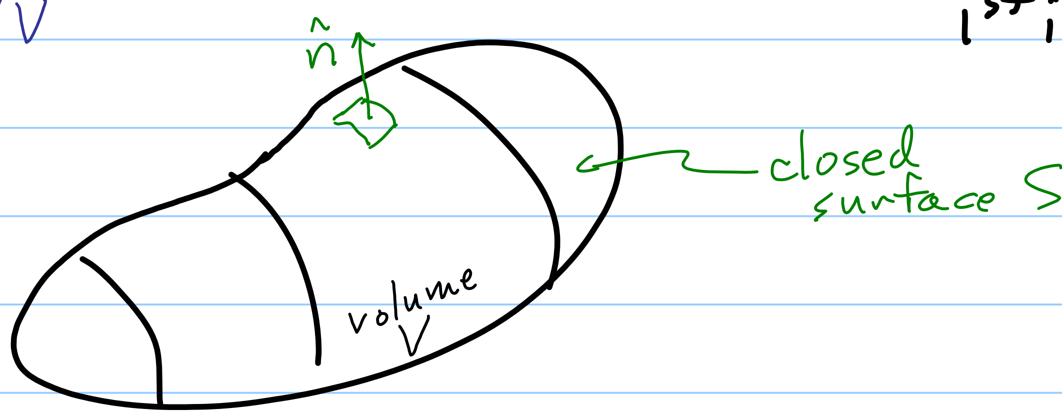
Next, integrate this over an arbitrary volume V bounded by a closed surface S:

outward normal partial derivative

$$\int_V \nabla\cdot(\phi\nabla\psi)\,d^3x \overset{\text{divergence theorem}}{=} \oint_S \phi\nabla\psi\cdot\hat{n}\,da \equiv \oint_S \phi\frac{\partial\psi}{\partial n}\,da$$

$$= \int_V \left(\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi\right)d^3x$$

= Green's 1st identity



II. Green's Theorem (2nd identity)

To derive it, just rewrite the 1st identity with ϕ, ψ interchanged, and subtract:

i.e.

$$\begin{pmatrix} \int_V (\cancel{\nabla\phi\cdot\nabla\psi} + \phi\nabla^2\psi)\,d^3x \\ - \int_V (\cancel{\nabla\psi\cdot\nabla\phi} + \psi\nabla^2\phi)\,d^3x \end{pmatrix} = \begin{pmatrix} \oint_S \phi\frac{\partial\psi}{\partial n}\,da \\ - \oint \psi\frac{\partial\phi}{\partial n}\,da \end{pmatrix}$$

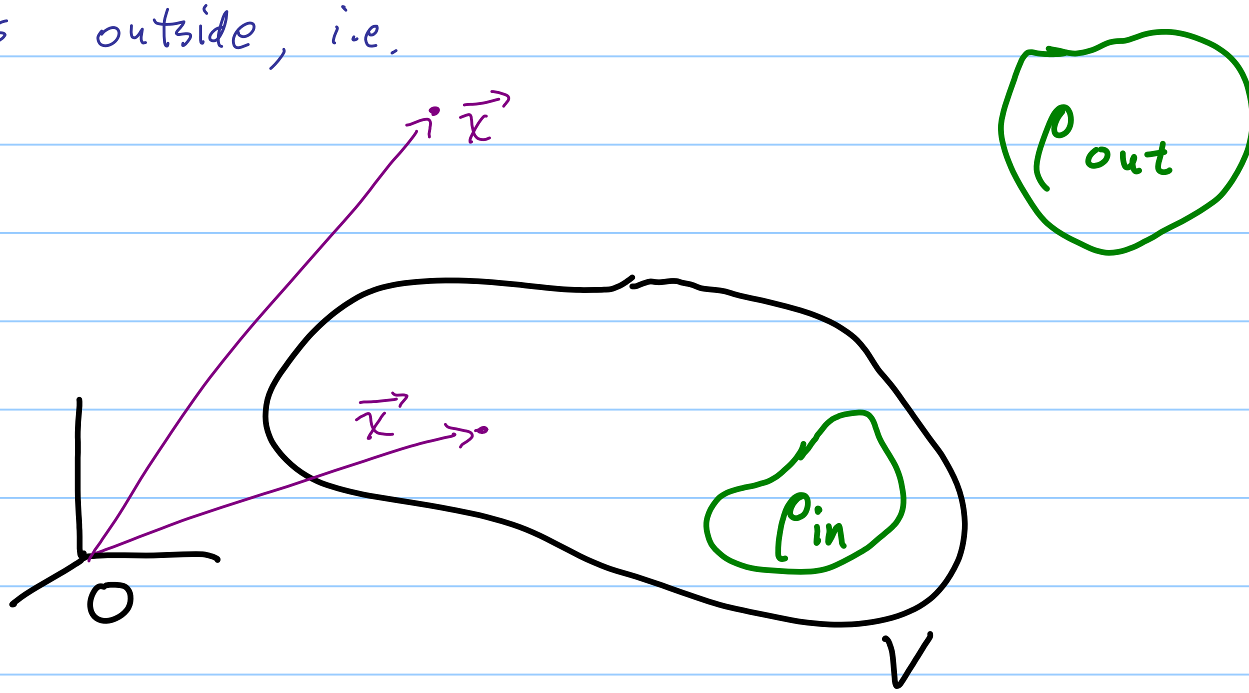
$$\Rightarrow \boxed{\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right)d^3x = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da}$$

Kewl!

Integral Equation Next we convert the Poisson PDE into an integral equation, starting from Poisson's eqn for $\Phi(\vec{x})$, written in terms of $\vec{x}'$ instead of $\vec{x}$:

$$\nabla'^2 \Phi(\vec{x}') = -\frac{\rho(\vec{x}')}{\epsilon_0}$$

We consider a volume V that contains a part ρ_{in} of this $\rho(\vec{x})$, while another part ρ_{out} is outside, i.e.



We will consider both cases, where the observation point is either inside or outside V.

Derivation In Green's Theorem let $\phi \to \Phi$, $\psi \to \frac{1}{|\vec{x}-\vec{x}'|}$ and let the integration variable be labelled as $\vec{x}'$

Also, use $\nabla'^2 \psi = -4\pi\,\delta(\vec{x}-\vec{x}')$

Then

$$\int_V \left[\Phi(\vec{x}\,') \overbrace{\nabla'^2 \frac{1}{|\vec{x}-\vec{x}\,'|}}^{-4\pi\delta(\vec{x}-\vec{x}\,')} - \frac{1}{|\vec{x}-\vec{x}\,'|} \overbrace{\nabla'^2 \Phi(\vec{x}\,')}^{-\rho(\vec{x}\,')/\epsilon_0} \right] d^3x'$$

$$= \oint_S \left[\Phi(\vec{x}\,') \underbrace{\frac{\partial}{\partial n'} \frac{1}{|\vec{x}-\vec{x}\,'|}}_{\hat{n}' \cdot \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}} - \frac{1}{|\vec{x}-\vec{x}\,'|} \frac{\partial}{\partial n'} \Phi(\vec{x}\,') \right] da'$$

To simplify the notation, define $R \equiv |\vec{x}-\vec{x}\,'|$.
Then if the point $\vec{x}$ lies within V, we have

$$-4\pi \int_V \Phi(\vec{x}\,') \delta(\vec{x}-\vec{x}\,') d^3x' = -4\pi \Phi(\vec{x})$$

$$= -\frac{1}{\epsilon_0} \int_V \frac{\rho(\vec{x}\,') d^3x'}{|\vec{x}-\vec{x}\,'|} + \oint_S \left[\Phi(\vec{x}\,') \frac{\partial}{\partial n'} \frac{1}{R} - \frac{1}{R} \frac{\partial \Phi(\vec{x}\,')}{\partial n'} \right] da'$$

OR

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}\,') d^3x'}{|\vec{x}-\vec{x}\,'|} + \frac{1}{4\pi} \oint_S \left[\frac{1}{R} \frac{\partial}{\partial n'} \Phi(\vec{x}\,') - \Phi(\vec{x}\,') \frac{\partial}{\partial n'} \frac{1}{R} \right] da'$$

notice that this integral only includes the charges INSIDE V!

observation If there are NO charges INSIDE V then $\Phi(\vec{x})$ is determined ENTIRELY by the values of Φ, $\frac{\partial \Phi}{\partial n'}$ on S.

However this overdetermines the solution of the problem, since these are then Cauchy Boundary Conditions

(we discuss this issue later)

Next, what if $\vec{x}$ lies outside of the volume V?

Of course the δ-function integral now vanishes, so we get a different identity, namely

$$0 = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}\,')\,d^3x'}{|\vec{x}-\vec{x}\,'|} + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}\,')\frac{\partial}{\partial n'}\frac{1}{R}\right]da',$$

which tells us nothing about $\Phi(\vec{x})$ outside!

Apparent error in Jackson, p. 37, end of top paragraph, where he writes that this result gives:
"... zero field and zero potential outside the volume V."

Variational Formulation

Consider the following functional $I[\Phi]$,

$$I[\Phi] \equiv \frac{1}{2}\int_V \nabla\Phi\cdot\nabla\Phi\, d^3x - \int_V \frac{\rho}{\epsilon_0}\Phi\, d^3x$$

$\Rightarrow$ We can prove that this expression is in fact a variational principle that can be used to determine Φ. Consider this expression with $\Phi \longrightarrow \Phi + \delta\Phi$ ← a "small deviation"

the exact solution

We can find δI by taking the 1st variation, i.e $\quad I[\Phi + \delta\Phi] \longrightarrow I[\Phi] + \delta I$

where $\displaystyle \delta I = \int_V \nabla\Phi \cdot \nabla(\delta\Phi)\, d^3x - \int_V \frac{\rho}{\epsilon_0}\delta\Phi\, d^3x$

application of Green's first identity gives

$$\delta I = \int_V \left(-\nabla^2\Phi - \frac{\rho}{\epsilon_0} \right)\delta\Phi\, d^3x + \oint_S \frac{\partial\Phi}{\partial n}\delta\Phi\, da$$

0 since Φ = exact solution

Hence $\delta I = 0$, i.e. the 1^{st} variation vanishes PROVIDED we only allow variations $\delta\Phi = 0$, that vanish on S.

This variational principle can be used by choosing a trial potential $\Phi_t[a, b, c, \ldots]$ that depends on some set of parameters $\{a, b, c, \ldots\}$, and then choosing those parameters that give an extremum for I.

Uniqueness - What is the minimum amount of information needed to uniquely specify Φ within a volume V.

We saw already that

$$\Phi(\vec{x}) = \int_V \frac{\rho(\vec{x}')\, d^3x'}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|} + \frac{1}{4\pi}\oint_S \left[\Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}'|}\frac{\partial}{\partial n'}\Phi(\vec{x}') \right] da'$$

But starting from BOTH Φ AND $\frac{\partial \Phi}{\partial n}$ on S overspecifies the solution,

$\Rightarrow$ it can lead to internal inconsistency in the solution

~~Ins~~ Instead, we consider 2 types of boundary conditions:

EITHER:

1) Specify $\Phi(\vec{x})$ on S $\Leftarrow$ "Dirichlet BCs"

(if $\Phi(\vec{x}) = 0$ on S, call them homogeneous-Dirichlet BCs)

2) Specify $\frac{\partial \Phi}{\partial n}$ on S $\Leftarrow$ "Neumann BCs"

(if $\frac{\partial \Phi}{\partial n} = 0$ on S $\Rightarrow$ homogeneous Neumann BCs)

THEOREM knowledge of ρ inside of V, and either Dirichlet OR Neumann BCs, is enough to specify Φ in V uniquely.

Proof Suppose Φ_1 and Φ_2 are 2 solutions obeying:

(i) $\nabla^2 \Phi_i = -\rho/\epsilon_0$ in V

(ii) $\Phi_1 = \Phi_2$ on S (Dirichlet case) OR

$\frac{\partial \Phi_1}{\partial n'} = \frac{\partial \Phi_2}{\partial n'}$ on S (Neumann case)