Physics 630
Classical electrodynamics
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Our goal this semester is to study the equations and phenomena of classical E+M, which follow from that tremendous synthesis into 4 vector equations, by James Clerk Maxwell. In vacuum, written in SI units, Maxwell's eqns are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (1)$$

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad (2)$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad (3)$$

$$\nabla \cdot \vec{B} = 0 \qquad (4)$$

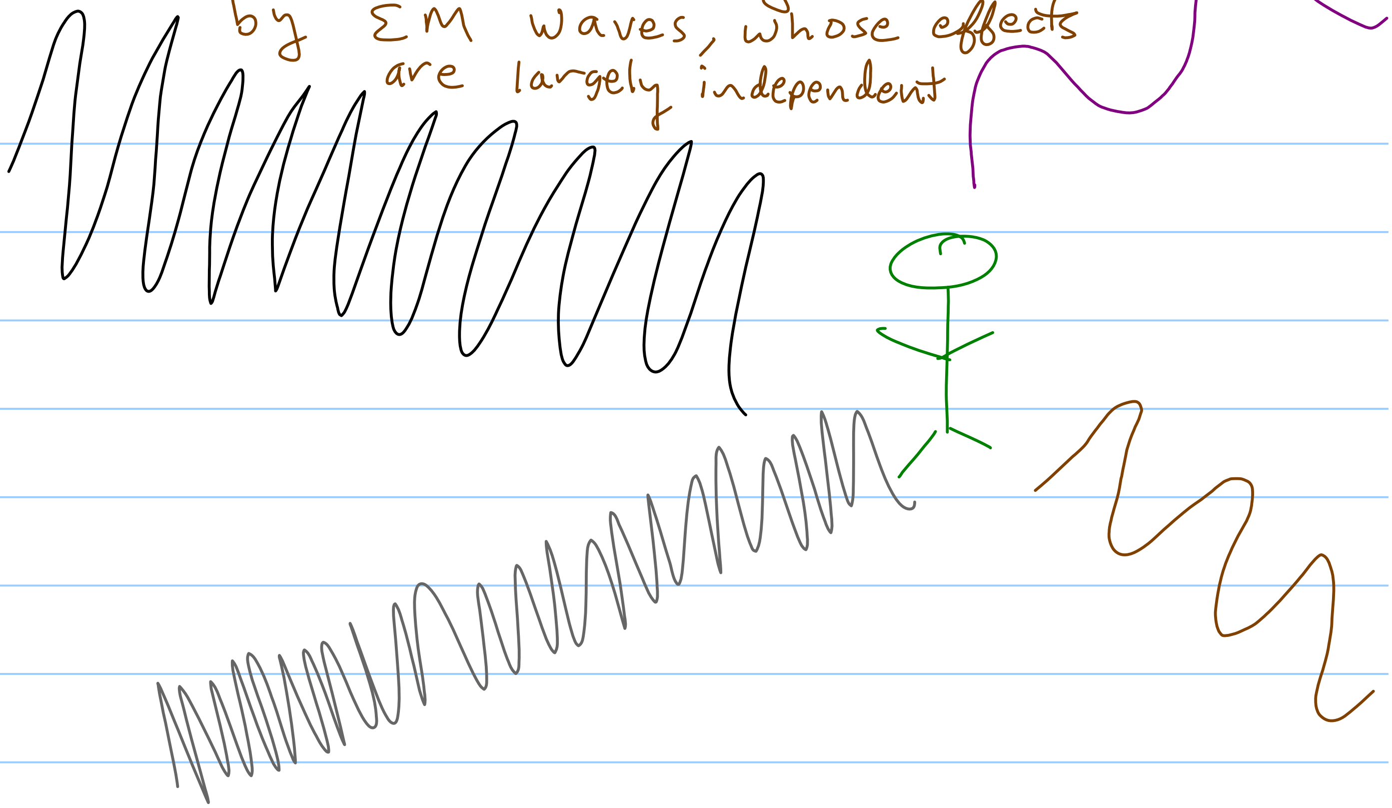
and these equations imply an auxiliary equation, derived by taking $\nabla \cdot (2)$ and plugging in (1):

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (5)$$

charge conservation law

IMPORTANT: Maxwell's equations are LINEAR, first-order differential equations

We are constantly bombarded by EM waves, whose effects are largely independent



We also must describe the effects of $\vec{E}, \vec{B}$ on the motion of a charged particle q. In classical mechanics this is the Lorentz force law,

$$\boxed{\vec{F} = q\vec{E}(\vec{x},t) + q\,\frac{d\vec{x}}{dt} \times \vec{B}(\vec{x},t)} \qquad (6)$$

$\vec{v}$

$\vec{E}(\vec{x},t)$ = electric field at position $\vec{x}$, time t, $\left(\frac{V}{m}\right)$ or $\left(\frac{N}{C}\right)$

$\vec{B}(\vec{x},t)$ = magnetic field " " ", time t, (T)

$\rho(\vec{x},t)$ = charge density, $\left(\frac{\text{Coulombs}}{m^3}\right)$

$\vec{J}(\vec{x},t)$ = current density, $\left(\frac{\text{Coulombs}}{m^2 \cdot s}\right)$

And in a linear, homogeneous, dielectric medium, $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$

whereby (1) becomes $\nabla \cdot \vec{D} = \rho$ (1′)

(2) becomes $\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$ (2′)

c = vacuum light speed $\equiv 299792458 \frac{m}{s}$, <u>EXACTLY</u>

$= \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

The first step in development of the subject of E & M was the discovery of Coulomb's Law: For 2 point charges in a vacuum, the Coulomb force on q_1 due to q_2 is

$$\vec{F}_{1,2} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

Since the force on q_1 is proportional to q_1 it is useful to define the <u>electric field</u> at $\vec{x}_1$, independently of whether q_1 is actually there, i.e. as

$$\vec{E}(\vec{x}_1) = \lim_{q_1 \to 0} \frac{\vec{F}_{1,2}}{q_1} = \frac{q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

If there are many charges, their net electric field at $\vec{x}$ is the sum

$$\vec{E}(\vec{x}) = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}$$

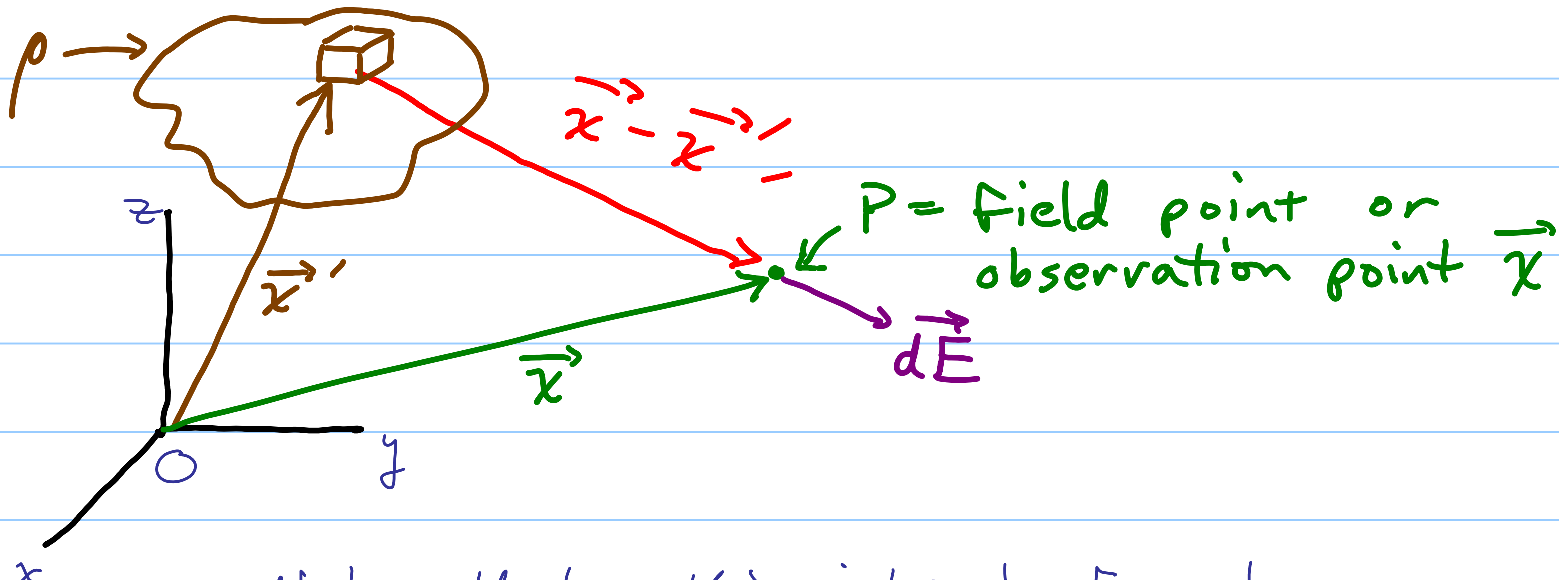
which generalizes for a <u>continuous</u> distribution of charges, through the substitutions

$$\vec{x}_i \longrightarrow \vec{x}\,'$$

$$q_i \longrightarrow dq' \equiv \rho(\vec{x}\,')\, d^3x',$$

giving the integral

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \rho(\vec{x}\,') \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3}}$$



Note that this integral formula can also be used for DISCRETE charges, provided we identify

$$\rho(\vec{x}\,') = \sum_i q_i \, \delta(\vec{x}\,' - \vec{x}_i)$$

Idealization Actually, <u>all</u> charges are discrete, so the most accurate picture would use <u>sums</u> only, but since there are often Avogadro's number of charges this is impractical. So approximating the sums by integrals is more practical and usually quite accurate.

Also, in classical E+M it is frequently stated that Gauss's Law implies that any net charge on a conductor in electrostatics must reside on its <u>outermost surface</u>, i.e. within an infinitesimally thin layer.

More realistic models including the quantum wavelike nature of the electrons shows (see Fig I.5) that the surface layer actually has a finite thickness of approximately $\Delta r \simeq 2\,\text{Å}$

While this is not infinitely thin, it is thin enough so that for most applications it can be treated that way, i.e.

$$\rho(\vec{x}') = \sigma\,\delta(x'-R)$$

Math Review: Dirac delta functions

(1) $$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \text{"}\infty\text{"}, & x = a \end{cases}$$

(2) $$\int_c^d \delta(x-a)\,dx = \begin{cases} 1, & \text{if } c<a<d \\ -1, & \text{if } c>a>d \\ 0, & \text{otherwise} \end{cases}$$

(Note: more careful consideration is needed if either $c=a$ or $d=a$.)

explicit representations

$$\frac{1}{2}\frac{d^2}{dx^2}|x| = \delta(x)$$

$$\delta(x) = \pi^{-\frac{1}{2}} \lim_{\epsilon \to 0^+} \epsilon^{-1/2} \exp(-x^2/\epsilon)$$

$$\delta(x) = \pi^{-1} \lim_{\epsilon \to 0^+} \frac{\epsilon}{x^2+\epsilon^2}$$

$$\delta(x) = \pi^{-1} \lim_{N \to \infty} \frac{\sin Nx}{x}$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

(3) $$\int_{-\infty}^{\infty} dx\, f(x)\,\delta(x-a)\,dx = f(a), \quad a = \text{real}$$

proof $$\int_{-\infty}^{\infty} dx\, f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-a)}\,dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, e^{-ika}\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\, e^{ikx} = f(a),$$

since this is the inverse Fourier transform of a Fourier transform.

(4) $$\int_{-\infty}^{\infty} f(x)\,\delta'(x-a)\,dx = f(x)\,\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x-a)\,dx$$

$$= -f'(a)$$

This result is sometimes written as

$$f(x)\frac{d}{dx}\delta(x-a) = -\delta(x-a)\frac{d}{dx}f(x)$$

(5) $$\delta(ax) = \frac{1}{|a|}\delta(x)$$

corollary $$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$$

provided $f(x)$ has only simple zeroes at x_i

e.g. $$\delta[(x-a)(x-b)] = \frac{1}{|a-b|}\left(\delta(x-a)+\delta(x-b)\right)$$

(6) δ-functions in multiple dimensions

$$\delta(\vec{x}-\vec{x}') = \delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(x_3-x_3')$$

(in Cartesian coordinates)

This vanishes except at the point $\vec{x}=\vec{x}'$

i.e. $$\int_V \delta(\vec{x}-\vec{x}')\,d^3x' = \begin{cases} 1 & \text{if } \vec{x} \text{ is in } V \\ 0 & \text{otherwise} \end{cases}$$

3-dimensional representation

$$\delta(\vec{x}-\vec{x}\,') = (2\pi)^{-3}\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}$$

If we consider general orthogonal, curvilinear coordinates, q_1, q_2, q_3
with differential arc length,

$$ds^2 = h_1^2\, dq_1^2 + h_2^2\, dq_2^2 + h_3^2\, dq_3^2$$

and volume element $d^3x = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$,

then
$$\delta(\vec{x}-\vec{x}\,') = \frac{1}{h_1 h_2 h_3}\,\delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

e.g. in spherical coordinates,

$$\delta(\vec{x}-\vec{x}\,') = \frac{1}{r^2\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

$$= \frac{1}{r^2}\,\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

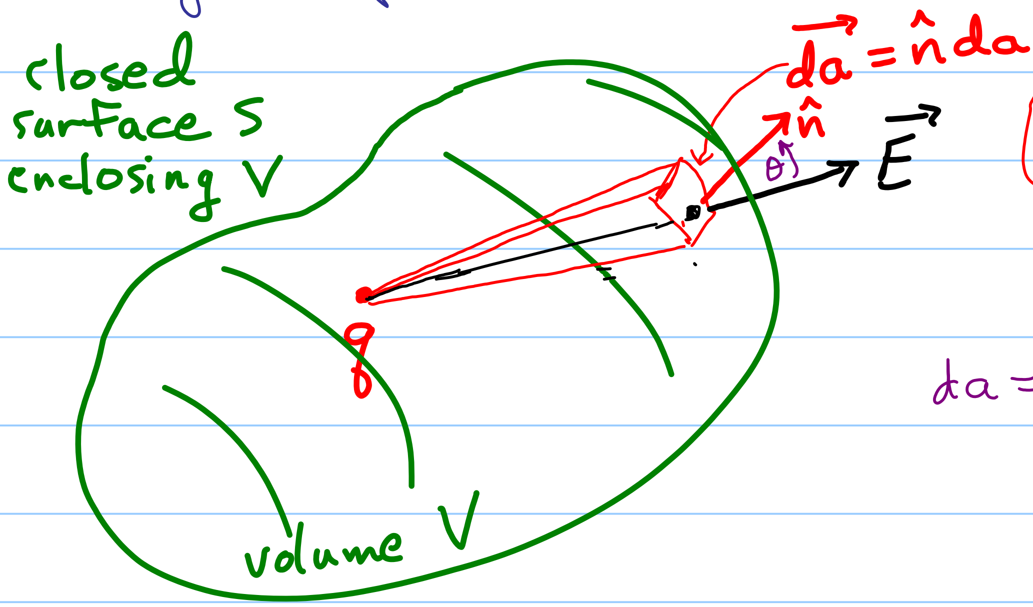
(7) another identity: $\nabla^2\frac{1}{r} = -4\pi\,\delta(\vec{r})$

whereby we see that

$$\frac{1}{r} = \underbrace{\left[4\pi\,(2\pi)^{-3}\right]}_{=\,(2\pi^2)^{-1}}\int d^3k\, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2}$$

Gauss's Law – This is often more convenient than evaluating a 3-dimensional Coulomb's Law integral. It also exemplifies the intuitive power and beauty of geometrically reinterpreting a math formula.

To demonstrate this, consider a point charge q at a point in space, and draw an imaginary closed surface S surrounding it:



$\hat{n}$ = outwardly directed unit vector normal to S

$da = |\vec{da}|$ = magnitude of differential area element

Now observe that $\vec{E} \cdot \vec{da} = \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{\cos\theta \, da}{r^2}\right)$

But $d\Omega = \cos\theta \dfrac{da}{r^2}$,

where $d\Omega$ = differential solid angle subtended by da from charge q.

So we can write

$$\oint_S \vec{E} \cdot \vec{da} = \oint_S \frac{q}{4\pi\epsilon_0} d\Omega = \begin{cases} q/\epsilon_0, & \text{if } q \text{ is in } V \\ 0, & \text{if } q \text{ is outside } V \end{cases}$$

Of course, this argument generalizes if there are many charges, i.e. by superposition which is permitted by the linearity of the equations, giving the key result:

$$\boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{q_{enclosed}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V d^3x' \rho(\vec{x}')}$$

where V = volume enclosed by S

Next, apply the divergence theorem:

$$\oint_S \vec{E}\cdot d\vec{a} = \overbrace{\int_V \nabla\cdot\vec{E}\, d^3x = \frac{1}{\epsilon_0}\int_V d^3x\, \rho(\vec{x})}$$

and since the second equality holds for an ARBITRARY surface + volume, the integrands must themselves be equal everywhere, i.e.

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$

is equivalent to $\oint_S \vec{E}\cdot d\vec{a} = \frac{q_{enclosed}}{\epsilon_0}$

= Gauss's Law in differential and integral form

Electrostatic Potential

Consider

$$\nabla_x \frac{1}{|\vec{x}-\vec{x}'|} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= \frac{\hat{x}(-\frac{1}{2})(2)(x-x') + \hat{y}(-\frac{1}{2})(2)(y-y') + \hat{z}(-\frac{1}{2})(2)(z-z')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

$$= -\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

(Aside: sometimes we will abbreviate $\nabla_x \equiv \nabla$, $\nabla_{x'} \equiv \nabla'$)

Then Coulomb's Law can be rewritten as:

$$\vec{E}(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\int d^3x' \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \nabla \frac{1}{|\vec{x}-\vec{x}'|}$$

or we can write $\boxed{\vec{E} = -\nabla \Phi(\vec{x})}$

where

$$\Phi(\vec{x}) = \int \frac{d^3x'}{4\pi\epsilon_0} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \equiv \text{Electrostatic Potential}$$

Comments ① since $\nabla \times \nabla f = 0$ <u>always</u> $\Rightarrow \nabla \times \vec{E} = 0$ in electrostatics, always!

② This integral expression for Φ is not the most general, since it has implicitly assumed the B.C. that $\Phi(\vec{x}) \to 0$ as $r \to \infty$, whenever ρ has a finite extent.

③ If $\Phi(\vec{x})$ is one solution, then an equally good solution is $\Phi(\vec{x})$ + constant, since both give the same $\vec{E}(\vec{x})$

④ Don't forget that you need BOTH

$$\nabla \cdot \vec{E} = \rho / \epsilon_0$$

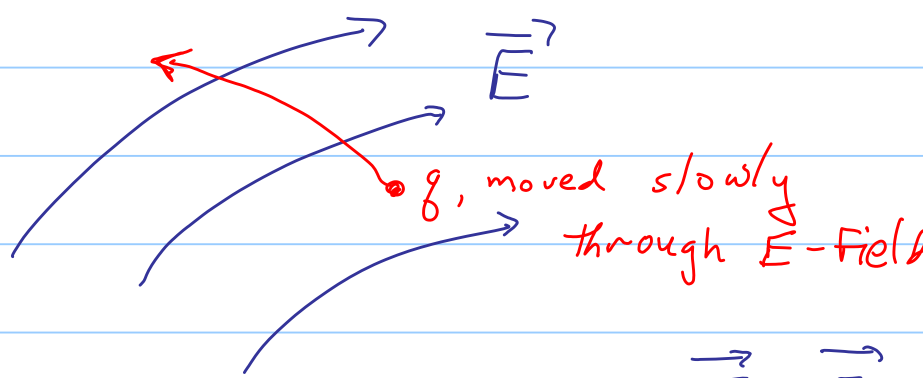
AND

$$\nabla \times \vec{E} = 0$$

to specify $\vec{E}$ completely, i.e. to within the gradient of a scalar function F that obeys Laplace's equation, $\nabla^2 F = 0$.

Relationship to energy

Let's try to discuss this carefully, since it is easy to make sign errors when discussing work and potential energy.



The electric force on a charge q is $\vec{F} = q\vec{E}$
so the force WE apply to hold q fixed, or to move it very slowly, is $\vec{F}_{us} = -q\vec{E}$

Thus the WORK that WE do in slowly moving q from $\vec{x}_1$ to $\vec{x}_2$ along some path $\mathcal{P}$ is

$$W_2 - W_1 = \int_{\vec{x}_1,\ \mathcal{P}}^{\vec{x}_2} \vec{F}_{us} \cdot \vec{dl} = - \int_{\vec{x}_1,\ \mathcal{P}}^{\vec{x}_2} \vec{F} \cdot \vec{dl}$$

or $$W_2 - W_1 = -q \int_{\vec{x}_1}^{\vec{x}_2} \vec{E} \cdot \vec{dl} = +q \int_{\vec{x}_1}^{\vec{x}_2} \nabla \Phi \cdot \vec{dl}$$

$$= q \int_{\vec{x}_1}^{\vec{x}_2} d\Phi = q\,\Phi(\vec{x}_2) - q\,\Phi(\vec{x}_1)$$

Thus we can interpret $q\,\Phi(\vec{x})$ as the

ELECTRIC POTENTIAL ENERGY

i.e. if we do positive work, the potential energy must increase if kinetic energy = constant

observations

① The work done is path-independent

$$-q \int_{\vec{x}_1}^{\vec{x}_2} \vec{E} \cdot \vec{dl} = q\,\Phi(\vec{x}_2) - q\,\Phi(\vec{x}_1)$$

② $$\int_{\vec{x}_1}^{\vec{x}_1} \vec{E}(\vec{x}) \cdot \vec{dl} = \oint_C \vec{E}(\vec{x}) \cdot \vec{dl} = 0$$

for ANY closed loop

③ $$\oint_C \vec{E} \cdot \vec{dl} = \int_S \nabla \times \vec{E} \cdot \vec{da} = 0$$ is consistent with $\nabla \times \vec{E} = 0$