Physics 630
Classical electrodynamics
Instructor: Chris Greene

Our goal this semester is to study the equations and phenomena of classical E+M, which follow from that tremendous synthesis into 4 vector equations, by James Clerk Maxwell. In vacuum, written in SI units, Maxwell's eqns are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (1)$$

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad (2)$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad (3)$$

$$\nabla \cdot \vec{B} = 0 \qquad (4)$$

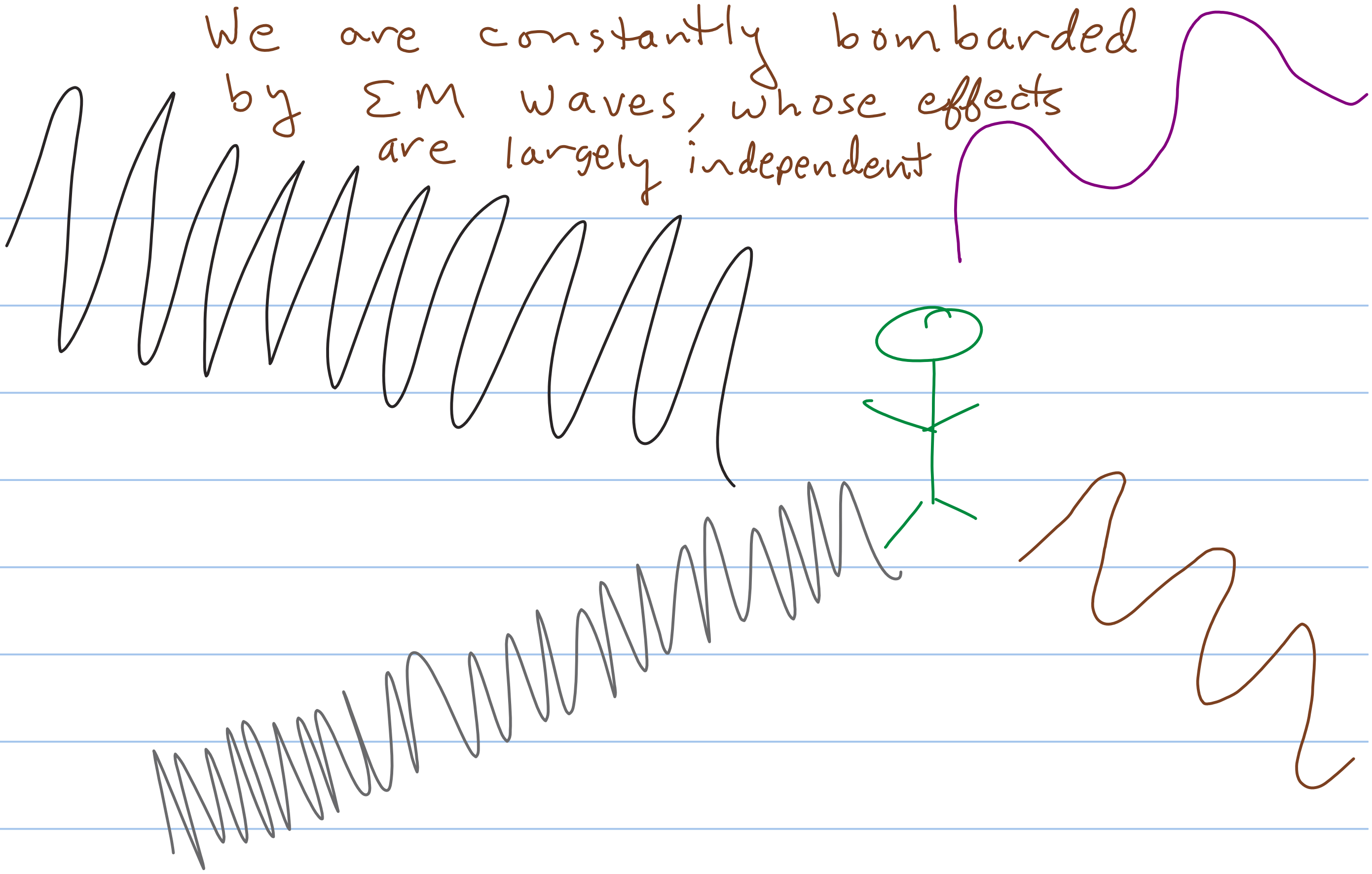
and these equations imply an auxiliary equation, derived by taking $\nabla \cdot (2)$ and plugging in (1):

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (5)$$

charge conservation law

<u>IMPORTANT</u>: Maxwell's equations are <u>LINEAR</u>, first-order differential equations

We are constantly bombarded by EM waves, whose effects are largely independent



We also must describe the effects of $\vec{E}, \vec{B}$ on the motion of a charged particle q.
In classical mechanics this is the Lorentz force law,

$$\boxed{\vec{F} = q\vec{E}(\vec{x},t) + q\underbrace{\frac{d\vec{x}}{dt}}_{\vec{v}} \times \vec{B}(\vec{x},t)} \qquad (6)$$

$\vec{E}(\vec{x},t)$ = electric field at position $\vec{x}$, time t, $\left(\frac{V}{m}\right)$ or $\left(\frac{N}{C}\right)$

$\vec{B}(\vec{x},t)$ = magnetic field " " ", time t, (T)

$\rho(\vec{x},t)$ = charge density, $\left(\frac{\text{Coulombs}}{m^3}\right)$

$\vec{J}(\vec{x},t)$ = current density, $\left(\frac{\text{Coulombs}}{m^2 \cdot s}\right)$

And in a linear, homogeneous, dielectric medium, $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$

whereby (1) becomes $\nabla \cdot \vec{D} = \rho$ (1′)

(2) becomes $\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$ (2′)

c = vacuum light speed $\equiv 299792458 \frac{m}{s}$, EXACTLY

$= \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

The first step in development of the subject of E + M was the discovery of Coulomb's Law: For 2 point charges in a vacuum, the Coulomb force on q_1 due to q_2 is

$$\vec{F}_{1,2} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

Since the force on q_1 is proportional to q_1 it is useful to define the electric field at $\vec{x}_1$, independently of whether q_1 is actually there, i.e. as

$$\vec{E}(\vec{x}_1) = \lim_{q_1 \to 0} \frac{\vec{F}_{1,2}}{q_1} = \frac{q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

If there are many charges, their net electric field at $\vec{x}$ is the sum

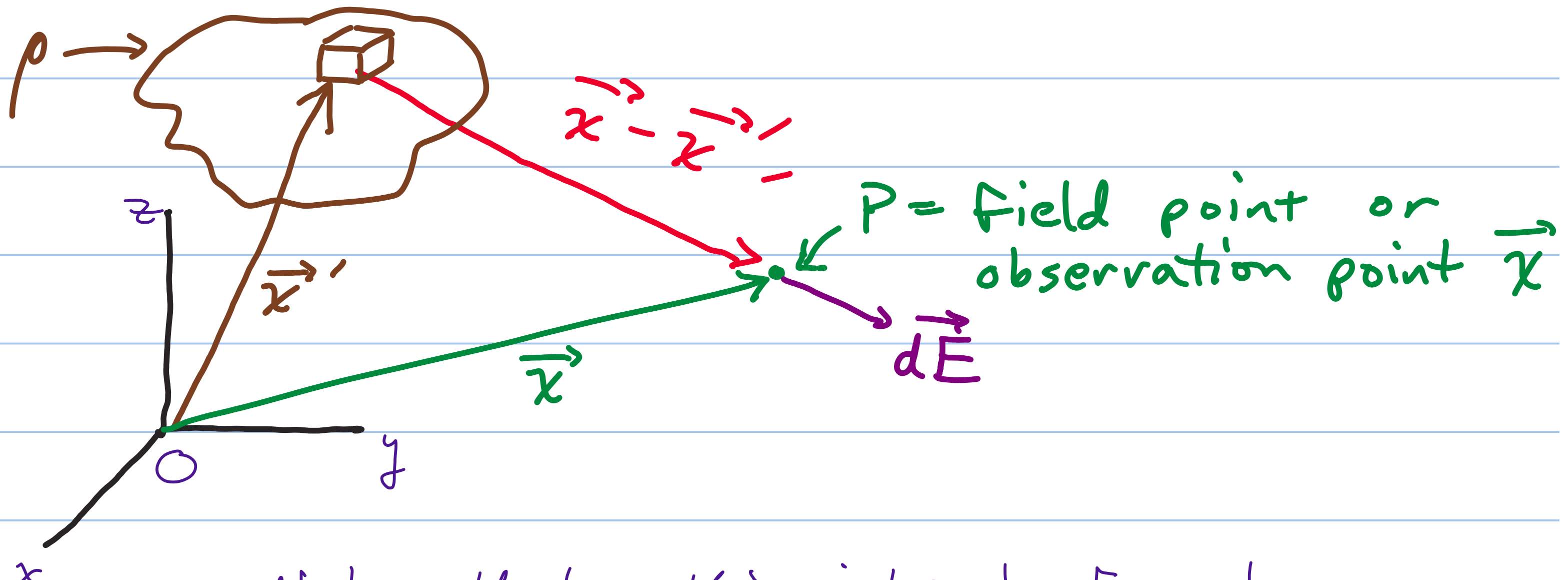
$$\vec{E}(\vec{x}) = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}$$

which generalizes for a continuous distribution of charges, through the substitutions

$$\vec{x}_i \longrightarrow \vec{x}\,'$$
$$q_i \longrightarrow dq' \equiv \rho(\vec{x}\,')\, d^3x',$$

giving the integral

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x'\, \rho(\vec{x}\,') \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}}$$



Note that this integral formula can also be used for DISCRETE charges, provided we identify

$$\rho(\vec{x}\,') = \sum_i q_i\, \delta(\vec{x}\,' - \vec{x}_i)$$

Idealization Actually, <u>all</u> charges are discrete, so the most accurate picture would use <u>sums</u> only, but since there are often Avogadro's number of charges this is impractical. So approximating the sums by integrals is more practical and usually quite accurate,

Also, in classical E+M it is frequently stated that Gauss's Law implies that any net charge on a conductor in electrostatics must reside on its <u>outermost</u> <u>surface</u>, i.e. within an infinitesimally thin layer.

More realistic models including the quantum wavelike nature of the electrons shows (see Fig I.5) that the surface layer actually has a finite thickness of approximately $\Delta r \simeq 2\,\mathring{A}$

while this is not infinitely thin, it is thin enough so that for most applications it can be treated that way, i.e,

$$\rho(\vec{x}') = \sigma\,\delta(x' - R)$$

Math Review: Dirac delta functions

(1) $$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \text{"}\infty\text{"}, & x = a \end{cases}$$

(2) $$\int_c^d \delta(x-a)\,dx = \begin{cases} 1, & \text{if } c<a<d \\ -1, & \text{if } c>a>d \\ 0, & \text{otherwise} \end{cases}$$

(Note: more careful consideration is needed if either $c=a$ or $d=a$.)

explicit representations

$$\frac{1}{2}\frac{d^2}{dx^2}|x| = \delta(x)$$

$$\delta(x) = \pi^{-\frac{1}{2}} \lim_{\epsilon\to 0^+} \epsilon^{-1/2} \exp\left(-x^2/\epsilon\right)$$

$$\delta(x) = \pi^{-1} \lim_{\epsilon\to 0^+} \frac{\epsilon}{x^2+\epsilon^2}$$

$$\delta(x) = \pi^{-1} \lim_{N\to\infty} \frac{\sin Nx}{x}$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

(3) $$\int_{-\infty}^{\infty} dx\; f(x)\,\delta(x-a)\,dx = f(a), \quad a = \text{real}$$

proof $$\int_{-\infty}^{\infty} dx\; F(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-a)}\,dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{-ika}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\; F(x)\,e^{ikx} = F(a),$$

since this is the inverse Fourier transform of a Fourier transform.

(4) $$\int_{-\infty}^{\infty} f(x)\,\delta'(x-a)\,dx = f(x)\,\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x-a)\,dx$$
$$= -f'(a)$$

This result is sometimes written as

$$f(x)\frac{d}{dx}\delta(x-a) = -\delta(x-a)\frac{d}{dx}f(x)$$

(5) $$\delta(ax) = \frac{1}{|a|}\delta(x)$$

corollary
$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$$
provided $f(x)$ has only simple zeroes at x_i

e.g. $$\delta[(x-a)(x-b)] = \frac{1}{|a-b|}\Big(\delta(x-a)+\delta(x-b)\Big)$$

(6) δ-functions in multiple dimensions

$$\delta(\vec{x}-\vec{x}\,') = \delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(x_3-x_3')$$
(in Cartesian coordinates)

This vanishes except at the point $\vec{x}=\vec{x}\,'$

i.e. $$\int_V \delta(\vec{x}-\vec{x}\,')\,d^3x' = \begin{cases} 1 & \text{if } \vec{x} \text{ is in } V \\ 0 & \text{otherwise} \end{cases}$$

3-dimensional representation

$$\delta(\vec{x}-\vec{x}') = (2\pi)^{-3} \int d^3k \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

If we consider general orthogonal, curvilinear coordinates, q_1, q_2, q_3 with differential arc length,

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$$

and volume element $d^3x = h_1 h_2 h_3 \, dq_1 dq_2 dq_3$,

then $$\delta(\vec{x}-\vec{x}') = \frac{1}{h_1 h_2 h_3} \delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

e.g. in spherical coordinates,

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2 \sin\theta} \delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

$$= \frac{1}{r^2} \delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

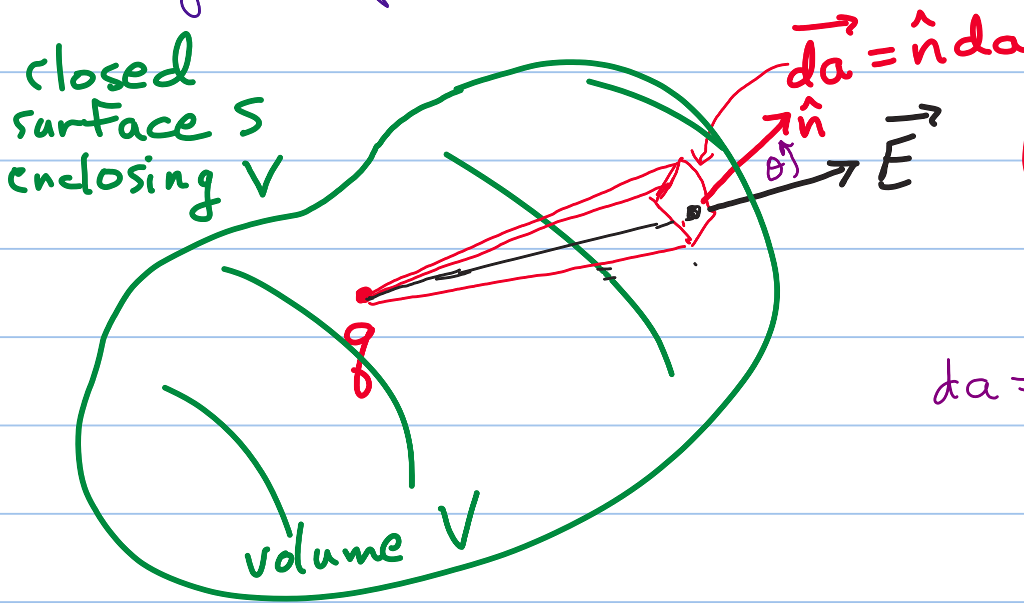
(7) another identity: $\nabla^2 \frac{1}{r} = -4\pi\,\delta(\vec{r})$

whereby we see that

$$\frac{1}{r} = \underbrace{\left[4\pi\,(2\pi)^{-3}\right]}_{=\,(2\pi^2)^{-1}} \int d^3k \, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2}$$

Gauss's Law – This is often more convenient than evaluating a 3-dimensional Coulomb's Law integral. It also exemplifies the intuitive power and beauty of geometrically reinterpreting a math formula.

To demonstrate this, consider a point charge q at a point in space, and draw an imaginary closed surface S surrounding it:



$\hat{n}$ = outwardly directed unit vector normal to S

$da = |\vec{da}|$ = magnitude of differential area element

Now observe that $\vec{E} \cdot \vec{da} = \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{\cos\theta\, da}{r^2}\right)$

But $d\Omega = \cos\theta \dfrac{da}{r^2}$,

where $d\Omega$ = differential solid angle subtended by da from charge q.

So we can write

$$\oint_S \vec{E} \cdot \vec{da} = \oint_S \frac{q}{4\pi\epsilon_0}\, d\Omega = \begin{cases} q/\epsilon_0, & \text{if } q \text{ is in } V \\ 0, & \text{if } q \text{ is outside } V \end{cases}$$

Of course, this argument generalizes if there are many charges, i.e. by superposition which is permitted by the linearity of the equations, giving the key result:

$$\boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{q_{\text{enclosed}}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V d^3x'\,\rho(\vec{x}')}$$

where V = volume enclosed by S

Next, apply the divergence theorem:

$$\oint_S \vec{E}\cdot d\vec{a} = \overbrace{\int_V \nabla\cdot\vec{E}\,d^3x = \frac{1}{\epsilon_0}\int_V d^3x\,\rho(\vec{x})}$$

and since the second equality holds for an ARBITRARY surface + volume, the integrands must themselves be equal everywhere, i.e.

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$

is equivalent to $\oint_S \vec{E}\cdot d\vec{a} = \dfrac{q_{\text{enclosed}}}{\epsilon_0}$

= Gauss's Law in differential and integral form

Electrostatic Potential

Consider

$$\nabla_x \frac{1}{|\vec{x}-\vec{x}'|} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= \frac{\hat{x}(-\frac{1}{2})(2)(x-x') + \hat{y}(-\frac{1}{2})(2)(y-y') + \hat{z}(-\frac{1}{2})(2)(z-z')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

$$= -\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

(Aside: sometimes we will abbreviate $\nabla_x \equiv \nabla$
$\nabla_{x'} \equiv \nabla'$)

Then Coulomb's Law can be rewritten as:

$$\vec{E}(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\int d^3x' \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \nabla \frac{1}{|\vec{x}-\vec{x}'|}$$

or we can write $\boxed{\vec{E} = -\nabla \Phi(\vec{x})}$

where

$$\Phi(\vec{x}) = \int \frac{d^3x'}{4\pi\epsilon_0} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \equiv \text{Electrostatic Potential}$$

Comments ① since $\nabla\times\nabla f = 0$ always $\Rightarrow \nabla\times\vec{E} = 0$ in electrostatics, always!

② This integral expression for Φ is not the most general, since it has implicitly assumed the B.C. that $\Phi(\vec{x}) \to 0$ as $r \to \infty$, whenever ρ has a finite extent.

③ If $\Phi(\vec{x})$ is one solution, then an equally good solution is $\Phi(\vec{x})$ + constant, since both give the same $\vec{E}(\vec{x})$

④ Don't forget that you need BOTH

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$

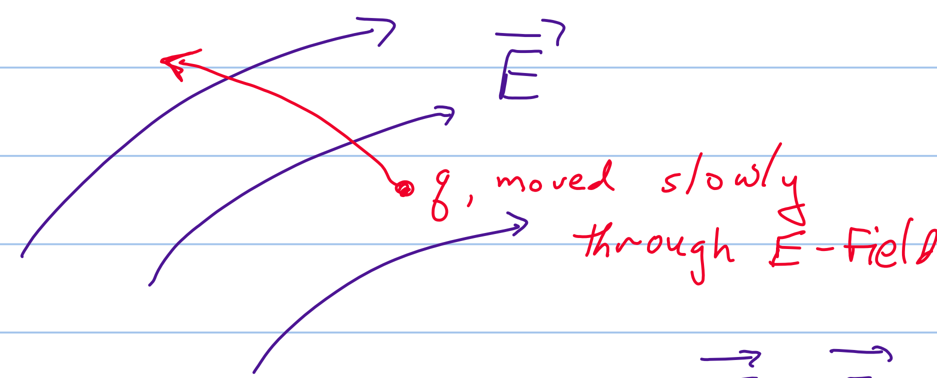
AND

$$\nabla\times\vec{E} = 0$$

to specify $\vec{E}$ completely, i.e. to within the gradient of a scalar function Φ that obeys Laplace's equation, $\nabla^2\Phi = 0$.

Relationship to energy

Let's try to discuss this carefully, since it is easy to make sign errors when discussing work and potential energy.



The electric force on a charge q is $\vec{F} = q\vec{E}$ so the force WE apply to hold q fixed, or to move it very slowly, is $\vec{F}_{us} = -q\vec{E}$

Thus the WORK that WE do in slowly moving q from $\vec{x}_1$ to $\vec{x}_2$ along some path $\mathcal{P}$ is

$$W_2 - W_1 = \int_{\vec{x}_1,\,\mathcal{P}}^{\vec{x}_2} \vec{F}_{us} \cdot \vec{d\ell} = -\int_{\vec{x}_1,\,\mathcal{P}}^{\vec{x}_2} \vec{F} \cdot \vec{d\ell}$$

or $$W_2 - W_1 = -q \int_{\vec{x}_1}^{\vec{x}_2} \vec{E} \cdot \vec{d\ell} = +q \int_{\vec{x}_1}^{\vec{x}_2} \nabla \Phi \cdot \vec{d\ell}$$

$$= q \int_{\vec{x}_1}^{\vec{x}_2} d\Phi = q\,\Phi(\vec{x}_2) - q\,\Phi(\vec{x}_1)$$

Thus we can interpret $q\,\Phi(\vec{x})$ as the

ELECTRIC POTENTIAL ENERGY

i.e. if we do positive work, the potential energy must increase if kinetic energy = constant

observations

① The work done is path-independent

$$-q \int_{\vec{x}_1}^{\vec{x}_2} \vec{E} \cdot \vec{d\ell} = q\,\Phi(\vec{x}_2) - q\,\Phi(\vec{x}_1)$$

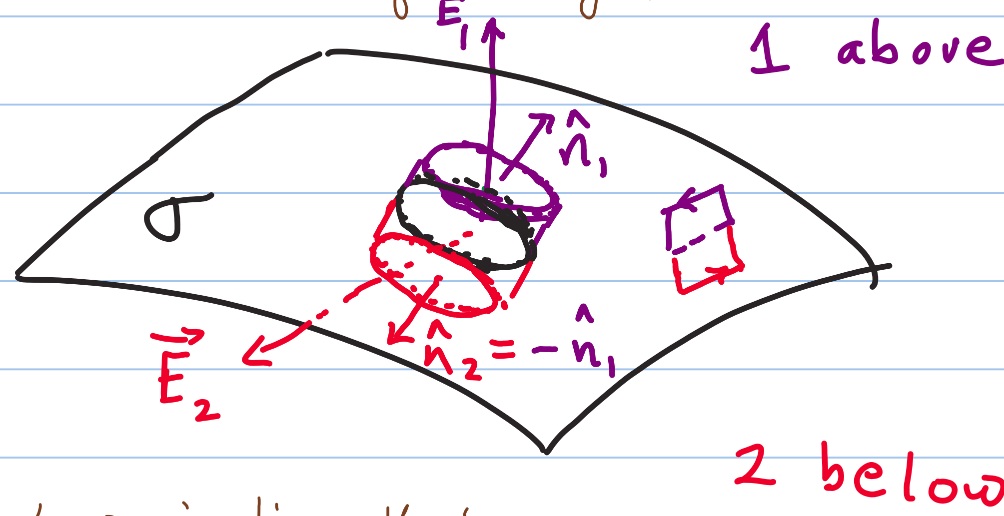
② $$\int_{\vec{x}_1}^{\vec{x}_1} \vec{E}(\vec{x}) \cdot \vec{d\ell} = \oint_C \vec{E}(\vec{x}) \cdot \vec{d\ell} = 0$$

for ANY closed loop

③ $$\oint_C \vec{E} \cdot \vec{d\ell} = \int_S \nabla \times \vec{E} \cdot \vec{da} = 0$$ is consistent with $\nabla \times \vec{E} = 0$

Implications of a surface charge distribution

Consider the electric field very close to a surface distribution of charge, σ



Now, Gauss's Law implies that

$$\oint \vec{E}\cdot\vec{da} = \left(\vec{E}_1\cdot\hat{n}_1 + \vec{E}_2\cdot\hat{n}_2\right) da = \frac{\sigma\, da}{\epsilon_0}$$

or $$\boxed{\left(\vec{E}_1 - \vec{E}_2\right)\cdot\hat{n}_1 = \frac{\sigma}{\epsilon_0}}$$

and since $\oint \vec{E}\cdot\vec{dl} = 0$, $\Rightarrow$ $\boxed{\vec{E}_1\cdot\vec{l}_t = \vec{E}_2\cdot\vec{l}_t}$

where $\vec{l}_t$ is any vector lying in the plane tangent to the surface at that point.

Similarly, consider the effect of a dipole layer

i.e. consider a layer of oriented dipole molecules on a surface, consisting of surface charge densities $+\sigma(\vec{x})$ and $-\sigma(\vec{x})$ separated by a tiny distance $a(\vec{x})$.

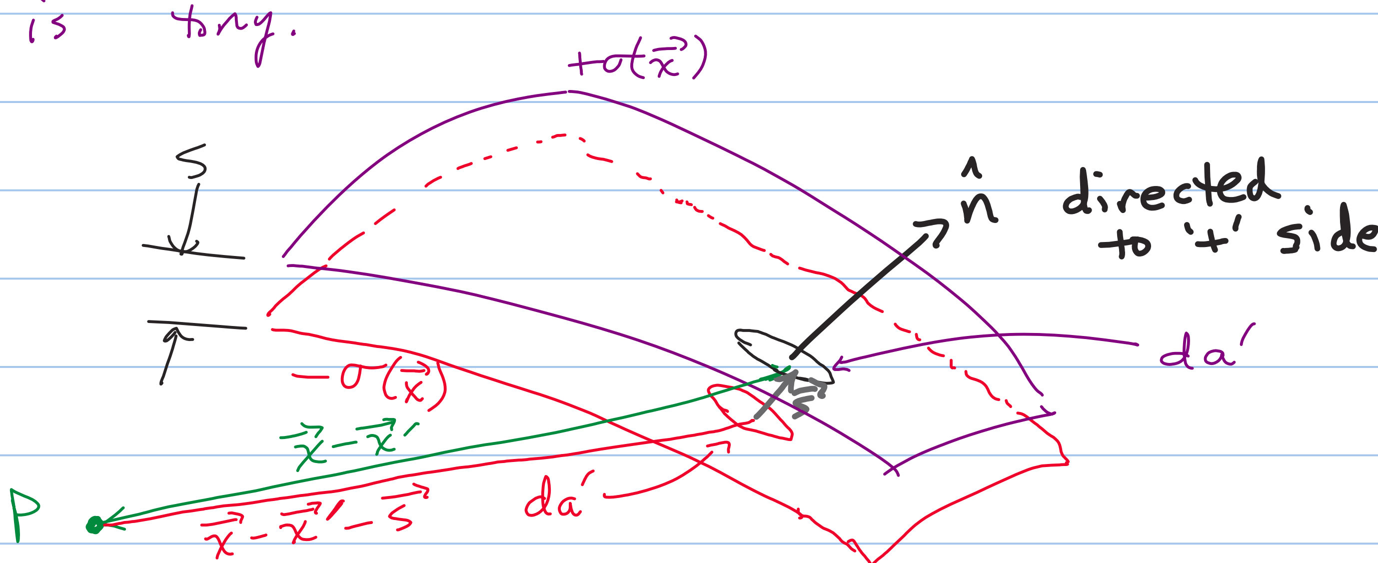
Then define $D(\vec{x}) = \sigma(\vec{x})\, s(\vec{x})$

Alternatively one can view a dipole layer as some $\frac{\#\text{ dipole molecules}}{\text{unit area}} \equiv n(\vec{x})$

each molecule having dipole moment $\vec{d} = e\vec{s}$

$$\Rightarrow \quad D(\vec{x}) = n(\vec{x})\,\vec{d} = n(\vec{x})\, e\vec{s}$$

Assume that the charge separation distance $\vec{s}$ is tiny.



Now, for a single layer of charge σ, the potential is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|}$$

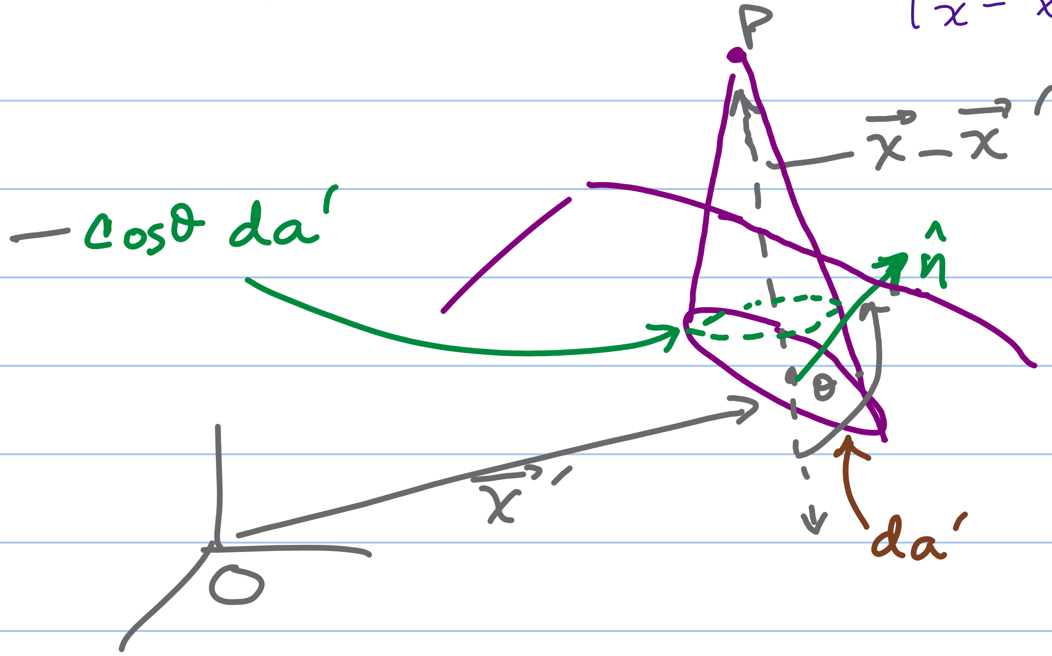
So for our dipole layer we have potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\int_{S'} \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|} - \int \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'+\vec{s}|}\right]$$

Now Taylor expand for small $\vec{s} \doteq \hat{n}s$

$$\frac{1}{|\vec{x}-\vec{x}'+\vec{s}|} \simeq \frac{1}{|\vec{x}-\vec{x}'|} + s\,\hat{n}\cdot\nabla' \frac{1}{|\vec{x}-\vec{x}'|} + O(s^2)$$

$$\Rightarrow \Phi(\vec{x}) = \cancel{-}\frac{1}{4\pi\epsilon_0}\int \sigma(\vec{x}\,')\, s\, \hat{n}_0 \cdot \frac{(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\, da' \,\cancel{(-1)}$$



This can be re-expressed as $d\Omega'$, the solid angle subtended by da' from P, i.e.

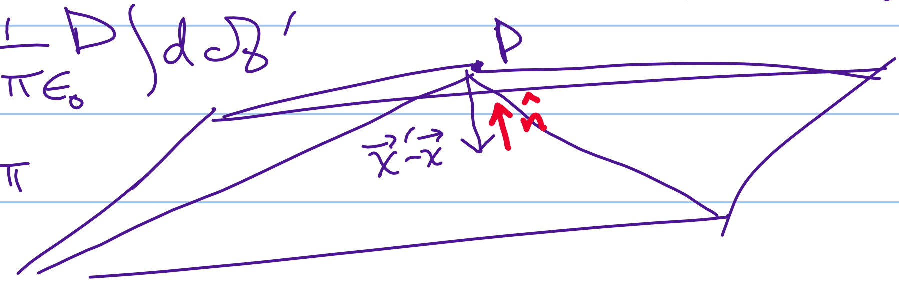
$$-\,d\Omega' = -\frac{(\vec{x}\,'-\vec{x}\,)\cdot\hat{n}\, da'}{|\vec{x}-\vec{x}\,'|^3} =$$ positive $(-d\Omega')$ is for $\vec{x}$ above plane

giving

$$\boxed{\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int D(\vec{x}\,')\, d\Omega'}$$

e.g. for a point just above a large dipolar plane, constant $D(\vec{x}) = D$,

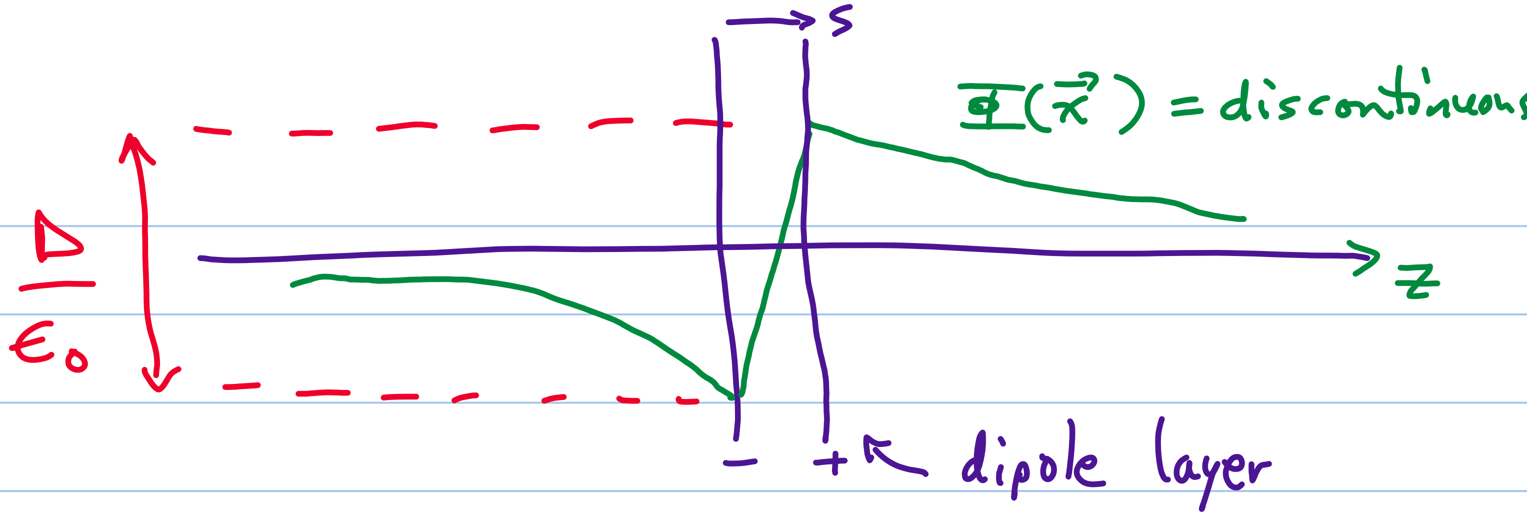
$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0} D \int d\Omega'$$

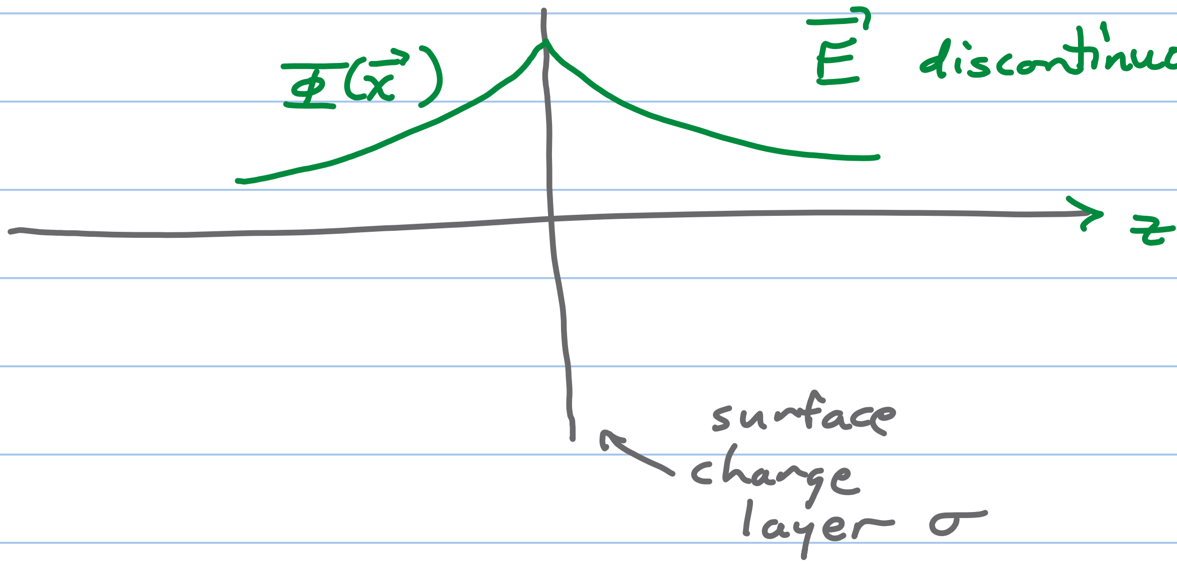


and $\int d\Omega' \to -2\pi$

so $\Phi^{\text{above}}(\vec{x}) \longrightarrow \dfrac{D}{2\epsilon_0}$, $\Phi^{\text{below}}(\vec{x}) \longrightarrow -\dfrac{D}{2\epsilon_0}$

and in general $\boxed{\Phi^{\text{above}}(\vec{x}) - \Phi^{\text{below}}(\vec{x}) = \dfrac{D}{\epsilon_0}}$





Poisson's and Laplace's equations

Again, a consequence of $\nabla \times \vec{E} = 0$ and $\nabla \cdot \vec{E} = \rho/\epsilon_0$ is that $\vec{E} = -\nabla \Phi$ where Φ obeys Poisson's eqn,

$$\boxed{\nabla^2 \Phi = -\rho/\epsilon_0}$$

A key special case is this equation in free space, with $\rho = 0$, where

$$\boxed{\nabla^2 \Phi = 0} = \text{Laplace's equation}$$

We will spend much time this year solving these kinds of equations.

An important check

Does $\Phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \frac{d^3x'}{|\vec{x}-\vec{x}'|}$

truly obey $\nabla^2 \Phi = -\rho/\epsilon_0$?

We can test this by taking derivatives, etc., everywhere except near the singularity at $\vec{x}=\vec{x}'$. Extra caution is always needed at singularities.

e.g., write $\nabla^2 \Phi = \left\{ \int_{V_1} + \int_{V_2} \right\} \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} d^3x'$

where V_2 = a tiny sphere of radius ϵ about $\vec{x}'=\vec{x}$

V_1 = everywhere else, outside of V_2, where $\vec{x}' \neq \vec{x}$

So within V_1 we just differentiate normally

$$\Rightarrow \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -\nabla \cdot \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right)$$

and with a variable change to $\vec{r} = \vec{x} - \vec{x}'$

$$\Rightarrow \nabla_x = \nabla_r \Rightarrow -\nabla_r \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3} \nabla_r \cdot \vec{r} + \vec{r} \cdot \nabla \frac{1}{r^3}$$

$$= \frac{3}{r^3} - \frac{3\,\vec{r}\cdot\vec{r}}{r^5} = 0 \quad \text{if } \vec{r} \neq 0\text{, true in } V_1$$

hence $\int_{V_1} d^3x' \rho(\vec{x}') \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = 0$!

Next, notice that $\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = \nabla'^2 \frac{1}{|\vec{x}-\vec{x}'|}$ ← prove to yourself!

and moreover, V_2 is so small that

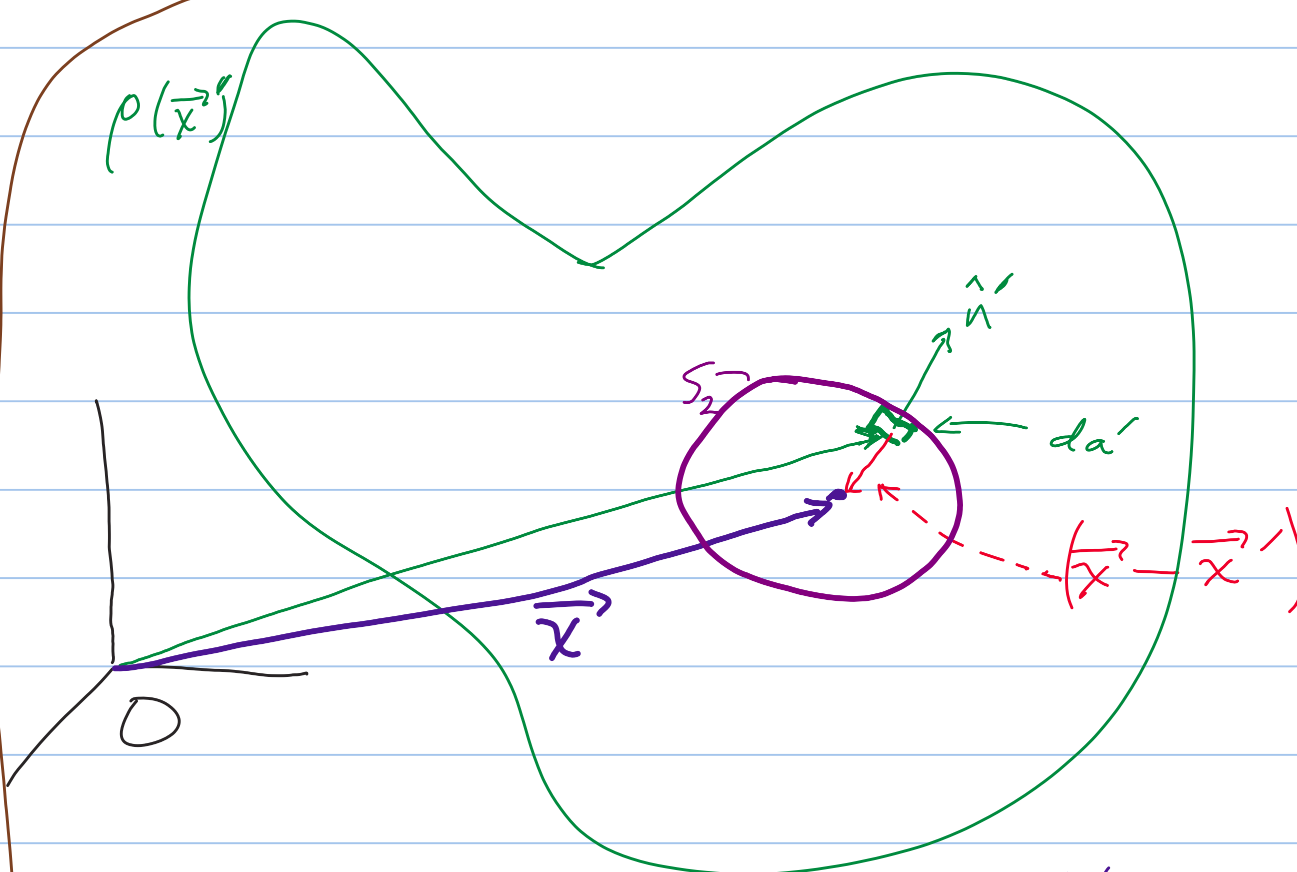
$\rho(\vec{x}\,') \simeq \rho(\vec{x})$ inside V_2,

i.e. approx. constant w.r.t. $\vec{x}\,'$

Thus $\displaystyle\int_{V_2} \frac{\rho(\vec{x}\,')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\vec{x}-\vec{x}\,'|}\, d^3x'$

$$\longrightarrow \frac{\rho(\vec{x})}{4\pi\epsilon_0} \int_{V_2} \nabla' \cdot \frac{(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\, d^3x'$$

$$= \frac{\rho(\vec{x})}{4\pi\epsilon_0} \oint_{S_2} \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3} \cdot \hat{n}'\, da' \quad \left(\begin{array}{c}\text{divergence}\\ \text{theorem}\end{array}\right)$$



Note from this diagram that $\hat{n}' \cdot (\vec{x}-\vec{x}\,') = -|\vec{x}-\vec{x}\,'|$

$$= -\frac{\rho(\vec{x})}{4\pi\epsilon_0} \oint_{S_2} d\Omega' = -\frac{\rho(\vec{x})}{\epsilon_0}$$

Conclusions: We have thus proved that

$$\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}$$

as was desired

Moreover, we have demonstrated that

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}')$$

GREEN'S THEOREM

In some problems we don't know ρ or σ everywhere, but we can still determine Φ, $\vec{E}$.
e.g. a conductor has $\vec{E}=0$ inside, whereby
$\Phi(\vec{x})$ = constant everywhere on + through it.
In such a problem, we might know Φ on the conductor, but not ρ.

Green's function methods are particularly useful for such problems.

I. 1^{st} Green Identity

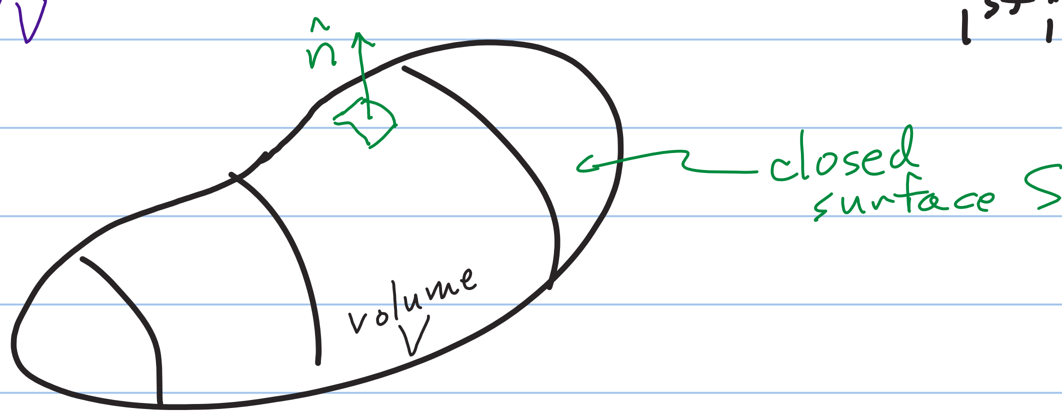
Consider two scalar functions ϕ, ψ. They obey the following identity

$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$$

Next, integrate this over an arbitrary volume V bounded by a closed surface S:

$$\int_V \nabla\cdot(\phi\nabla\psi)\,d^3x \overset{\text{divergence theorem}}{=} \oint_S \phi\nabla\psi\cdot\hat{n}\,da \equiv \oint_S \phi\,\underbrace{\frac{\partial\psi}{\partial n}}_{\text{outward normal partial derivative}}\,da$$

$$= \int_V \left(\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi\right)d^3x \quad = \text{Green's } 1^{st} \text{ identity}$$



II. Green's Theorem (2^{nd} identity)

To derive it, just rewrite the 1^{st} identity with ϕ, ψ interchanged, and subtract:

i.e.

$$\begin{pmatrix} \int_V (\cancel{\nabla\phi\cdot\nabla\psi} + \phi\nabla^2\psi)\,d^3x \\ - \int_V (\cancel{\nabla\psi\cdot\nabla\phi} + \psi\nabla^2\phi)\,d^3x \end{pmatrix} = \begin{pmatrix} \oint_S \phi\frac{\partial\psi}{\partial n}\,da \\ - \oint \psi\frac{\partial\phi}{\partial n}\,da \end{pmatrix}$$

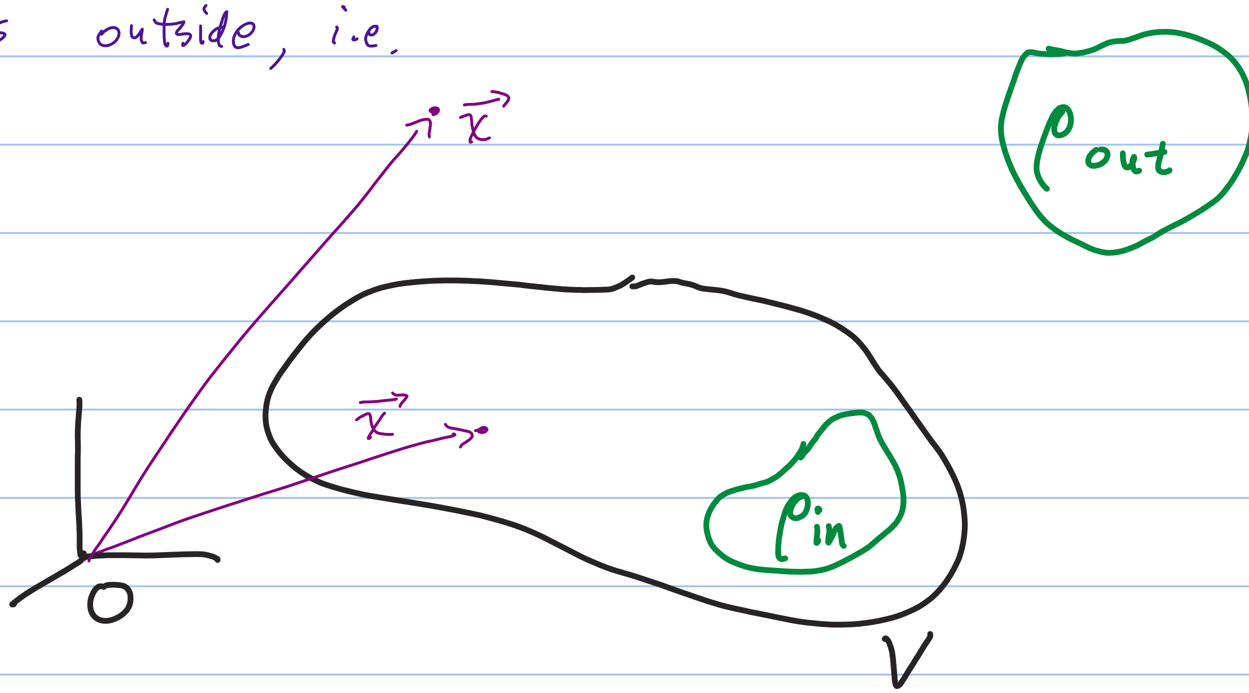
$$\Rightarrow \boxed{\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da}$$

Kewl!

Integral Equation Next we convert the Poisson PDE into an integral equation, starting from Poisson's eqn for $\Phi(\vec{x})$, written in terms of $\vec{x}\,'$ instead of $\vec{x}$:

$$\nabla'^2 \Phi(\vec{x}\,') = -\frac{\rho(\vec{x}\,')}{\epsilon_0}$$

We consider a volume V that contains a part ρ_{in} of this $\rho(\vec{x})$, while another part ρ_{out} is outside, i.e.



We will consider both cases, where the observation point is either inside or outside V.

Derivation In Green's Theorem let $\phi \to \Phi$, $\psi \to \frac{1}{|\vec{x}-\vec{x}\,'|}$

and let the integration variable be labelled as $\vec{x}\,'$

Also, use $\nabla'^2 \psi = -4\pi\,\delta(\vec{x}-\vec{x}\,')$

Then

$$\int_V \Big[\Phi(\vec{x}')\, \overbrace{\nabla'^2 \frac{1}{|\vec{x}-\vec{x}'|}}^{-4\pi\delta(\vec{x}-\vec{x}')} - \frac{1}{|\vec{x}-\vec{x}'|} \overbrace{\nabla'^2 \Phi(\vec{x}')}^{-\rho(\vec{x}')/\epsilon_0} \Big] d^3x'$$

$$= \oint_S \Big[\Phi(\vec{x}')\, \underbrace{\frac{\partial}{\partial n'} \frac{1}{|\vec{x}-\vec{x}'|}}_{\hat{n}'\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}} - \frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial}{\partial n'}\Phi(\vec{x}') \Big] da'$$

To simplify the notation, define $R \equiv |\vec{x}-\vec{x}'|$.

Then <u>if the point $\vec{x}$ lies within V</u>, we have

$$-4\pi \int_V \Phi(\vec{x}')\,\delta(\vec{x}-\vec{x}')\, d^3x' = -4\pi\, \Phi(\vec{x})$$

$$= -\frac{1}{\epsilon_0}\int_V \frac{\rho(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|} + \oint_S \Big[\Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{R} - \frac{1}{R}\frac{\partial \Phi(\vec{x}')}{\partial n'}\Big] da'$$

OR

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi}\oint_S \Big[\frac{1}{R}\frac{\partial}{\partial n'}\Phi(\vec{x}') - \Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{R}\Big] da'}$$

notice that this integral only includes the charges <u>INSIDE</u> V!

<u>**observation**</u> If there are NO charges INSIDE V then $\Phi(\vec{x})$ is determined ENTIRELY by the values of Φ, $\frac{\partial \Phi}{\partial n'}$ on S.

However this <u>overdetermines</u> the solution of the problem, since these are then <u>Cauchy Boundary Conditions</u>

(we discuss this issue later)

Next, what if $\vec{x}$ lies outside of the volume V?

Of course the δ-function integral now vanishes, so we get a different identity, namely

$$0 = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}\,')\,d^3x'}{|\vec{x}-\vec{x}\,'|} + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}\,')\frac{\partial}{\partial n'}\frac{1}{R}\right]da',$$

which tells us nothing about $\Phi(\vec{x})$ outside!

Apparent error in Jackson, p.37, end of top paragraph, where he writes that this result gives: "... zero field and zero potential outside the volume V."

Variational Formulation

Consider the following functional $I[\Phi]$,

$$I[\Phi] \equiv \frac{1}{2}\int_V \nabla\Phi\cdot\nabla\Phi\, d^3x - \int_V \frac{\rho}{\epsilon_0}\Phi\, d^3x$$

$\Rightarrow$ We can prove that this expression is in fact a variational principle that can be used to determine Φ. Consider this expression with $\Phi \longrightarrow \Phi + \delta\Phi$ ← a "small deviation"

the exact solution

We can find δI by taking the 1st variation, i.e $\quad I[\Phi + \delta\Phi] \longrightarrow I[\Phi] + \delta I$

where $\quad \delta I = \int_V \nabla\Phi \cdot \nabla(\delta\Phi)\, d^3x - \int_V \frac{\rho}{\epsilon_0}\, \delta\Phi\, d^3x$

application of Green's first identity gives

$$\delta I = \int_V \left(-\nabla^2\Phi - \frac{\rho}{\epsilon_0}\right)\delta\Phi\, d^3x + \oint_S \frac{\partial\Phi}{\partial n}\,\delta\Phi\, da$$

0 since Φ = exact solution

Hence $\delta I = 0$, i.e. the 1^{st} variation vanishes PROVIDED we only allow variations $\delta\Phi = 0$, that vanish on S.

This variational principle can be used by choosing a trial potential $\Phi_t[a,b,c,\ldots]$ that depends on some set of parameters $\{a,b,c,\ldots\}$, and then choosing those parameters that give an extremum for I.

Uniqueness - What is the minimum amount of information needed to uniquely specify Φ within a volume V.

We saw already that

$$\Phi(\vec{x}) = \int_V \frac{\rho(\vec{x}')\, d^3x'}{4\pi\epsilon_0|\vec{x}-\vec{x}'|} + \frac{1}{4\pi}\oint_S \left[\Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial}{\partial n'}\Phi(\vec{x}')\right] da'$$

But starting from BOTH Φ AND $\frac{\partial \Phi}{\partial n}$ on S overspecifies the solution,

$\Rightarrow$ it can lead to internal inconsistency in the solution

~~Ins~~tead, we consider 2 types of boundary conditions:

EITHER:

1) Specify $\Phi(\vec{x})$ on S $\Leftarrow$ "Dirichlet BCs"

(if $\Phi(\vec{x}) = 0$ on S, call them homogeneous-Dirichlet BCs)

2) Specify $\frac{\partial \Phi}{\partial n}$ on S $\Leftarrow$ "Neumann BCs"

(if $\frac{\partial \Phi}{\partial n} = 0$ on S $\Rightarrow$ homogeneous Neumann BCs)

THEOREM knowledge of ρ inside of V, and either Dirichlet OR Neumann BCs, is enough to specify Φ in V uniquely.

Proof Suppose Φ_1 and Φ_2 are 2 solutions obeying:

(i) $\nabla^2 \Phi_i = -\rho/\epsilon_0$ in V

(ii) $\Phi_1 = \Phi_2$ on S (Dirichlet case) OR

$\frac{\partial \Phi_1}{\partial n'} = \frac{\partial \Phi_2}{\partial n'}$ on S (Neumann case)

Now define $U \equiv \Phi_1 - \Phi_2$

$\Rightarrow \nabla^2 U = 0$ in V, and $U = 0$ on S

OR $\frac{\partial U}{\partial n} = 0$ on S.

Start from the 1^{st} Green identity

with $\phi \to U$, $\psi \to U$

$$\Rightarrow \int_V [U \underbrace{\nabla^2 U}_{0} + \nabla U \cdot \nabla U] d^3x = \oint_S \underbrace{U \frac{\partial U}{\partial n}}_{0} da$$

0 for either BC!

$$\Rightarrow \int_V |\nabla U|^2 d^3x = 0 \quad \text{for either B.C.}$$

But since $|\nabla U|^2 \geq 0$ everywhere, the only possibility is that $\nabla U = 0$ everywhere in V.

$$\Rightarrow U = \Phi_1 - \Phi_2 = \text{constant in } V$$

meaning that Φ_1 and Φ_2 are equivalent, as the physics is unchanged by $\Phi \to \Phi + \text{constant}$.

Moreover, since <u>either</u> Dirichlet <u>or</u> Neumann BCs gives a unique solution, one sees that arbitrarily specifying BOTH Φ AND $\left.\frac{\partial \Phi}{\partial n}\right|_S$ will not be consistent.

To learn more about this issue, e.g. that Cauchy B.C.'s are problematic in this type of PDE, see Morse + Feshbach, volume I, or Arfken + Weber.

Green's Function solution of Poisson's equation

This method is especially useful and practical for solution of the following two types of problems:

(i) Potentials are specified on a boundary S

OR

(ii) The charge distribution ρ is given in a region that is inside of a conducting boundary

The Green's function can be viewed as the electrostatic potential at an observation point $\vec{x}$ due to a unit point charge at a source point $\vec{x}'$, subject to Dirichlet or Neumann boundary conditions on the relevant surfaces.

Mathematically, the Green's function $G(\vec{x}, \vec{x}')$ obeys

$$\boxed{\nabla^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')}$$

We already know a particular solution to be:

$$G(\text{particular}) = \frac{1}{|\vec{x} - \vec{x}'|}$$

since $\nabla^2 \dfrac{1}{|\vec{x} - \vec{x}'|} = -4\pi\delta(\vec{x} - \vec{x}')$

But this is not the most general solution. More generally we should write

$$G(\vec{x}, \vec{x}\,') = \frac{1}{|\vec{x} - \vec{x}\,'|} + F(\vec{x}, \vec{x}\,')$$

where F obeys Laplace's equation in V, i.e.

$$\nabla^2 F(\vec{x}, \vec{x}\,') = 0$$

By choosing F appropriately, we can impose specific BCs on G, using tricks or more sophisticated math techniques.

One example of such a "trick" is the METHOD OF IMAGES, which determines F by placing "FAKE CHARGES" external to the volume V of interest.

Question How can we use $G(\vec{x}, \vec{x}\,')$ to obtain the desired unique solution for $\Phi(\vec{x})$?

Solution Choose G (i.e. specify F) to eliminate one of the two surface integrals in the general formula for $\Phi(\vec{x})$.

e.g. return to Green's theorem, setting now

$$\phi \to \Phi(\vec{x}\,') \text{ and } \psi \to G(\vec{x}, \vec{x}\,')$$

when we want to specify particular BCs on G, we use the abbreviations:

$$G_D(\vec{x}, \vec{x}') = \text{Dirichlet Green's Function}$$
(vanishes on surface(s) S)

$$G_N(\vec{x}, \vec{x}') = \text{Neumann BC Green's Function}$$
($\frac{\partial G_N}{\partial n'}$ specified on S)

$$\Rightarrow \int_V \Big[\Phi(\vec{x}') \underbrace{\nabla'^2 G(\vec{x}', \vec{x})}_{=-4\pi\delta(\vec{x}-\vec{x}')} - G(\vec{x}', \vec{x}) \underbrace{\nabla'^2 \Phi(\vec{x}')}_{=-\frac{\rho(\vec{x}')}{\epsilon_0}} \Big] d^3x'$$

$$= \oint_S \Big[\Phi(\vec{x}') \frac{\partial G(\vec{x}', \vec{x})}{\partial n'} - G(\vec{x}', \vec{x}) \frac{\partial \Phi(\vec{x}')}{\partial n'} \Big] da'$$

$$= -4\pi \Phi(\vec{x}) + \int_V \frac{\rho(\vec{x}')}{\epsilon_0} G(\vec{x}', \vec{x}) d^3x'$$

Aside - We will see later that $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$

$\Rightarrow$ also $\frac{\partial G}{\partial n'}(\vec{x}', \vec{x}) = \frac{\partial}{\partial n'} G(\vec{x}, \vec{x}')$

Thus we arrive at the key equation for Φ:

$$\boxed{\begin{aligned}\Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0} \int_V G(\vec{x}, \vec{x}') \rho(\vec{x}') d^3x' \\ &+ \frac{1}{4\pi} \oint_S \Big[G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} - \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \Phi(\vec{x}') \Big] da'\end{aligned}}$$

Apply this formula to the 2 standard cases:

Dirichlet BCs — $\Phi(\vec{x})$ given on S

$\Rightarrow$ choose $G_D(\vec{x},\vec{x}\,') = 0$ for $\vec{x}$ on S
or for $\vec{x}\,'$ on S

$$\Rightarrow \Phi(\vec{x}) = \int_V G_D(\vec{x},\vec{x}\,')\frac{\rho(\vec{x}\,')}{4\pi\epsilon_0}d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial G_D(\vec{x},\vec{x}\,')}{\partial n'}\Phi(\vec{x}\,')\,da'$$

For this Green's function choice, $\frac{\partial \Phi}{\partial n'}$ is irrelevant and not used!

Neumann BCs — $\frac{\partial \Phi}{\partial n'}$ is given on S

$\Rightarrow$ Specify $\frac{\partial G(\vec{x},\vec{x}\,')}{\partial n'} = \frac{-4\pi}{S}$, a constant over S,

where $S = \oint da'$ = total area of surface S

Note If we had chosen $\left.\frac{\partial G}{\partial n'}\right|_S = 0$, this would cause inconsistencies,

since $\oint_S \frac{\partial G_N}{\partial n'}(\vec{x},\vec{x}\,')\,da' = \int_V \nabla'^2 G_N(\vec{x},\vec{x}\,')\,d^3x'$
$= -4\pi$

$\Rightarrow$ our BC for G_N is consistent with this!

So in the Neumann BC problem,

$$\Phi(\vec{x}) = \int_V G_N(\vec{x},\vec{x}\,')\frac{\rho(\vec{x}\,')}{4\pi\epsilon_0}d^3x'$$
$$+ \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}\,')\frac{\partial\Phi}{\partial n'}(\vec{x}\,')\,da' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}\,')\frac{\partial G}{\partial n'}(\vec{x},\vec{x}\,')\,da'$$

and the last term here is

$$-\frac{1}{4\pi}\left(\frac{4\pi}{S}\right)\oint_S \Phi(\vec{x}\,')da' = \langle \Phi \rangle_S$$

= <u>mean</u> value of Φ on S

= a constant that is usually dropped whereby we obtain

$$\boxed{\Phi(\vec{x}) = \int_V G_N(\vec{x},\vec{x}\,')\frac{\rho(\vec{x}\,')}{4\pi\epsilon_0}d^3x' + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}\,')\frac{\partial\Phi(\vec{x}\,')}{\partial n'}da'}$$

A <u>third</u> possible choice of Green's function, not considered by Jackson but sometimes useful, (instead of Dirichlet or Neumann BCs), is to simply apply the <u>same</u> BCs to G that Φ must obey on all boundaries S. For this choice, all surface integrals vanish and we obtain simply

$$\boxed{\Phi(\vec{x}) = \int_V G(\vec{x},\vec{x}\,')\frac{\rho(\vec{x}\,')}{4\pi\epsilon_0}d^3x'}$$

<u>GF symmetry</u> (1) can show that $G_D(\vec{x},\vec{x}\,') = G_D(\vec{x}\,',\vec{x})$ is automatically true, $\Rightarrow$ complete symmetry under interchange of source and observation points

(2) However, $G_N(\vec{x}, \vec{x}')$ need not automatically be equal to $G_N(\vec{x}', \vec{x})$, but one can demand this symmetry as a separate requirement, without loss of generality.

We will see examples of using these formulas in Chapters 2, 3.

Electrostatic Potential Energy

** Reading Assignment: Read through Jackson Sec. 2.6

The energy needed to take N charges,

$$q_1, q_2 \ldots q_N$$

from infinity, and assemble them at respective positions $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_N$, is called the POTENTIAL ENERGY of the system, U.

$\Rightarrow$ This energy can be calculated by bringing in each charge, one-by-one:

e.g. $U_1 = 0$ $\qquad U_2 = q_2 \left(\dfrac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_2|} \right)$

$$U_3 = q_3 \left(\frac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_3|} + \frac{q_2}{4\pi\epsilon_0 |\vec{x}_2 - \vec{x}_3|} \right)$$

$$\vdots$$

$$U_N = q_N \left(\frac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_N|} + \cdots + \frac{q_{N-1}}{4\pi\epsilon_0 |\vec{x}_{N-1} - \vec{x}_N|} \right)$$

and $U = \sum_{i=1}^{N} U_i = \dfrac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \dfrac{q_i q_j}{4\pi\epsilon_0 |\vec{x}_i - \vec{x}_j|}$

This result can be rewritten as

$$\boxed{U = \frac{1}{8\pi\epsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \qquad (1)}$$

implies that "self-energy" terms are omitted w/ $i=j$

The most logical generalization of this expression (1) to a continuous distribution of charge $\rho(\vec{x})$ is evidently

$$U = \frac{1}{8\pi\epsilon_0} \int \frac{\rho(\vec{x})\;\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x\, d^3x'$$

or alternatively, using $\Phi(\vec{x}) = \int \frac{\rho(\vec{x}\,')\, d^3x'}{4\pi\epsilon_0 |\vec{x} - \vec{x}\,'|}$

$$\Rightarrow U = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x$$

Yet another form is $U = -\frac{\epsilon_0}{2} \int_{\text{all space}} \Phi(\vec{x}) \nabla^2 \Phi(\vec{x})\, d^3x$

and after applying Green's 1st identity with all surface terms at ∞ neglected,

$$\Rightarrow U = \frac{\epsilon_0}{2} \int_{\substack{\text{all}\\ \text{space}}} |\nabla u|^2 d^3x$$

or finally $\boxed{U = \frac{\epsilon_0}{2} \int_{\substack{\text{all}\\ \text{space}}} |\vec{E}|^2 d^3x}$ (2)

This result is usually interpreted as implying that the $\vec{E}$-field at any point $\vec{x}$ in space stores an energy density equal to $u(\vec{x}) = \frac{\epsilon_0}{2} |\vec{E}|^2$ (i.e. J/m^3)

Observe: We have an apparent inconsistency in our derivation!

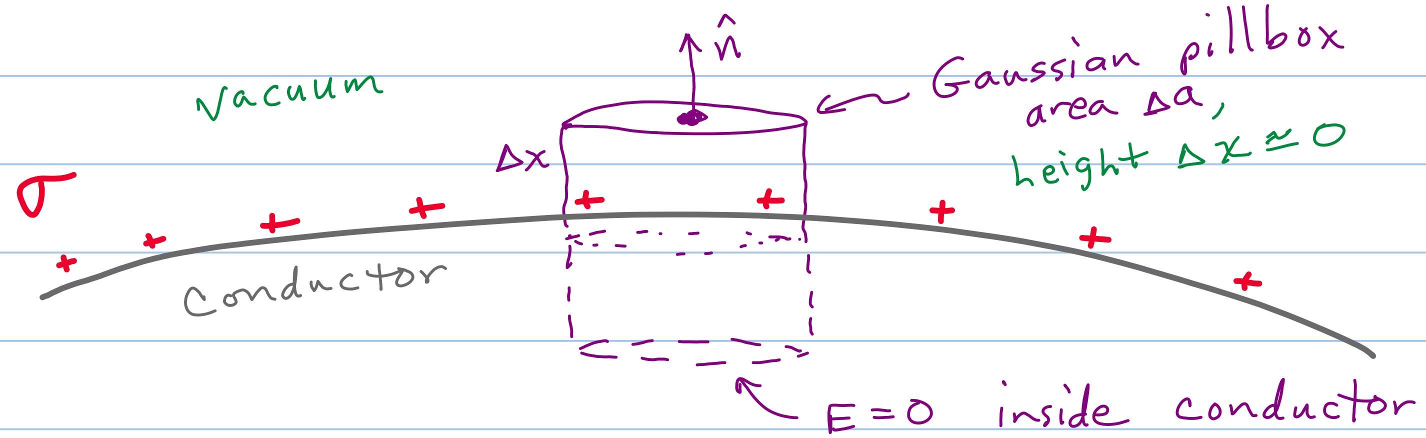
Namely, U in Eq. (1) can be positive OR negative, but U in Eq. (2) is positive definite!

Resolution This is because Eq. 2 includes the "self-energy" terms, i.e. when $\vec{x} = \vec{x}'$, whereas Eq. (1) excludes them!

Electrostatic Pressure

Consider the situation where charge density σ is on the surface of a conductor.

$\Rightarrow$ Electrostatic forces cause an outward pressure, as we now explore in detail:



Gauss's Law

$$\Rightarrow \oint \vec{E} \cdot \vec{da} = E \Delta a = \frac{\sigma \Delta a}{\epsilon_0}$$

$\Rightarrow \vec{E} = \frac{\sigma}{\epsilon_0} \hat{n} \Rightarrow$ at the surface there is an energy density

$$u(\vec{x}) = \frac{\epsilon_0}{2} \frac{\sigma^2}{\epsilon_0^2} = \frac{\text{energy}}{\text{unit volume}}$$

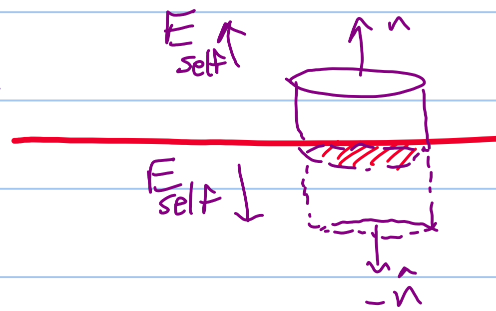
Suppose we now move a differential surface area element outward, in the direction $\hat{n}$, by a small distance Δx

$\Rightarrow$ the E-field does work

$$\Delta W = F_{other}\,\Delta x = (\sigma\,\Delta a)\,E_{other}\,\Delta x$$

In order to deduce what "E_{other}" is for this problem, consider what the electric field produced by this σ would be if this thin layer of charge were <u>not</u> on a conductor

$\Rightarrow$ Then



$$\Rightarrow E_{self}\,\Delta a + E_{self}\,\Delta a = \frac{\sigma}{\epsilon_0}\,\Delta a$$

$$\Rightarrow \vec{E}_{self} = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\hat{n}, & \text{above} \\ -\dfrac{\sigma}{2\epsilon_0}\hat{n}, & \text{below} \end{cases}$$

The <u>other</u> charges in the universe must therefore produce an electric field $\vec{E}_{other} = \dfrac{\sigma}{2\epsilon_0}\hat{n}$, which cancels $\vec{E}_{self}$ below the surface (inside the conductor), and doubles the total field outside. We are interested only in the force due to those "other" charges, since charge cannot exert a net force on itself.

$\Rightarrow \vec{F}_{other} = (\sigma\,\Delta a)\dfrac{\sigma}{2\epsilon_0}\hat{n}$, or the $\dfrac{\text{force}}{\text{unit area}}$ = PRESSURE

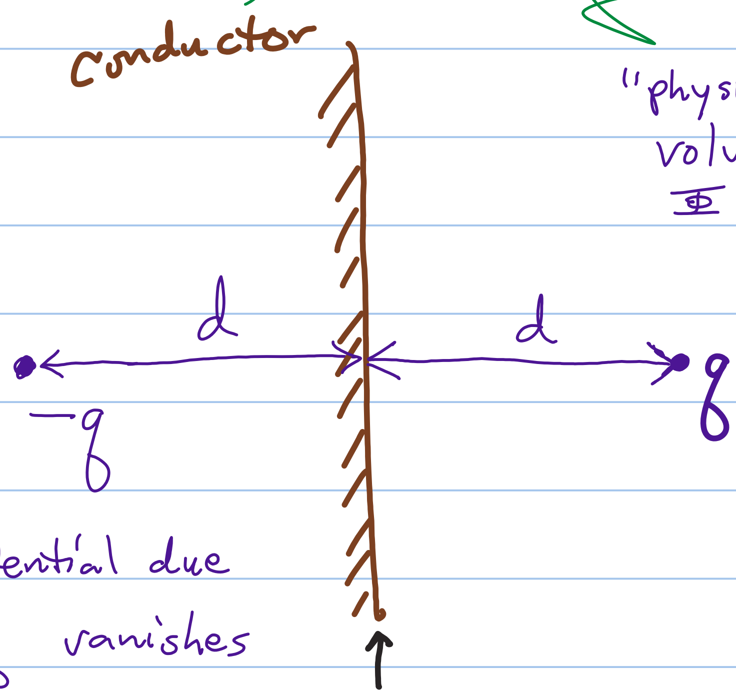
is $\boxed{\dfrac{\Delta\vec{F}}{\Delta a} = \dfrac{\sigma^2}{2\epsilon_0}\hat{n}}$

Chapter 2 Boundary value problems in electrostatics, part I

Sec. 2.1 For problems with charges near (a) boundary surface(s), we may use appropriately-placed "image charges" <u>outside</u> the volume V of interest, to simulate the boundary conditions.

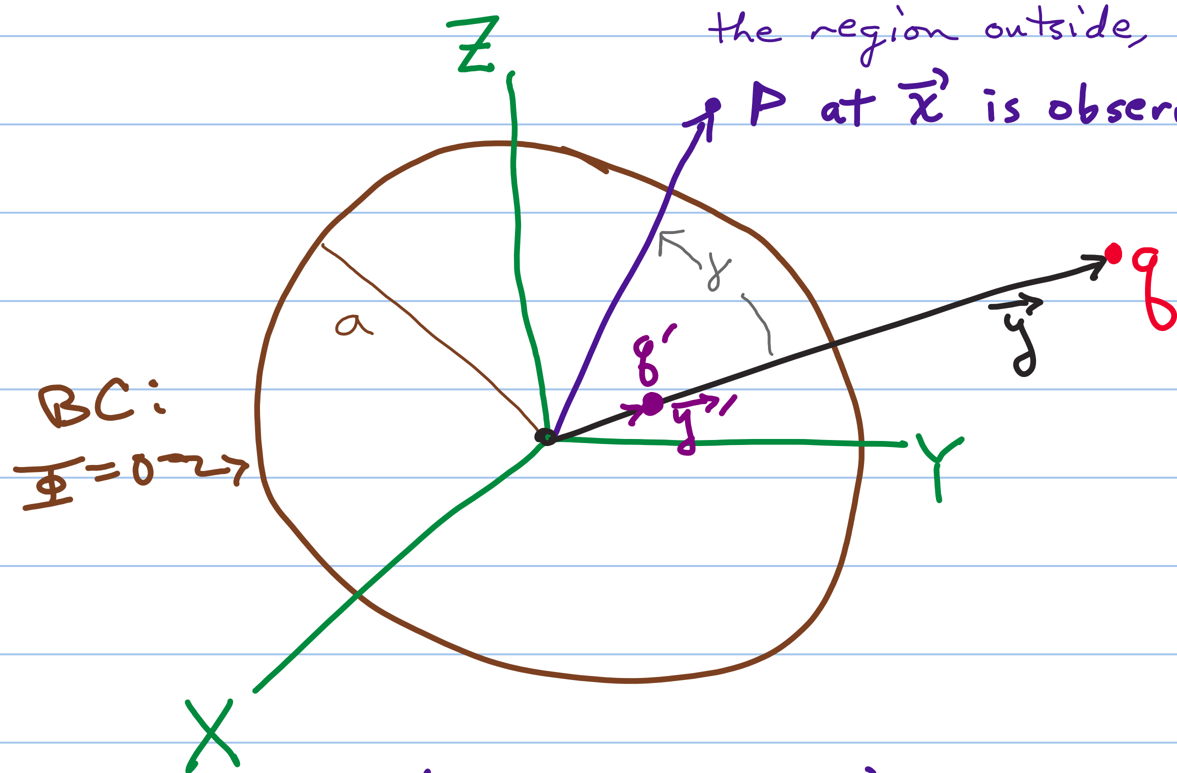
<u>Note</u> a charge placed <u>outside</u> of V will obey Laplace's equation inside V, since $\nabla^2 \dfrac{1}{|\vec{x}-\vec{x}_0|} = 0$ except near $\vec{x} \simeq \vec{x}_0$

<u>e.g.</u> the grounded, conducting plane



Clearly the potential due to q and $-q$ vanishes at $z=0$, so $\Phi_{(right)}$ can be calculated using q and $(-q)$ as point charges, only!

Sec. 2.2 Consider a point charge q at position $\vec{y}$ outside a grounded, conducting sphere of radius a. The volume V of interest is the region outside, at $|\vec{y}| > a$.



Idea: place an image charge q' INSIDE the sphere at $\vec{y}'$, and choose $q', \vec{y}'$ such that $\Phi = 0$ on the sphere surface.

By symmetry, of course, $\vec{y}'$ must be along $\vec{y}$, i.e. $\vec{y}' \propto \vec{y}$, pointing towards q. So if we set $\vec{x} = x\hat{n}$, $\vec{y} = y\hat{n}'$, $\vec{y}' = y'\hat{n}'$, with $\hat{n}$ and $\hat{n}'$ unit vectors, the total potential in V is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right)$$

Now demand that

$$4\pi\epsilon_0 \Phi(x=a) = 0 = \frac{q}{|a\hat{n}-y\hat{n}'|} + \frac{q'}{|a\hat{n}-y'\hat{n}'|}$$

or

$$0 = \frac{q}{(a^2+y^2-2ay\cos\gamma)^{1/2}} + \frac{q'}{(a^2+y'^2-2ay'\cos\gamma)^{1/2}}$$

where $\hat{n}\cdot\hat{n}' \equiv \cos\gamma$

$$\Rightarrow \frac{q'}{q} = -\frac{(a^2+y'^2-2ay'\cos\gamma)^{1/2}}{(a^2+y^2-2ay\cos\gamma)^{1/2}} \Rightarrow \frac{q'}{q} < 0$$

square this and collect terms

$$\Rightarrow \left[\left(\frac{q'}{q}\right)^2(a^2+y^2)-(a^2+y'^2)\right] + \cos\gamma\left[-\left(\frac{q'}{q}\right)^2(2ay)+2ay'\right] = 0$$

Since this equation holds for ALL γ, both terms in [ ] must separately vanish, giving 2 eqns and 2 unknowns (q'/q) and y':

SOLUTION:

$$\boxed{\frac{q'}{q} = -\frac{a}{y}, \quad y' = \frac{a^2}{y}}$$

Note that $y' < y$, so $\vec{y}'$ is outside of V, a necessary condition for an image charge.

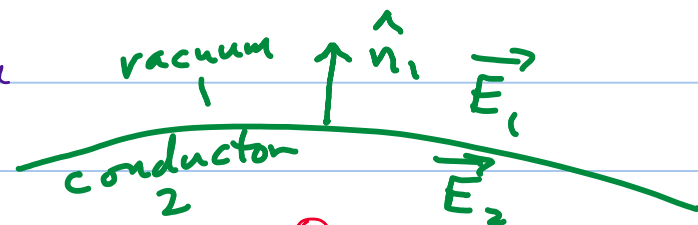
We can now determine any desired observable, e.g.

$$\Phi(x,\gamma) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{(x^2+y^2-2xy\cos\gamma)^{1/2}} - \frac{q\,a/y}{\left(x^2+\left(\frac{a^2}{y}\right)^2-2x\frac{a^2}{y}\cos\gamma\right)^{1/2}}\right\}$$

which is valid at all $x > a$, $y > a$.

Charge Density One quantity often of interest is the charge density on any conducting surface.

For the sphere problem



we saw previously that $(\vec{E}_1 - \cancelto{0}{\vec{E}_2})\cdot\hat{n}_1 = \dfrac{\sigma}{\epsilon_0}$

So here,

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a^+}$$

or $$\boxed{\sigma = -\frac{q}{4\pi a^2}\,\frac{a}{y}\,\frac{(1-a^2/y^2)}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma\right)^{3/2}}}$$

Consider the limit as $y \gg a$

$$\Rightarrow \sigma \xrightarrow[y\gg a]{} \frac{-q}{4\pi a^2}\,\frac{a}{y}\left(1+\left(-\tfrac{3}{2}\right)\left(-2\tfrac{a}{y}\right)\cos\gamma + O\!\left(\tfrac{a^2}{y^2}\right)\right)$$

$$\hookrightarrow \frac{-q}{4\pi a y} - \frac{3q}{4\pi y^2}\cos\gamma + O\!\left(\frac{a}{y^3}\right),$$

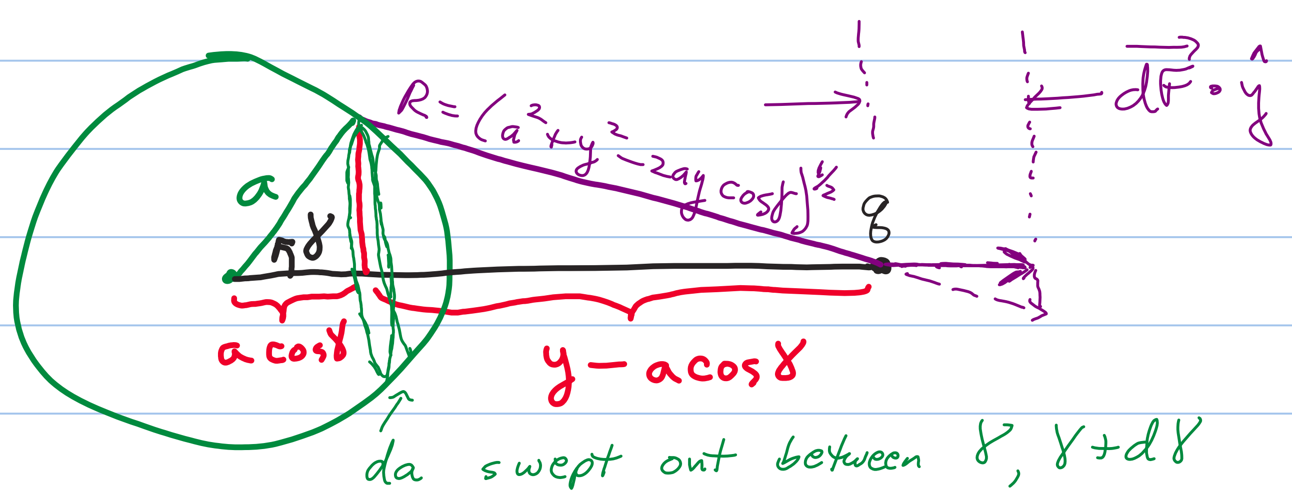
which looks approximately like a small net charge and a small electric dipole moment as $y \to \infty$.

Observe that $\oint \sigma\, da = q' = -q\frac{a}{y}$ at all y, which makes sense that the magnitude of charge on the surface equals the image charge magnitude!

Force on the charge q

Let's calculate directly the force on q, using the "real charges" we have just derived on the surface of the sphere. This is worth studying from different viewpoints, because with force and energy there can be factors of 2 or $\frac{1}{2}$ lurking.

method 1 integrate force from each $\sigma da'$:



So $$\vec{dF} \cdot \hat{y} = \left(\frac{y - a\cos\gamma}{R} \right) \frac{q}{4\pi\epsilon_0 R^2}\, \sigma(\gamma)\, 2\pi a^2 \sin\gamma\, d\gamma$$

$$\equiv dF_y$$

$$\Rightarrow dF_y = \frac{2\pi a^2 q}{4\pi\epsilon_0}\left\{\underbrace{\frac{-q}{4\pi a y}\left(1-\frac{a^2}{y^2}\right)\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{-3/2}}_{\sigma(\gamma)}\right\}$$

$$\times \frac{\sin\gamma \, d\gamma}{(y^2+a^2-2ay\cos\gamma)^{3/2}}(y-a\cos\gamma)\cdot\frac{y^3}{(y^2)^{3/2}}$$

$$= \frac{-q^2}{8\pi\epsilon_0}\frac{a}{y}\left(1-\frac{a^2}{y^2}\right)y^3 \frac{\sin\gamma\,(y-a\cos\gamma)}{(a^2+y^2-2ay\cos\gamma)^3}$$

Now integrate this from $\gamma=0$ to π, giving

$$\vec{F} = \frac{-q^2}{8\pi\epsilon_0}a(y^2-a^2)\left[\frac{2y}{(y^2-a^2)^3}\right]\hat{y}$$ Mathematica

or finally $$\vec{F} = -\hat{y}\frac{q^2}{4\pi\epsilon_0}\frac{ay}{(y^2-a^2)^2}$$

method 2 interestingly, the above result agrees exactly with the trivial method of just computing the force on q due to the point change q' at y'!

$\Rightarrow$ much simpler to compute that way!

method 3 Use Newton's 3rd law, compute the force on the sphere due to q, then multiply by (-1) to get the force on q. This gives the same answer also!

Section 2.3 Next treat a point charge q outside a charged, insulated, conducting sphere carrying total charge Q

Concept First solve for the potential in the case where q is at distance $y > a$ from a grounded sphere.

$\Rightarrow$ We saw previously that this induces a total charge $q' = -q\frac{a}{y}$ on the sphere, in order for the sphere to be an equipotential.

$\Rightarrow$ Start from that solution and add a small additional amount of charge

CLAIM: The additional charge must spread over the sphere surface symmetrically, because the sphere is an equipotential. If we keep adding charge this way until the total amount of charge is Q, this means we must add a total amount of extra charge equal to

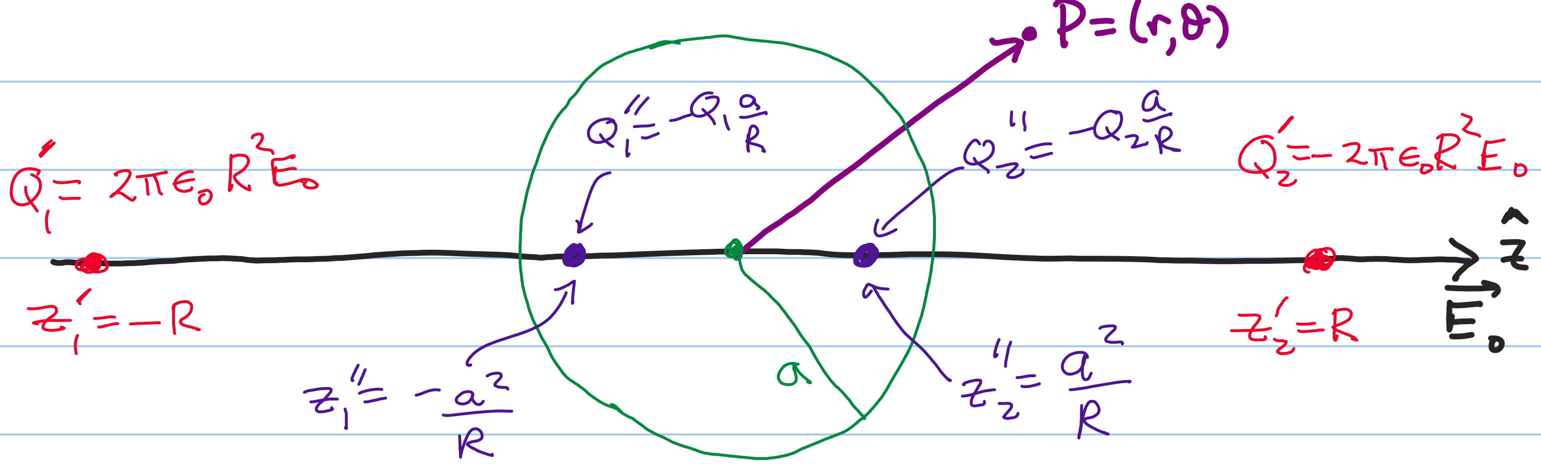
$$Q_{NEW} = Q - q' = Q + \frac{a}{y}q$$

We also know that the potential outside a spherically-symmetric distribution of charge Q_{NEW} is the same as if it were all at the center!

Therefore the total potential outside is equal to

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{y}\frac{1}{|\vec{x}-\frac{a^2}{y^2}\vec{y}|} + \frac{Q+\frac{a}{y}q}{|\vec{x}|}\right\}}$$

Section 2.5 Consider a grounded conducting sphere in a uniform external E-field, $\vec{E}_0 = E_0\hat{z}$



We solve this as a limiting case, with two charges far away $(R\to\infty)$ that set up the uniform electric field. The solution is again carried out using image charges.

$\Rightarrow$ The potential at all points outside the sphere is then

$$\Phi(r,\theta) = \lim_{R\to\infty} \frac{1}{4\pi\epsilon_0}\frac{Q_1'}{[R^2+r^2-2rR\cos(\pi-\theta)]^{1/2}}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{(-aQ_1'/R)}{[(a^2/R)^2+r^2-2r(a^2/R)\cos(\pi-\theta)]^{1/2}} + \frac{1}{4\pi\epsilon_0}\frac{Q_2'}{[R^2+r^2-2rR\cos\theta]^{1/2}}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{(-aQ_2'/R)}{[(a^2/R)^2+r^2-2r(a^2/R)\cos\theta]^{1/2}}$$

It is easiest to find the large $R \to \infty$ limit if we first make a binomial expansion

i.e. $(1+\epsilon)^{\gamma} = 1 + \gamma\epsilon + \frac{\gamma(\gamma-1)}{2!}\epsilon^2 + \ldots$

... skipping a few lines of algebra ...

$$\Rightarrow \Phi(r,\theta) = \lim_{R\to\infty} \frac{E_0 R}{2}\left[1 - \frac{r}{R}\cos\theta - 1 - \frac{r}{R}\cos\theta\right]$$

$$- \frac{E_0 R}{2}\frac{a}{r}\left[1 - \frac{a^2}{rR}\cos\theta - 1 - \frac{a^2}{rR}\cos\theta\right]$$

$$+ \text{ terms that vanish at } R \to \infty$$

or $$\boxed{\Phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta}$$

Aside: observing that this Φ vanishes at $\theta = \frac{\pi}{2}$, we notice that we have solved another problem too, namely a grounded sphere + a grounded conducting plane at $z=0$.

<u>Induced</u> <u>surface</u> <u>charge</u> on the sphere:

$$\sigma(a,\theta) = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \epsilon_0 E_0\cos\theta + \epsilon_0 E_0 \frac{2a^3}{a}\cos\theta$$

or $$\boxed{\sigma(a,\theta) = 3\epsilon_0 E_0 \cos\theta}$$

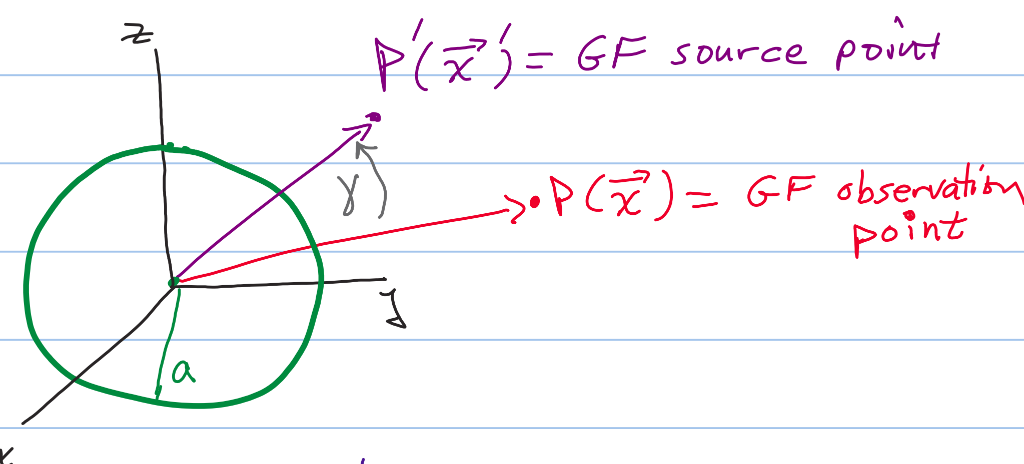
and since the integral vanishes, this means there is NO NET CHARGE on the sphere.

Dirichlet Green's Function for a sphere

Using the Green's function, we could solve many different problems. Recall that the Dirichlet GF for the exterior of a sphere must vanish at $r=a$, and it must obey $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$ where $\vec{x}', \vec{x}$ are arbitrary positions outside the sphere, i.e. $G(|\vec{x}|=a, \vec{x}') = 0$.

This GF can be found using the method of images now, simply by inspection:

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|}$$



Making this more explicit yields

$$\boxed{G(\vec{x},\vec{x}') = (x^2+x'^2-2xx'\cos\gamma)^{-1/2} - \left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{-1/2}}$$

To solve a general Dirichlet problem we will sometimes need the normal derivatives of G, i.e.

$$\boxed{\frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\Big|_{x'=a} = -\frac{\partial G(\vec{x},\vec{x}')}{\partial x'}\Big|_{x'=a} = \frac{a^2-x^2}{a(a^2+x^2-2ax\cos\gamma)^{1/2}}}$$

Next – A typical application: Find Φ outside a sphere whose two conducting hemispheres are held at equal + opposite potentials $\pm V$.

e.g. top is at $+V$ $(0 \le \theta \le \pi/2)$, (sphere radius a)

bottom (insulated from top) is at $-V$ $(\frac{\pi}{2} < \theta \le \pi)$

Solution Here $\rho = 0$ outside so there are only surface integrals in this Dirichlet BC problem.

$$\Rightarrow \Phi(r,\theta,\phi) = \frac{1}{4\pi}\underbrace{\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' a^2}_{\int da'} \frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\Big|_{x'=a} V(\theta',\phi')$$

$$= \frac{a^2 V}{4\pi}\int_0^{2\pi} d\phi' \left[\int_0^1 d(\cos\theta') - \int_{-1}^0 d(\cos\theta')\right] \times$$

$$\times \left\{\frac{a^2-x^2}{a(a^2+x^2-2ax\cos\gamma)^{3/2}}\right\}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

which has a complicated dependence on $\theta, \phi, \theta', \phi'$.

Jackson works this out for a special case, $\theta = 0 = \phi$.

Some mathematical preliminaries or review

⇒ Orthogonal functions and expansions

- Expansions can provide an excellent method for electrostatic potential problems, and in other physics subfields, especially QM.

The simplest and usual case works with an orthonormal set of functions, say,

$$\{u_n(\xi)\}, \quad n = 1, 2, \ldots \infty$$

with the independent variable ξ having a finite range $\quad a \leq \xi \leq b$

with orthogonality relation written as:

$$\int_a^b u_n^*(\xi)\, u_m(\xi)\, d\xi = \delta_{nm}$$

Assume for now that the $u_n(\xi)$ are square-integrable, i.e.

$$N^2 \equiv \int_a^b |\bar{u}_n(\xi)|^2 d\xi < \infty$$

whereby one can construct normalized $u_n(\xi)$ as

$$u_n(\xi) = \frac{1}{N}\bar{u}_n(\xi)$$

Usually the set $\{u_n(\xi)\}$ is also <u>complete</u>, in which case any square-integrable function $f(\xi)$ over $a \leq \xi \leq b$ can be expanded as a linear combination, $f(\xi) = \sum_{n=1}^{\infty} a_n u_n(\xi)$

When $f(\xi)$ is known, the a_n can be determined by what Griffiths calls "Fourier's trick"

$$a_n = \int_a^b u_n^*(\xi) f(\xi)\, d\xi$$

$$= \int_a^b u_n^*(\xi') f(\xi')\, d\xi'$$

Plugging this last expression into the $f(\xi)$ expansion gives

$$f(\xi) = \int_a^b d\xi' \left[\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi)\right] f(\xi')$$

This implies the completeness or closure relation,

$$\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) = \delta(\xi' - \xi)$$

Using notation slightly different from Jackson, here is one example of a Fourier series over a symmetric interval $-\frac{a}{2} \le x \le \frac{a}{2}$

$$U_m^e(x) = \sqrt{\frac{2-\delta_{m,0}}{a}} \cos\left(\frac{2\pi m x}{a}\right), \quad m = 0, 1, 2, \ldots$$

$$U_m^o(x) = \sqrt{\frac{2}{a}}\, \sin\left(\frac{2\pi m x}{a}\right), \quad m = 1, 2, 3, \ldots$$

where the combination of both the even (e) and odd (o) sets gives a complete, orthonormal set, $\{U_m^e(x), U_m^o(x)\}$, can write for any $f(x)$,

$$\Rightarrow f(x) = \sum_{m=0}^{\infty} A_m U_m^e(x) + \sum_{m=1}^{\infty} B_m U_m^o(x)$$

where $$A_m = \left(\frac{2-\delta_{m,0}}{a}\right)^{1/2} \int_{-a/2}^{a/2} \cos\frac{2\pi m x}{a} f(x)\,dx$$

$$B_m = \left(\frac{2}{a}\right)^{1/2} \int_{-a/2}^{a/2} \sin\frac{2\pi m x}{a} f(x)\,dx$$

and these relations generalize to 2 or more dimensions in the natural way, e.g.

$$F(\xi,\eta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, u_m(\xi)\, v_n(\eta),$$

and, e.g. $$A_{mn} = \int_a^b d\xi \int_c^d d\eta\, u_m^*(\xi) v_n^*(\eta) F(\xi,\eta)$$

etc.

Generalization to an INFINITE INTERVAL:

We can rewrite the above sin, cos example in terms of exponentials, e.g.

let $$U_n(x) = a^{-1/2} e^{\frac{2i\pi n x}{a}}, \quad n = 0, \pm1, \pm2, \ldots$$

$$\Rightarrow \quad F(x) = \sum_{n=-\infty}^{\infty} A_n \frac{1}{\sqrt{a}} e^{2i\pi n x/a}$$

where $$A_n = \int_{-\frac{a}{2}}^{a/2} \frac{1}{\sqrt{a}} e^{-\frac{2i\pi n x}{a}} F(x)\,dx$$

We take the $a \longrightarrow \infty$ limit, defining:

$$k \equiv \frac{2\pi n}{a}$$

$$\sum_m \longrightarrow \int dm = \frac{a}{2\pi}\int dk$$

$$A_m \rightarrow \left(\frac{2\pi}{a}\right)^{1/2} A(k)$$

Then (a) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$

(b) $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x')\, dx'$

and

(c) $f(x) = \int_{-\infty}^{\infty} dx'\, f(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} \right]$

which again implies a completeness relation,

(d) $\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x')$

and we will need the "orthonormality relation",

(e) $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik'x}\, e^{ikx}\, dx = \delta(k-k')$

These are the basic equations of Fourier Analysis

Solution of partial differential equations by separation of variables

IDEA when possible, there is great simplification if a PDE can be solved by the separable product form

e.g. $\Phi(x,y,z) = X(x)\, Y(y)\, Z(z)$

For any specific problem the solution may not have such a simple form, but it is still often useful to solve the PDE in this form as an initial step in the solution process.

Let's use this ansatz to solve

$$\nabla^2 \Phi(x,y,z) = 0 = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Plug in the above separable form, divide the entire equation by $X(x)\,Y(y)\,Z(z)$

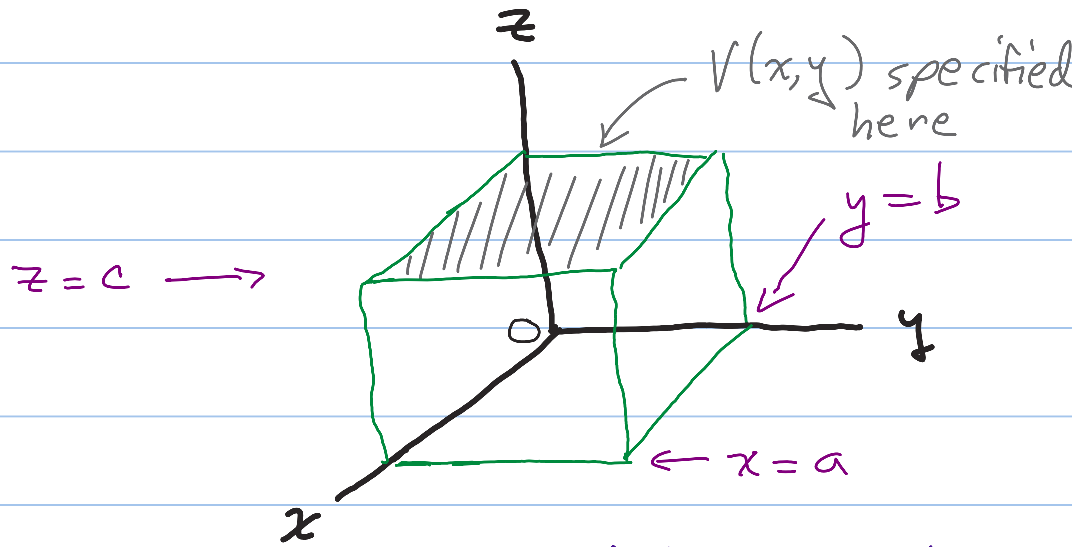
$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

and since this must hold for <u>all</u> x, y, z, each term must separately be a constant, i.e.

$$\frac{X''(x)}{X(x)} = -\alpha^2, \quad \frac{Y''(y)}{Y(y)} = -\beta^2$$

$$\frac{Z''(z)}{Z(z)} = \alpha^2 + \beta^2$$

separation constants



A useful rule of thumb for problems like this: Choose the sign of the separation constants to produce <u>oscillatory</u> solutions in any dimension having a

$\Phi = 0$ boundary condition at both limits.

e.g. suppose we are given the BCs to be

$$\Phi(0,y,z) = \Phi(a,y,z) = 0$$
$$\Phi(x,0,z) = \Phi(x,b,z) = 0$$
$$\Phi(x,y,0) = 0; \ \Phi(x,y,c) = V(x,y)$$

$\Rightarrow$ the above choice of signs of α, β works well for this PDE. Assume initially that α, β = real, but keep open the possibility that they could later be found to be imaginary or complex.

$\Rightarrow$ $X(x)$ = linear combinations of $e^{\pm i\alpha x}$

$Y(y)$ = " " " $e^{\pm i\beta y}$

$Z(z)$ = " " $e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$

By inspection the linear combinations needed in x, y are:

$$X_m(x) = \left(\frac{2}{a}\right)^{1/2} \sin\frac{m\pi x}{a}, \quad 0 \le x \le a,$$
$$\left(\text{i.e. } \alpha = \frac{m\pi}{a}, \ m = 1, 2, 3, \ldots\right)$$

and

$$Y_n(y) = \left(\frac{2}{b}\right)^{1/2} \sin\frac{n\pi y}{b}, \quad 0 \le y \le b$$
$$\left(\beta = \frac{n\pi}{b}, \ n = 1, 2, \ldots\right)$$

And the B.C. $Z(0) = 0$ implies that the only solution in z must be proportional to

$$Z_{m,n}(z) = \sinh\left[\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right]$$

for $0 \le z \le c$

Hence an infinite set of product solutions that obey all BC's in the problem $X_m(x)\,Y_n(y)\,Z_{mn}(z)$ EXCEPT at $z = c$, where we must still find a way to satisfy the B.C.:

$$\Phi(x,y,c) = V(x,y)$$ ← specified + assumed to be known

Using the linearity of the PDE, we can superpose this set with coefficients A_{mn} still to be determined to match the BCs:

$$\Rightarrow \Phi(x,y,z) = \sum_{m,n=1}^{\infty} A_{mn} \left(\frac{2}{a}\right)^{1/2} \left(\frac{2}{b}\right)^{1/2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh(\gamma_{mn} z)$$

where $\gamma_{mn} \equiv \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\;\pi$

DEMAND

$$\xrightarrow[z \to c]{} V(x,y)$$

Use the orthonormality of the $\left(\frac{2}{a}\right)^{1/2} \sin\frac{m\pi x}{a}$ and $\left(\frac{2}{b}\right)^{1/2} \sin\frac{n\pi y}{b}$

$\left[\text{i.e. } \int_0^a dx \frac{2}{a} \sin\frac{m\pi x}{a} \sin\frac{m'\pi x}{a} = \delta_{mm'}\right]$

Now apply "Fourier's Trick", mutiplying both sides by $\left(\frac{2}{a}\right)^{1/2} \sin \frac{m'\pi x}{a} \left(\frac{2}{b}\right)^{1/2} \sin \frac{n'\pi y}{b}$

and integrate, giving

$$A_{mn} = \frac{\int_0^a dx \int_0^b dy \left(\frac{4}{ab}\right)^{1/2} V(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{\sinh \gamma_{mn} c}, \quad m = 1, 2, \ldots \infty, \quad n = 1, 2, \ldots \infty$$

$$\gamma_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\, \pi$$

Now when these constant coefficients are used in the infinite expansion above, then the above expression for $\Phi(x,y,z)$ will obey all the BCs and the PDE $\nabla^2 \Phi = 0$.

<u>Other little tricks:</u>

- If $\Phi \neq 0$ is specified on any of the other sides, simply solve for one nonzero side at a time, but then use linearity and add them all up to make the final solution.

- If $\rho \neq 0$ inside as well, then we must use an appropriate Green's Function inside, as discussed in Chapter 3.

Chapter 3 Boundary value problems in electrostatics II

IMPORTANT: it is crucial whenever possible to solve a PDE in a coordinate system adapted to the shapes of the boundary surfaces.

Laplace's equation in spherical coordinates

$$\nabla^2 \Phi(r,\theta,\phi) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$= 0$$

separable spherical solutions are found by setting $\Phi(r,\theta,\phi) = \frac{u(r)}{r} P(\theta) Q(\phi)$

$$\Rightarrow r^2\sin^2\theta\left\{\frac{u''(r)}{u(r)} + \frac{1}{r^2\sin\theta}\frac{(\sin\theta\, P'(\theta))'}{P(\theta)}\right\} + \frac{Q''(\phi)}{Q(\phi)} = 0 \qquad (*)$$

$\Rightarrow \frac{Q''(\phi)}{Q(\phi)} = -m^2 = \text{constant}$ since $(*)$ must hold for all r, θ, ϕ

$$\Rightarrow Q(\phi) = e^{\pm im\phi}$$

and if the full angular range $0 \leq \phi < 2\pi$ is allowed, $Q(\phi)$ must be single-valued,

$$\Rightarrow m = \text{any integer}$$

Returning to the r, θ equations,

$$\Rightarrow \underbrace{r^2\frac{u''(r)}{u(r)}}_{\text{all } r\text{-dependence}} + \underbrace{\frac{1}{\sin\theta}\frac{(\sin\theta\, P'(\theta))'}{P(\theta)} - \frac{m^2}{\sin^2\theta}}_{\text{all } \theta\text{-dependence}} = 0$$

$\Rightarrow$ The two circled terms must equal constants that sum to 0, call them $\pm\, \ell(\ell+1)$

$$\Rightarrow \quad -\frac{1}{\sin\theta}\left(\sin\theta\, P'(\theta)\right)' + \frac{m^2}{\sin^2\theta} P(\theta) = \ell(\ell+1)\, P(\theta) \qquad (**)$$

and then
$$u''(r) - \frac{\ell(\ell+1)}{r^2}\, u(r) = 0 \qquad (***)$$

For any given (ℓ, m) there are 2 linearly independent solutions called Associated Legendre Functions

$$P_{\ell m}(\cos\theta)\,, \quad Q_{\ell m}(\cos\theta)$$

LegendreP[ℓ, m, Cos[θ]], in Mathematica
or Q

- If $m =$ integer, only the solutions $P_{\ell m}(\cos\theta)$ are regular at $\theta = 0$, and these will not be well-behaved at $\theta = \pi$ unless $\ell =$ integer $\geq |m|$
i.e. $\ell = |m|, |m|+1, \ldots$ etc.

- If the electrostatic potential is known to be azimuthally symmetric about the z-axis, (meaning independent of ϕ), then only $m = 0$ can be present. The associated Legendre functions for $m=0$ are simpler + are called Legendre polynomials, $P_\ell(x)$ or LegendreP[ℓ, x] ($x = \cos\theta$ here)

Legendre Polynomial properties:

- Differential equation:

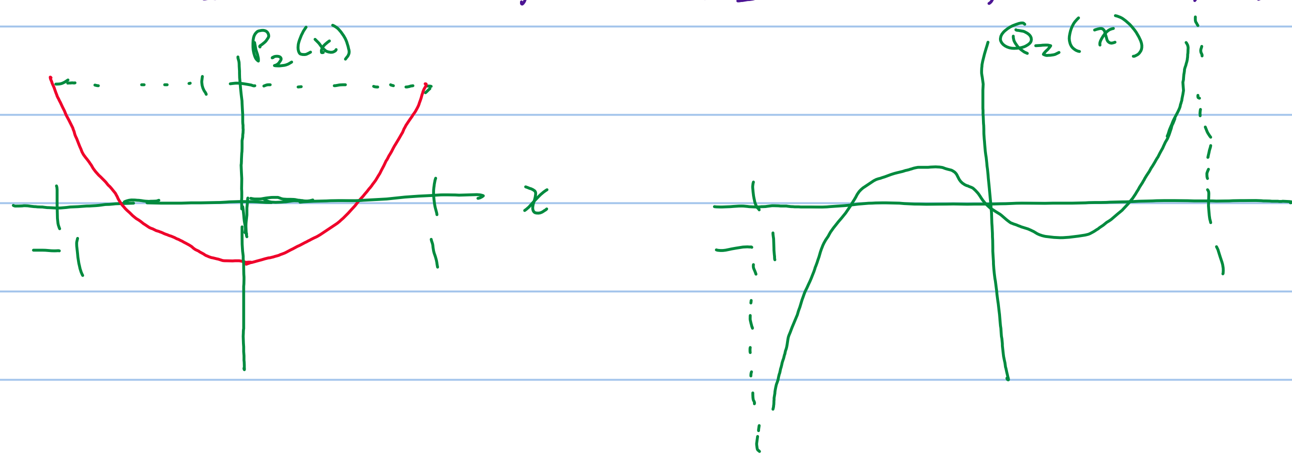
$$\frac{d}{dx}(1-x^2)\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0$$

to solve it, this is simple via power series, or it can be coaxed out of Mathematica, e.g.

```
DSolve[(1-x^2)P''[x]-2xP'[x]+l(l+1)P[x]==0, P[x], x]
```

returns

```
P[x] -> C[1] LegendreP[l,x] + C[2] LegendreQ[l,x]
```



Things you should know about Legendre polynomials:

- First few:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$\vdots$$

- Highest power of x in $P_\ell(x)$ is x^ℓ
- Parity under $x \to -x$ is $(-1)^\ell$, i.e. $P_\ell(-x) = (-1)^\ell P_\ell(x)$

- Rodriguez' formula $$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

- Orthogonality $$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

(can prove this using Rodriguez' formula, then integrate by parts)

- Completeness: over $-1 \le x \le 1$, can expand "any" function, as
$$F(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$
where $$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} P_\ell(x) F(x)\, dx$$

- Recurrence relations, e.g.

(a) $P'_{\ell+1} - P'_{\ell-1} = (2\ell+1) P_\ell$

(b) $(\ell+1) P_{\ell+1} - (2\ell+1) x P_\ell + \ell P_{\ell-1} = 0$

(c) $P'_{\ell+1} - x P'_\ell - (\ell+1) P_\ell = 0$

(d) $(x^2-1) P'_\ell - x\ell P_\ell + \ell P_{\ell-1} = 0$

for others, see Jackson Eq. 3.29
use such relations to evaluate integrals like
$$\int_{-1}^{1} x^k P_\ell(x) P_{\ell'}(x)\, dx, \text{ etc.}$$

Finally, the radial equation is solved as:

$$r^2 \frac{u''(r)}{u(r)} = \ell(\ell+1) \Rightarrow u(r) = r^{\ell+1} \text{ or } r^{-\ell}$$

$$\Rightarrow \frac{u(r)}{r} = A_\ell r^\ell + B_\ell r^{-\ell-1}$$ is the general solution.

Azimuthally-symmetric boundary value problems

For problems having no ϕ-dependence, the general solution to $\nabla^2 \Phi = 0$ takes the form:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

where the constants A_ℓ, B_ℓ are found by imposing boundary conditions.

EXAMPLE Suppose a sphere of radius a has a specified surface potential equal to

$$V(\theta) = V_0 \cos^2\theta$$

BCs: Require $\Phi \xrightarrow[r\to 0]{}$ finite

and $\Phi \xrightarrow[r\to\infty]{} 0$ ← can demand this only for finite charge distributions

$\Rightarrow B_\ell \xrightarrow[\text{inside}]{} 0$, $A_\ell \xrightarrow[\text{outside}]{} 0$

i.e. $\Phi_{inside}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

$\Phi_{outside}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

Since there is no dipole layer, Φ = continuous,

$\Rightarrow$ $\Phi_{inside}(a,\theta) = V(\theta) = \Phi_{outside}(a,\theta)$

SOLUTION use these and orthonormality of $P_l(\cos\theta)$ to find A_l, B_l:

$$\text{e.g. } \sum_l A_l a^l P_l(\cos\theta) = V_0 \cos^2\theta$$

mutiply by $P_{l'}(\cos\theta) \equiv P_{l'}(x)$ and integrate:

$$\int_{-1}^{1} dx\, P_{l'}(x) \sum_l A_l a^l P_l(x) = V_0 \int_{-1}^{1} P_{l'}(x)\, x^2 dx$$

or $$\frac{2}{2l+1} A_l a^l = V_0 \int_{-1}^{1} P_l(x)\, x^2 dx$$

But notice that $x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$

$$\Rightarrow \int_{-1}^{1} P_l(x)\, x^2 dx = \frac{2}{5}\left(\frac{2}{3}\right)\delta_{l,2} + \frac{1}{3}(2)\,\delta_{l,0}$$

$$\Rightarrow A_2 = \frac{2}{3}\frac{V_0}{a^2}, \quad A_0 = \frac{1}{3}V_0$$

and similarly $\frac{2}{5} B_2 a^{-3} = V_0 \frac{4}{15}$ or $B_2 = \frac{2}{3} a^3 V_0$

and $B_0 = \frac{1}{3} a V_0$

So finally

$$\boxed{\begin{aligned}\Phi_{inside}(r,\theta) &= \tfrac{1}{3}V_0 + \tfrac{2}{3}V_0\left(\tfrac{r}{a}\right)^2 P_2(\cos\theta)\\ \Phi_{outside}(r,\theta) &= \tfrac{1}{3}V_0\tfrac{a}{r} + \tfrac{2V_0}{3}\tfrac{a^3}{r^3} P_2(\cos\theta)\end{aligned}}$$

EXAMPLE Construct the potential everywhere in space for an azimuthally-symmetric problem, starting from the solution known only on the symmetry axis.

e.g. since we know that in a charge-free region of space, the general ϕ-independent potential is

$$\Phi(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

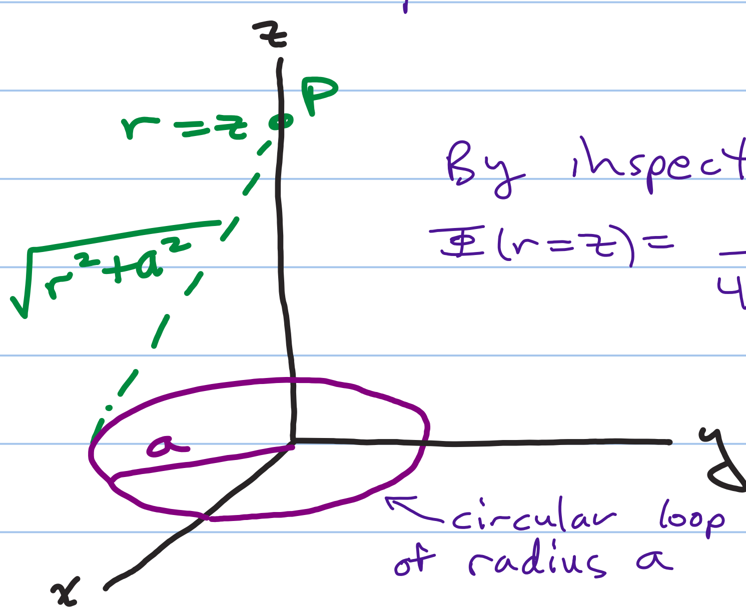
we can evaluate this on axis where $\theta = 0$, $P_\ell(1) = 1$,

$$\Rightarrow \quad \Phi(r,0) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)$$

We can turn this logic around, as follows:

Suppose we know $\Phi(r=z)$ along the z-axis for an azimuthally-symm. problem. If we expand $\Phi(r=z)$ in a series, we can deduce $\Phi(r,\theta)$ everwhere.

To be concrete, consider a ring of total charge Q in the xy-plane,



By inspection

$$\Phi(r=z) = \frac{Q}{4\pi\epsilon_0 \sqrt{r^2+a^2}}$$

Next, expand Φ at $r>a$ as a binomial expansion:

$$\Phi(r=z) = \frac{Q}{4\pi\epsilon_0 r}\left(1+\frac{a^2}{r^2}\right)^{-1/2}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{a^2}{r^2}\right)^2 + \dots\right\}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\frac{a^4}{r^4} - \frac{15}{48}\frac{a^6}{r^6} + \dots\right\}$$

$$= \sum_\ell \frac{B_\ell}{r^{\ell+1}} \quad \text{with } B_0 = \frac{Q}{4\pi\epsilon_0}, \; B_1 = 0$$

$$B_2 = \frac{-Qa^2}{8\pi\epsilon_0}, \text{ etc...}$$

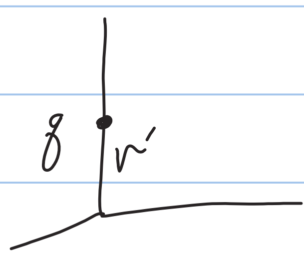
Then by the above logic, the potential off-axis at $r>a$ must be

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{r} - \frac{a^2}{2r^3}P_2(\cos\theta) + \frac{3}{8}\frac{a^4}{r^5}P_4(\cos\theta) + \dots\right\}$$

Another application of this logic:

The potential at $\vec{x}$ due to a point charge q at $\vec{x}'$ is $\Phi(\vec{x},\vec{x}') = \dfrac{q}{4\pi\epsilon_0|\vec{x}-\vec{x}'|}$

If we choose $\vec{x}'$ to lie <u>on</u> <u>the</u> <u>z-axis</u> then we have axial symmetry, i.e. $r'=z'$



Now, the potential at any point $\vec{x} = r\hat{z}$ on the z-axis is of course

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0 |r-r'|} = \frac{q}{4\pi\epsilon_0} \begin{cases} \frac{1}{r}\left(1-\frac{r'}{r}\right)^{-1}, & r'<r \\ \frac{1}{r'}\left(1-\frac{r}{r'}\right), & r'>r \end{cases}$$

or in a more succinct notation, <u>define</u>

$$r_< = \min(r, r')$$

$$r_> = \max(r, r')$$

So in general, on axis,

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0} \frac{1}{r_>} \left(1 - \frac{r_<}{r_>}\right)^{-1}$$

and performing a binomial expansion gives

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0} \frac{1}{r_>} \left\{ 1 + \frac{r_<}{r_>} + \frac{(-1)(-2)}{2!}\left(\frac{r_<}{r_>}\right)^2 + \cdots \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^{\,l}}{r_>^{\,l+1}}$$

and as before, this result generalizes off-axis to

$$\Phi(r, \theta; r', 0) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^{\,l}}{r_>^{\,l+1}} P_l(\cos\theta)$$

Finally, noting that there was nothing special about choosing $\vec{x}'$ on the z-axis, we have generally

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^{\,l}}{r_>^{\,l+1}} P_l(\cos\gamma)$$

where γ = angle between $\vec{x}$ and $\vec{x}'$

This last result is also equal to $G(\vec{x}, \vec{x}')$, the free space Green's function with no finite boundaries.

Note that we can write $\cos\gamma$ in terms of the separate spherical coordinates of $\vec{x}$ and $\vec{x}'$, as follows:

Start from $\hat{x} \cdot \hat{x}' = \cos\gamma$

but $\vec{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\vec{x}' = r'(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$

and $\hat{x} \cdot \hat{x}' = \sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi') + \cos\theta\cos\theta'$

or $$\boxed{\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')}$$

useful!

Read sec. 3.4 on your own.

Next, suppose the system has <u>no azimuthal symmetry</u> $\Rightarrow$ NEED SPHERICAL HARMONICS!

The angular diff. equation is now

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0$$

& solutions are associated Legendre functions,

$$P_l^m(x), \quad x = \cos\theta, \quad -1 \le x \le 1$$

and for $l, m =$ integers, an explicit formula is

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

Other relations: $P_\ell^m(1) = \delta_{m,0}$

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell, \text{ for } -\ell \le m \le \ell$$

and also, $$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

Important! The $P_\ell^m(x)$ with the SAME m but different ℓ are orthogonal and complete, i.e.

$$\int_{-1}^{1} P_\ell^m(x)\, P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell,\ell'}$$

Recalling that the ϕ-dependence for the Laplace equation solutions must be $e^{im\phi}$, the full solution in any charge-free region can be written (assuming no angular boundaries) as:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right) P_\ell^m(\cos\theta)\, e^{im\phi}$$

But it is usually more convenient if we work instead with spherical harmonics, having unit normalization, $Y_{\ell m}(\theta,\phi)$, obeying orthonormality,

$$\int d\Omega\, Y_{\ell' m'}^*(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\, \delta_{mm'}$$

$$\Rightarrow Y_{lm}(\vartheta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\vartheta)\, e^{im\phi}$$

whose completeness relation reads:

$$\sum_{lm} Y^*_{lm}(\vartheta',\phi')\, Y_{lm}(\vartheta,\phi) = \delta(\phi-\phi')\,\frac{\delta(\vartheta-\vartheta')}{\sin\vartheta}$$

$$= \delta(\phi-\phi')\,\delta(\cos\vartheta-\cos\vartheta')$$

or with the useful abbreviations in spherical coordinates, $\hat{x}\equiv(\vartheta,\phi)$, $\hat{x}'\equiv(\vartheta',\phi')$

$$\Rightarrow \sum_{l,m} Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x}) = \delta^{(2)}(\hat{x}-\hat{x}')$$

Miscellaneous useful relations:

$$Y^*_{lm}(\vartheta,\phi) = (-1)^m Y_{l,-m}(\vartheta,\phi)$$

and $$Y_{l0}(\vartheta,\phi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\vartheta)$$

and in Mathematica,

SphericalHarmonicY[L, M, θ, φ]

Spherical Harmonic Addition Theorem:

important! $$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\vartheta',\phi')\, Y_{lm}(\vartheta,\phi)$$

where $\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\phi-\phi')$,

i.e. γ = angle between $\hat{x}$ and $\hat{x}'$

or $$P_l(\hat{x}\cdot\hat{x}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x})$$

Read Jackson's Proof in Sec. 3.6 and/or test it, e.g using Mathematica.

$$\Rightarrow \boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x})}$$

memorize this!

Next: Green's function construction

Again, for the Dirichlet problem, we pick $G \to 0$ on boundary surfaces, and then

$$\Phi(\vec{x}) = \int_V G(\vec{x},\vec{x}') \frac{\rho(\vec{x}')}{4\pi\epsilon_0} d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\,\Phi(\vec{x}')\,da'$$

when $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

For many problems with spherical boundary surfaces, it is convenient to expand G into spherical harmonics, starting from:

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

$$= \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\hat{x}')\,Y_{lm}(\hat{x})$$

Strategies to construct multidimensional G:

(I) Eigenfunction in ALL coordinates

e.g., to solve

$$\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi\,\delta(\vec{x} - \vec{x}\,'), \qquad (1)$$

subject to some BCs, such as homogeneous Dirichlet BCs $\Rightarrow$ $G(\vec{x}, \vec{x}\,') = 0$ for $\vec{x}$ on S,

we can consider a companion problem, the homogeneous eigenvalue problem for eigenfunctions $\psi_n(\vec{x})$ obeying the SAME BCs as $G(\vec{x}, \vec{x}\,')$, i.e. consider

$$-\nabla^2 \psi_n(\vec{x}) = \lambda_n \psi_n(\vec{x}) \qquad (2)$$

Now, one can show that these $\psi_n(\vec{x})$ are eigenfunctions of a Hermitian operator and they can be chosen to be orthonormal and complete, i.e.

$$\int_V \psi_n^*(\vec{x})\, \psi_{n'}(\vec{x})\, d^3x = \delta_{nn'} \qquad (3)$$

and

$$\sum_n \psi_n^*(\vec{x}\,')\, \psi_n(\vec{x}) = \delta(\vec{x} - \vec{x}\,') \qquad (4)$$

So expanding G into this complete set looks like

$$G(\vec{x}, \vec{x}\,') = \sum_n c_n(\vec{x}\,')\, \psi_n(\vec{x}) \qquad (5)$$

and plug this into (1) to find the expansion coefficients $c_n(\vec{x}\,')$:

i.e. $-\nabla^2 \sum_n c_n(\vec{x}\,') \psi_n(\vec{x}) = +4\pi\,\delta(\vec{x}-\vec{x}\,')$

$$\hookrightarrow = \sum_n c_n(\vec{x}\,') \lambda_n \psi_n(\vec{x}) = 4\pi \sum_n \psi_n^*(\vec{x}\,') \psi_n(\vec{x})$$

$$\Rightarrow \quad \lambda_n c_n(\vec{x}\,') = 4\pi \psi_n^*(\vec{x}\,')$$

Hence finally

$$\boxed{G(\vec{x},\vec{x}\,') = 4\pi \sum_n \frac{\psi_n^*(\vec{x}\,')\psi_n(\vec{x})}{\lambda_n}} \quad (6)$$

The case of a continuous eigenvalue spectrum

$\Rightarrow$ if λ_n are continuous, we must generalize the $\sum_n$ to an integral, e.g.

$$-\nabla^2 \psi_{\vec{k}}(\vec{x}) = k^2 \psi_{\vec{k}}(\vec{x})$$

If $V=$ infinite free space, the solutions are

$$\psi_{\vec{k}}(\vec{x}) = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{x}}$$

And when the eigenvalue spectrum is continuous, must generalize the normalization condition to

$$\delta(\vec{k}-\vec{k}\,') = \int_V \psi_{\vec{k}\,'}^*(\vec{x})\,\psi_{\vec{k}}(\vec{x})\,d^3x$$

and the above derivation for G generalizes to

$$G(\vec{x},\vec{x}\,') = 4\pi \int d^3k\, \frac{\psi_{\vec{k}}^*(\vec{x}\,')\psi_{\vec{k}}(\vec{x})}{k^2}$$

It will be instructive, especially later in the course, to evaluate this integral. We can use a "trick", namely to choose the z-axis in k-space to lie along the direction of $\vec{x} - \vec{x}'$. We may do this because we're integrating over all k-space.

$$G(\vec{x},\vec{x}') = \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{k^2 dk}{k^2} \int_{-1}^{1} d(\cos\theta_k) \int_0^{2\pi} d\phi_k \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

where θ_k = angle between $\vec{k}$ and $(\vec{x}-\vec{x}')$

$$= \frac{1}{2\pi^2}(2\pi) \int_0^\infty dk \int_{-1}^{1} d(\cos\theta_k) \, e^{ikR\cos\theta_k}$$

where $R = |\vec{x}-\vec{x}'|$

$$\Rightarrow G(\vec{x},\vec{x}') = \frac{1}{\pi}\int_0^\infty dk \, \frac{e^{ikR} - e^{-ikR}}{ikR}$$

$$= \frac{2}{\pi}\int_0^\infty dk \, \frac{\sin kR}{kR} = \frac{2}{\pi R}\overbrace{\int_0^\infty du \, \frac{\sin u}{u}}^{\text{“}\frac{\pi}{2}\text{”}}$$

So finally $\boxed{G(\vec{x},\vec{x}') = \dfrac{1}{|\vec{x}-\vec{x}'|}}$ = an alternative way to find this!

Method II Eigenfunction expansion in all coordinates EXCEPT one.

e.g. for a spherical problem, write an ansatz that is motivated by the spherical harmonic completeness relation, namely

$$G(\vec{x}, \vec{x}') = \sum_{l,m} g_l(r,r') \, Y^*_{lm}(\theta',\phi') \, Y_{lm}(\theta,\phi)$$

Now use ∇^2 in the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\vec{L}^2(\theta,\phi)}{r^2}, \quad \text{where}$$

$$\vec{L}^2(\theta,\phi) = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

and recall that $\vec{L}^2 Y_{lm}(\theta,\phi) = l(l+1)\,Y_{lm}(\theta,\phi)$

Thus we can find the equation $g_l(r,r')$ obeys,

$$\nabla^2 G(\vec{x},\vec{x}') = \sum_{lm} Y^*_{lm}(\hat{x}')\,Y_{lm}(\hat{x})\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] g_l(r,r')$$

$$= -4\pi \frac{\delta(r-r')}{r^2}\,\delta(\cos\theta - \cos\theta')\,\delta(\phi-\phi')$$

and this equation will be satisfied if

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] g_l(r,r') = -4\pi\frac{\delta(r-r')}{r^2}$$

$\Rightarrow$ We must solve this equation, subject to the appropriate BCs, over $a \le r \le b$, and then we will have $G(\vec{x},\vec{x}')$

Solution of the radial equation $= 2^{nd}$-order linear diff. eqn.

Think of $g_l(r,r')$ = function of r, for fixed r'.

$\Rightarrow$ At any $r \ne r'$, the most general solution in r must be a linear combination of 2 independent solutions of the HOMOGENEOUS DIFF. EQN.

i.e. $g_\ell(r,r') = \begin{cases} A r^\ell + B r^{-\ell-1}, & r<r' \\ C r^\ell + D r^{-\ell-1}, & r>r' \end{cases}$

where A, B, C, D all depend on ℓ and r' but not r.

Dirichlet case Must satisfy the following 4 equations

① $g_\ell(a,r') = 0 = A a^\ell + B a^{-\ell-1}$

② $g_\ell(b,r') = 0 = C b^\ell + D b^{-\ell-1}$

③ Continuity at $r=r'$

$\Rightarrow \quad A r'^\ell + B r'^{-\ell-1} = C r'^\ell + D r'^{-\ell-1}$

④ Derivative discontinuity at $r=r'$, caused by $\delta(r-r')$

To derive the precise form of this deriv. discontinuity integrate the diff. eqn. for $g_\ell(r,r')$ over r from $r=r'-\epsilon$ to $r=r'+\epsilon$, then take $\lim_{\epsilon\to 0^+}$

i.e.

$$\int_{r=r'-\epsilon}^{r=r'+\epsilon} r^2 dr \left\{ \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial g(r,r')}{\partial r} - \cancel{\frac{\ell(\ell+1)}{r^2} g_\ell(r,r')} \right\}$$

$$= -4\pi \int_{r'-\epsilon}^{r'+\epsilon} r^2 dr \, \frac{\delta(r-r')}{r^2}$$

0 since continuous

$$\Rightarrow r^2 \frac{\partial g_\ell(r,r')}{\partial r}\Bigg|_{r=r'-\epsilon}^{r=r'+\epsilon} = -4\pi$$

or $$\boxed{\frac{\partial g_\ell(r,r')}{\partial r}\Bigg|_{r=r'+\epsilon} - \frac{\partial g_\ell(r,r')}{\partial r}\Bigg|_{r=r'-\epsilon} = \frac{-4\pi}{r'^2}}$$

This gives the 4^{th} equation needed to find the 4 constants A, B, C, D, namely:

$$l C r'^{\,l-1} + (-l-1) D r'^{\,-l-2} - \left[l A r'^{\,l-1} - (l+1) B r'^{\,-l-2} \right] = -\frac{4\pi}{r'^2}$$

=> Solution of these 4 equations and 4 unknowns gives the solution in the form written as Jackson Eq. 3.125, namely for a Dirichlet BC problem with spherical conducting boundaries at $r=a$ and $r=b$, the GF at $a \le r \le b$ is

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x}) \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1} \right]}$$

and it is readily seen that the free space GF is recovered upon setting $a \to 0$, $b \to \infty$.

1-dimensional Green Functions – general treatment for the Sturm-Liouville operator.

(see, e.g. Arfken + Weber, Chap. 9)

This 2nd-order differential operator looks in general like

$$\hat{\mathcal{L}}_x = \frac{d}{dx}\left(p(x) \frac{d}{dx} \right) + q(x)$$

Let's find the Dirichlet GF obeying

$$\hat{\mathcal{L}}_x\, g(x, x') = -\frac{\delta(x - x')}{\beta(x')}$$

over $a \le x \le b$, subject to $g(a,x') = 0 = g(b,x')$

Again, there are two linearly-independent solutions to the homogeneous differential equation, $\boxed{\hat{\mathcal{L}}_x u(x) = 0}$ (*)

Call $u_1(x)$ = a solution of (*) obeying the LEFT boundary condition, $u_1(a) = 0$

and $u_2(x)$ = a solution obeying $u_2(b) = 0$

Then the GF must have the form

$$g(x,x') = \begin{cases} C_1 u_1(x), & a \le x \le x' \\ C_2 u_2(x), & x' \le x \le b \end{cases}$$

where C_1, C_2 depend on x'.

Next impose continuity at $x=x'$

$$\Rightarrow \boxed{C_1 u_1(x') = C_2 u_2(x')} \quad (**)$$

Next find the derivative discontinuity at $x=x'$, by integrating:

$$\lim_{\epsilon \to 0^+} \int_{x=x'-\epsilon}^{x=x'+\epsilon} \hat{\mathcal{L}}_x g(x,x')\,dx = \int_{x'-\epsilon}^{x'+\epsilon} \left(-\frac{\delta(x-x')}{\beta(x')}\right) dx$$

$$\Rightarrow \boxed{p(x') \frac{dg(x,x')}{dx}\Bigg|_{x=x'+\epsilon} - p(x')\frac{dg(x,x')}{dx}\Bigg|_{x=x'-\epsilon} = \frac{-1}{\beta(x')}} \quad (***)$$

Solve this along with (**) to get $C_1(x'), C_2(x')$

$$\Rightarrow \text{Find } \; C_1(x') = \frac{-u_2(x')}{\beta(x')p(x')[u_1(x')u_2'(x') - u_1'(x')u_2(x')]}$$

$$C_2(x') = \frac{-u_1(x')}{\beta(x')p(x')[u_1(x')u_2'(x') - u_1'(x')u_2(x')]}$$

Notational aside: an important quantity in the theory of differential equations is the Wronskian: $W(u_1, u_2) \equiv u_1(x)u_2'(x) - u_1'(x)u_2(x)$

For our Sturm–Liouville operator one can prove that, for any 2 solutions $u_1(x), u_2(x)$ of the homogeneous diff. equation, $\underline{p\,W(u_1,u_2) = \text{constant}}$

Thus in concise form, we have

$$\boxed{g(x,x') = \frac{-1}{\beta(x')\,p\,W(u_1,u_2)}\,u_1(x_<)\,u_2(x_>) \quad \text{is the GF for} \quad \frac{d}{dx}p(x)\frac{dg}{dx} + q(x)\,g = -\frac{\delta(x-x')}{\beta(x')}}$$

extremely useful!

Application to the radial GF for Laplace's equation in spherical coordinates

The diff. equation is:

$$\frac{d}{dr}r^2\frac{d}{dr}g_\ell(r,r') - \ell(\ell+1)\,g_\ell(r,r') = -4\pi\,\delta(r-r')$$

which matches our Sturm–Liouville eqn if we identify

$$p(r) = r^2$$
$$q = -\ell(\ell+1)$$
$$\beta(r) = \frac{1}{4\pi}$$

$$a \le r \le b$$

$$\Rightarrow \quad g_\ell(r,r') = \frac{-4\pi\; u_1(r_<)\,u_2(r_>)}{r'^2\,W[u(r'),u_2(r')]}$$

where $u_i(r)$ obey the homogeneous ~~diff~~ eqn,

$$\frac{d}{dr} r^2 \frac{du_i(r)}{dr} - \ell(\ell+1) u_i(r) = 0$$

and $u_1(a) = 0$, $u_2(b) = 0$

Aside: if you forget what these solutions are, you can always TRY a power law, e.g. try $u(r) = r^\gamma$ and see whether it works, e.g. here

$$\frac{d}{dr} r^2 (\gamma r^{\gamma-1}) - \ell(\ell+1) r^\gamma = 0 = [\gamma(\gamma+1) - \ell(\ell+1)] r^\gamma = 0$$

$\Rightarrow$ this solution works provided
$\gamma(\gamma+1) - \ell(\ell+1) = 0$, i.e. $\gamma = \ell$ or $-\ell-1$

$\Rightarrow$ the general solution is $u(r) = Ar^\ell + Br^{-\ell-1}$

So the linear combinations that obey homogeneous Dirichlet BCs are

$$u_1(r) = \left(\frac{r}{a}\right)^\ell - \left(\frac{a}{r}\right)^{\ell+1}, \quad u_2(r) = \left(\frac{r}{b}\right)^\ell - \left(\frac{b}{r}\right)^{\ell+1}$$

whose p×Wronskian works out to be

$$r'^2 W(u_1, u_2) = (2\ell+1)\left(\frac{b^{\ell+1}}{a^\ell} - \frac{a^{\ell+1}}{b^\ell}\right)$$

giving the radial GF,

$$g_\ell(r, r') = \frac{-4\pi}{(2\ell+1)\left(\frac{b^{\ell+1}}{a^\ell} - \frac{a^{\ell+1}}{b^\ell}\right)} \left[\left(\frac{r_<}{a}\right)^\ell - \left(\frac{a}{r_<}\right)^{\ell+1}\right]\left[\left(\frac{r_>}{b}\right)^\ell - \left(\frac{b}{r_>}\right)^{\ell+1}\right]$$

and $G(\vec{x}, \vec{x}') = \sum_{\ell m} g_\ell(r, r') Y^*_{\ell m}(\hat{x}') Y_{\ell m}(\hat{x})$

Application Consider a uniform thin ring of total charge Q, of radius R, inside a grounded, conducting sphere of radius b. Find $\Phi(\vec{r})$ inside.

We can write this in terms of the Dirichlet GF that was just found, as

$$\Phi(\vec{x}) = \int_V \frac{G(\vec{x},\vec{x}')\rho(\vec{x}')}{4\pi\epsilon_0} d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} \cancel{\Phi(\vec{x}')}\, da'$$

$\Phi(\vec{x}') \to 0$ since grounded

Observe that we could use $G(\vec{x},\vec{x}')$ from our preceding derivation, and set $a \to 0$, or else we could use G from our method of images derivation in Chapter 2. Their equality implies an identity,

$$G(\vec{x},\vec{x}') = \sum_{l,m} \left(\frac{-4\pi}{2l+1}\right) \frac{r_<^l}{b^{l+1}} \left[\left(\frac{r_>}{b}\right)^l - \left(\frac{b}{r_>}\right)^{l+1}\right] Y_{lm}^*(\hat{x}') Y_{lm}(\hat{x})$$

$$= \frac{1}{|\vec{x}-\vec{x}'|} - \frac{b}{r'\left|\vec{x} - \frac{b^2}{r'^2}\vec{x}'\right|}$$

The charged ring has charge density

$$\rho(\vec{x}') = \frac{Q}{2\pi R^2}\delta(r'-R)\,\delta(\cos\theta)$$

For the solution, let's write separate formulas for $r<R$ and for $r>R$

e.g., for $r<R$, $\Rightarrow r_< = r$, $r_> = R = r'$ (owing to $\delta(r'-R)$)

$$\Phi(r,\theta,\phi) = \sum_{lm} \frac{Q}{2\pi R^2}\frac{1}{4\pi\epsilon_0}\left(\frac{4\pi}{2l+1}\right)\frac{r^l}{b^{l+1}} Y_{lm}(\theta,\phi)$$

$$\times \int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta')\, Y^*_{lm}(\theta',\phi')\,\delta(\cos\theta')$$

$$\times \int_0^b r'^2 dr'\, \delta(r'-R)\left[\left(\frac{r'}{b}\right)^l - \left(\frac{b}{r'}\right)^{l+1}\right]$$

$$= 2\pi\,\delta_{m,0}\, Y_{lm}\left(\frac{\pi}{2},0\right) = \delta_{m,0}\left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(0)$$

$$R^2\left[\left(\frac{R}{b}\right)^l - \left(\frac{b}{R}\right)^{l+1}\right]$$

And the final answer, for $r<R$, is:

$$\Phi(r,\theta,\phi) = \frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty} r^l\left(\frac{1}{R^{l+1}} - \frac{R^l}{b^{2l+1}}\right) P_l(0)\, P_l(\cos\theta)$$

and one could simplify this further if it is desired, using, e.g.

$$P_{2n+1}(0) = 0, \quad P_{2n}(0) = \frac{(-1)^n}{2^n n!}(2n-1)!!$$

Laplace's equation in cylindrical coordinates

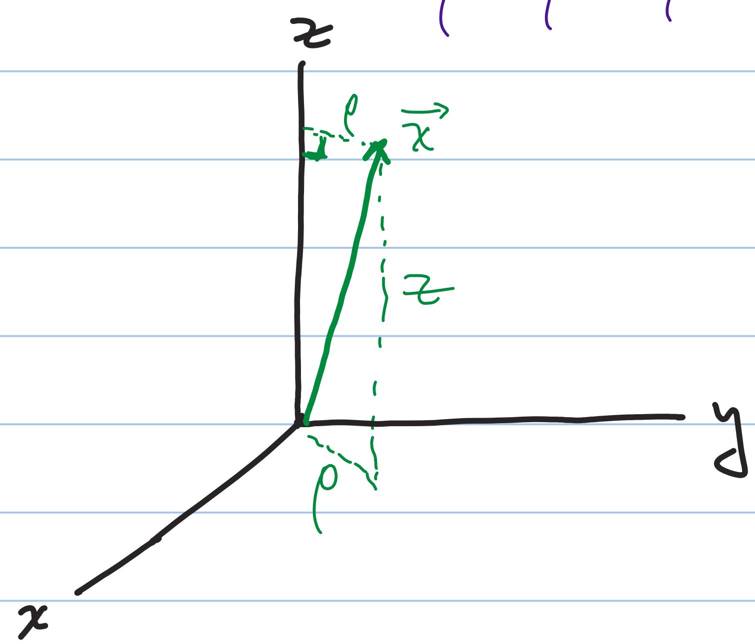
Now $(\rho, \phi, z) \equiv (q_1, q_2, q_3)$

and $h_\rho = 1, \; h_\phi = \rho, \; h_z = 1$

So recalling that

$$\nabla^2 \Phi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\right) + \text{cyclic permutations} \ldots$$

$$\Rightarrow \nabla^2 \Phi(\rho, \phi, z) = 0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$



Again solve this PDE by separation of variables. That is, we initially look for separable solutions, of the form

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z)$$

which gives the separated equations,

$$\frac{d^2 Q(\phi)}{d\phi^2} + \nu^2 Q(\phi) = 0 \Rightarrow e^{\pm i\nu\phi} \text{ are the 2 solns when } \nu \neq 0$$

and
$$e^{i\nu\phi}\Big|_{\nu\to 0} \longrightarrow A + B\phi$$

Next,
$$\frac{d^2 Z(z)}{dz^2} - k^2 Z(z) = 0$$
$$\Rightarrow Z(z) = e^{\pm kz}$$
or $\sinh kz$, $\cosh kz$...

and finally in ρ,
$$\Rightarrow \frac{d^2 R(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R(\rho) = 0$$
or dividing this last equation by k^2
and calling $x = k\rho$, gives
$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

This can be solved by power series (Fröbenius method), be sure you know how to do this! The solution is
$$J_\nu(x) = \left(\frac{x}{2}\right)^{\nu}\sum_{j=0}^{\infty}\frac{(-1)^j\,(x/2)^{2j}}{j!\,\Gamma(\nu+j+1)}$$

Observation, since the differential equation depends only on ν^2, we know immediately that
$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,\Gamma(-\nu+j+1)}\left(\frac{x}{2}\right)^{2j}$$
is ALSO a solution to the diff. equation!

Theory of the Neumann Bessel Function.

These two functions $J_\nu(x)$, $J_{-\nu}(x)$ could serve as our two linearly-independent solutions, provided they are in fact independent.

We can assess their independence by looking at their asymptotic forms (see Arfken + Weber, or Morse + Feshbach):

$$J_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

and

$$J_{-\nu}(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

This is proportional to $J_\nu(x)$ when ν=integer

i.e. this shows that $J_{-m}(x) = (-1)^m J_m(x)$ when m=integer.

However, if we could define a solution $N_\nu(x)$ that behaves asymptotically like

$$N_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$

then it would be linearly-independent of $J_\nu(x)$ at all values of ν, x! Actually, we can find such a solution by starting from the identity,

$$\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\cos\pi\nu - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$
$$= \cos\left(x + \frac{\nu\pi}{2} - \pi/4\right)$$

i.e. $J_\nu(x)\cos\pi\nu - N_\nu(x)\sin\pi\nu = J_{-\nu}(x)$

and this in turn suggests that we should define

$$\boxed{N_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}}$$

which is called the "Neumann function", or Bessel function of the 2^{nd} kind, sometimes written instead as $Y_\nu(x)$

Through this construction $J_\nu(x)$ and $N_\nu(x)$ are now linearly-independent for ALL values of ν, including $\nu = m = $ integer!

For electrostatics problems, these Bessel functions are most useful when $\Phi \to 0$ boundary conditions are imposed on a surface in ρ, because they are oscillatory in ρ for k=real, i.e. when $k^2 > 0$. It is also crucial to recognize that for small distances,

$$J_\nu(k\rho) \xrightarrow[k\rho\to 0]{} \text{regular for } \nu \geq 0$$

$$N_\nu(k\rho) \xrightarrow[k\rho\to 0]{} \text{IRREGULAR at all } \nu \geq 0$$

See, e.g., Jackson Eqs. 3.89, 3.90

Also, in some problems we will the values of k at which $J_\nu(k\rho_0)$ vanishes, i.e. the roots of

$$J_\nu(x_{\nu n}) = 0$$

with $x_{\nu n} = n^{th}$ zero of $J_\nu(x)$

Tables of these $x_{\nu n}$ can be found, e.g., in the NIST Tables, the old Abramowitz + Stegun, but also Mathematica can provide them with the statement BesselJZeros[ν, n]

first n zeroes of $J_\nu(x_{\nu n}) = 0$.

Example application Find the potential Φ inside a right-circular cylinder of radius a, height L, all of whose walls are held at $\Phi = 0$ (think of a grounded conductor), Except for the top cap at $z = L$, which is instead held at $V(\rho, \phi) = V_0 \frac{\rho}{a} w(\phi)$ and $w(\phi)$ is some known function of ϕ.

Solution Start from a sum over the separable Bessel solutions, regular at $\rho = 0$, which vanish at $\rho = a$:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(\frac{A_{m,n} e^{im\phi} + A_{-m,n} e^{-im\phi}}{1 + \delta_{m0}} \right) \times J_m\left(x_{mn} \frac{\rho}{a}\right) \sinh\left(x_{mn} \frac{z}{a}\right)$$

i.e. write $k = \frac{x_{mn}}{a}$ in our earlier separable solution.

This has been designed to satisfy the PDE

$$\nabla^2 \Phi = 0$$

and all BCs in the problem except one, the one at $z = L$. To find the A_{mn} such that this last BC is obeyed, use the orthogonality relation obeyed by

$$e^{im\phi} J_m\left(x_{mn} \frac{\rho}{a}\right)$$

For a derivation, see Jackson pp. 114-115, namely,

$$\int_0^a J_m\left(x_{mn'} \frac{\rho}{a}\right) J_m\left(x_{mn} \frac{\rho}{a}\right) \rho \, d\rho = \delta_{nn'} \frac{a^2}{2} \left[J_{m+1}(x_{mn})\right]^2$$

Another useful aside: from the last result, notice that the functions

$$u_n^{(m)}(\rho) = \frac{\sqrt{2}}{a J_{m+1}(x_{mn})} J_m\left(x_{mn} \frac{\rho}{a}\right)$$

form a complete, orthonormal set, with weight function ρ, over $0 \le \rho \le a$, for each $m = 0, 1, 2, \ldots$

i.e. $\int_0^a u_n^{(m)}(\rho)\, u_{n'}^{(m)}(\rho)\, \rho \, d\rho = \delta_{nn'}$

and $\sum_{n=1}^{\infty} u_n^{(m)}(\rho)\, u_n^{(m)}(\rho') = \frac{\delta(\rho - \rho')}{\rho}$

for $0 \le \rho \le a$ and $0 \le \rho' \le a$

To complete this solution, use "Fourier's Trick" again, using orthogonality in ρ, ϕ, i.e.

$$A_{\pm m,n} = \frac{1}{2\pi} \frac{1}{\sinh\left(\frac{x_{mn} L}{a}\right)} \frac{2}{a^2 \left[J_{m+1}(x_{mn})\right]^2}$$

$$\times \int_0^{2\pi} e^{\mp i m \phi} d\phi \int_0^a \rho \, d\rho \, J_m\left(x_{mn} \frac{\rho}{a}\right) V(\rho, \phi)$$

$$= \frac{V_0}{\pi a^2 \left(J_{m+1}(x_{mn})\right)^2} \frac{\int_0^{2\pi} e^{\mp i m \phi} w(\phi)\, d\phi}{\sinh\left(x_{mn} \frac{L}{a}\right)}$$

$$\times \left(\int_0^a \frac{\rho^2}{a} J_m\left(x_{mn} \frac{\rho}{a}\right) d\rho\right)$$

An aside on orthogonality + completeness

$\Rightarrow$ **Read this on your own!**

We have already commented that eigenfunctions $u_n(x)$ of the Sturm–Liouville operator are orthogonal in general, if the $u_n(x)$ all obey the same BCs on $\frac{u'}{u}$ at the boundaries.

Careful statement: Let $\{u_n(x)\}$ be the set of all eigenfunctions of

can prove that the λ_n are real

$$\hat{\mathcal{L}} u_n(x) + \lambda_n w(x) u_n(x) = 0,$$

over $a \le x \le b$, where $w(x)$ is a nonnegative weight function, and where each $u_n(x)$ obeys the BCs $\frac{u_n'(a)}{u_n(a)} = \alpha$, $\frac{u_n'(b)}{u_n(b)} = \beta$

Then one can show (see, e.g. Arfken + Weber, Chap. 9):

ORTHOGONALITY $\int_a^b u_n^*(x)\, u_m(x)\, w(x)\, dx = 0$

if $\lambda_n \neq \lambda_m$. Moreover, if the eigenvalue spectrum is discrete (countable), then it is possible to normalize the eigenfunctions to unity, as

$$\int_a^b |u_n(x)|^2 w(x)\, dx = 1$$

and we assume hereafter that this has been done $\Rightarrow \int_a^b u_n^*(x)\, u_m(x)\, w(x)\, dx = \delta_{nm}$

Completeness One can also show that the $\{u_n(x)\}_{n=1,2,\ldots\infty}$ are complete. To derive the actual completeness relation, start from the following expansion of an arbitrary function $F(x)$, over $a \le x \le b$: $F(x) = \sum_n c_n u_n(x)$ (I)

The c_n obtained from "Fourier's Trick" are found the usual way (multiply by $u_m^*(x)\, w(x)$ and integrate)

$$\Rightarrow c_m = \int_a^b u_m^*(x')\, F(x')\, w(x')\, dx'$$

use x' as dummy integration variable.

Now plug this back into (I)

$$\Rightarrow F(x) = \int_a^b \left(\sum_n u_n^*(x')\, u_n(x)\right) F(x')\, w(x')\, dx'$$

$\Rightarrow$ For this to hold at all x, we MUST have:

$$\sum_n u_n^*(x')\, u_n(x) = \delta(x-x')/w(x')$$

Back to cylindrical electrostatics problems

For some problems we will need the <u>other</u> family of Laplace equation solutions, which are oscillatory in z rather than ρ.

i.e. where we chose $\frac{Z''(z)}{Z(z)} = k^2$, we <u>could</u> have chosen $-\kappa^2$

$\Rightarrow$ This new family of solutions can be obtained by analytical continuation, which means setting $k \to i\kappa$ with $\kappa =$ real, ≥ 0 now.

$\Rightarrow$ solutions for $Z(z)$ become $\sinh(i\kappa z) = i \sin \kappa z$

$\cosh(i\kappa z) = \cos \kappa z$

whereas the Bessel solutions become

$$J_\nu(i\kappa\rho),\ N_\nu(i\kappa\rho),$$

and both of these <u>diverge</u> <u>exponentially</u> at $\kappa\rho \to \infty$ analogous to $\sinh \kappa\rho$, $\cosh \kappa\rho$.

Because of this divergence, it is often convenient to work with a different 2^{nd} solution called $K_\nu(\kappa\rho)$ instead of with $N_\nu(i\kappa\rho)$, where $K_\nu(\kappa\rho)$ is defined to be regular at $\kappa\rho \to \infty$. (<u>Analogy</u>: this is similar to the fact that <u>one</u> linear combination of $\sinh \kappa z$, $\cosh \kappa z$, namely $e^{-\kappa z}$, is regular at ∞.)

Similarly, it is conventional to introduce a new solution $I_\nu(k\rho)$ proportional to $J_\nu(ik\rho)$ such that $I_\nu(k\rho)$ is real for real $k\rho > 0$, and still regular at $k\rho \to 0$

Definitions of non-oscillatory Bessel functions:

$$I_\nu(x) \equiv i^{-\nu} J_\nu(ix) = \text{real for real } x$$

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + i N_\nu(ix) \right]$$

$$= \frac{\pi}{2} i^{\nu+1} \underbrace{H_\nu^{(1)}(ix)}$$

← Hankel function of the 1st kind.

Terminology and limiting forms

$I_\nu(x)$ and $K_\nu(x)$ are often called

MODIFIED BESSEL FUNCTIONS

Small x $\quad I_\nu(x) \xrightarrow[|x| \to 0]{} \frac{1}{\nu!} \left(\frac{x}{2}\right)^\nu$ = regular at 0 if $\nu \geq 0$

$$K_\nu(x) \xrightarrow[|x| \to 0]{} \begin{cases} -\ln x - \gamma + \ln 2 + \cdots, & \nu = 0 \\ 2^{-\nu-1} (\nu - 1)!\, x^{-\nu}, & \nu > 0 \end{cases}$$

= IRREGULAR at 0

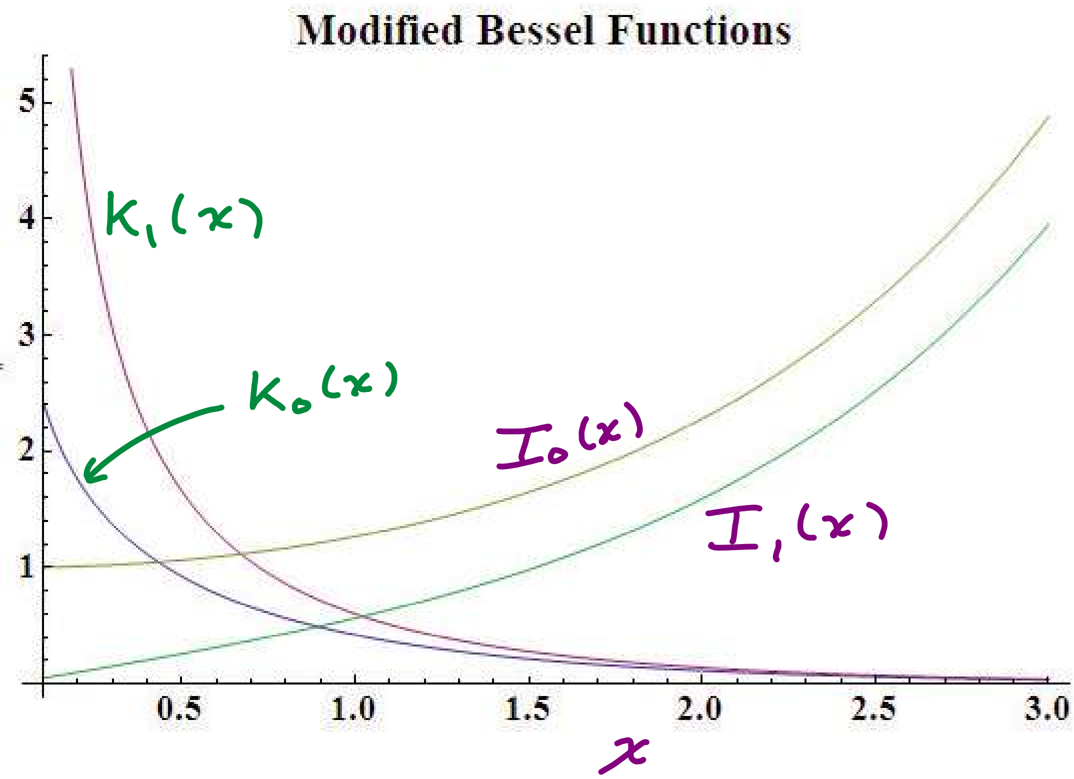
Large $|x|$ $\quad I_\nu(x) \xrightarrow[|x| \to \infty]{} \text{const} \times \frac{e^x}{\sqrt{x}}$, ⇒ IRREGULAR at ∞

$$K_\nu(x) \xrightarrow[|x| \to \infty]{} \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \Rightarrow \text{REGULAR at } \infty$$

```
In[3]:= $DefaultFont = {"HelveticaBold", 16};

In[6]:= Plot[{BesselK[0, x], BesselK[1, x], BesselI[0, x], BesselI[1, x]}, {x, 0.1, 3},
          BaseStyle → {FontWeight → "Bold", FontSize → 18},
          PlotLabel → "Modified Bessel Functions"]
```

Out[6]=



Example This class of solutions is useful for problems, especially if the top ($z=L$) and bottom ($z=0$) caps of a cylinder are held at $\Phi = 0$. Then we can utilize a complete, orthonormal set of solutions in z that vanish at $z=0, L$, with wavenumbers $k = \frac{n\pi}{L}$: $\left(\frac{2}{L}\right)^{1/2} \sin \frac{n\pi z}{L}$,

$n = 1, 2, 3, \ldots$

The solutions inside this cylinder are then

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{im\phi} \left(\frac{2}{L}\right)^{1/2} \sin \frac{n\pi z}{L} \, I_m\!\left(\frac{n\pi\rho}{L}\right)$$

- K_m cannot be present if $\rho = 0$ is within the volume V since K_m = irregular at $\rho = 0$
- Note that $I_{-m}(x) = I_m(x)$ for integer m.

Green's Function in Cylindrical Coordinates

<u>Problem</u> Find $G(\vec{x}, \vec{x}')$ in free space, using oscillatory solutions in ϕ and z, i.e. e^{ikz}

In this infinite space volume V, we want to solve

$$\nabla_x^2 G(\vec{x}, \vec{x}') = -4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi') \delta(z-z')$$

<u>IDEA</u> Start from the completeness relations in ϕ and z, namely

$$\text{(i)} \quad \delta(z-z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z-z')} dk$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \cos k(z-z') \, dk$$

$$\text{(ii)} \quad \delta(\phi-\phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

Then building the Green's function out of these forms, with a <u>still-unknown</u> ρ, ρ'-dependence,

$$\Rightarrow G(\vec{x}, \vec{x}') = \frac{1}{2\pi^2} \sum_m \int_0^{\infty} dk \cos k(z-z') \, e^{im(\phi-\phi')} g_m(\rho, \rho'; k)$$

Plug this into the PDE to find the equation obeyed by $g_m(\rho, \rho'; k)$:

i.e.

$$\nabla^2 G(\vec{x},\vec{x}\,') = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) G(\vec{x},\vec{x}\,')$$

$$= \frac{1}{2\pi^2}\sum_m \int_0^\infty dk\, \cos k(z-z')\, e^{im(\phi-\phi')} \left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - k^2 - \frac{m^2}{\rho^2}\right\} g_m(\rho,\rho';k)$$

$$= -4\pi\frac{\delta(\rho-\rho')}{\rho}\left\{\frac{1}{2\pi^2}\sum_m e^{im(\phi-\phi')}\int_0^\infty \cos k(z-z')\,dk\right\}$$

Clearly, this last equation will be satisfied iff g_m obeys:

$$\cancel{\frac{1}{\rho}}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} g_m(\rho,\rho';k) - \left(k^2+\frac{m^2}{\rho^2}\right) g_m(\rho,\rho';k) = -4\pi\frac{\delta(\rho-\rho')}{\cancel{\rho}}$$

and observe that this fits the Sturm-Liouville equation discussed previously, namely

$$\frac{d}{dx}p(x)\frac{d}{dx}g(x,x') + q(x)\,g(x,x') = -\frac{\delta(x-x')}{\beta(x')}$$

where we derived $g(x,x') = \dfrac{-u_1(x_<)u_2(x_>)}{\beta(x')p(x')\,W[u_1,u_2]\big|_{x'}}$

So to apply this here, we must let

$x \equiv \rho \qquad p \longrightarrow \rho\,, \quad \beta \rightarrow \frac{1}{4\pi}$

And the appropriate solutions of the homogeneous differential equation are:

$u_1(\rho) = I_m(k\rho)$, obeys the B.C. at $\rho\to 0$

and $u_2(\rho) = K_m(k\rho)$, obeys the B.C. at $\rho\to\infty$

-and $\rho'\,W(I_m,K_m)\big|_{\rho'}$ can be evaluated at ANY ρ' since it is constant.

I will evaluate it at $\rho \to 0$ where

$$I_m(k\rho) \xrightarrow[\rho \to 0]{} \frac{1}{|m|!}\left(\frac{k\rho}{2}\right)^{|m|}$$

$$K_m(k\rho) \xrightarrow[\rho \to 0]{} 2^{|m|-1}(|m|-1)!\,(k\rho)^{-|m|}, \quad \text{for } m \neq 0$$

and thus $\rho' W(I_m, K_m)_{\rho'} = \lim\limits_{\rho' \to 0} \frac{\rho'}{2|m|}\left(\frac{-|m|}{\rho'} - \frac{|m|}{\rho'}\right) = -1$

and this result TURNS OUT to hold <u>also</u> for $m=0$. (Prove this on your own!)

Therefore, finally, $g_m(\rho,\rho';k) = 4\pi I_m(k\rho_<)K_m(k\rho_>)$ and another expression for the Green's function with NO BOUNDARIES PRESENT is:

$$G(\vec{x},\vec{x}') = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk \cos k(z-z')\, e^{im(\phi-\phi')} I_m(k\rho_<)K_m(k\rho_>)$$

$$= \frac{1}{|\vec{x}-\vec{x}'|}$$

Example problem 3.16

(a) δ-function orthogonality proof

Consider the 2-dim^l Fourier transform of a function $f(x,y)$:

$$F(s,\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, f(x,y)\, e^{i(sx+\sigma y)}$$

and now specialize to a particular $f(x,y)$ written in polar coordinates,

$f(x,y) \to f(r)\, e^{im\theta}$, where $x = r\cos\theta$, $y = r\sin\theta$

and call $s = \rho\cos\alpha$, $\sigma = \rho\sin\alpha$ (*)

$$\Rightarrow F(\rho,\alpha) = \frac{1}{2\pi}\int_0^\infty r\,dr\int_0^{2\pi} d\theta\; f(r)\exp[ir\rho(\underbrace{\cos\theta\cos\alpha + \sin\theta\sin\alpha}_{=\cos(\theta-\alpha)}) + im\theta]$$

and the inverse of this Fourier transform is

$$f(r)\, e^{im\theta} = \frac{1}{2\pi}\int_0^\infty \rho\, d\rho\int_0^{2\pi} d\alpha\; F(\rho,\alpha)\, e^{-ir\rho\cos(\theta-\alpha)} \qquad (**)$$

Refer now to Gradsteyn & Ryzhik 8.411.1,

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta + iz\sin\theta}\, d\theta$$

and set $\beta \equiv \theta - \frac{\pi}{2}$ here

$$\Rightarrow J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos\beta - in\beta - in\frac{\pi}{2}}\, d\beta$$

comparing this with our original transform (*) gives

$$F(\rho,\alpha) = e^{im(\alpha+\frac{\pi}{2})}\int_0^\infty r f(r)\, J_m(\rho r)\, dr$$

and similarly (**) gives

$$f(r)\, e^{im\theta} = \frac{1}{2\pi}\int_0^\infty \rho\, d\rho\int_0^{2\pi} d\alpha\left\{\frac{1}{2\pi}\int_0^\infty r'dr'\int_0^{2\pi} d\theta'\right.$$

$$\left.\left[f(r')\, e^{i(r'\rho\cos(\theta'-\alpha) + m\theta')}\, e^{-ir\rho\cos(\theta-\alpha)}\right]\right.$$

$$= \frac{1}{2\pi} \int_0^\infty \rho\, d\rho \int_0^\infty r'\, dr' \int_0^{2\pi} d\alpha\; e^{-ir\rho\cos(\theta-\alpha)}$$

$$\times \left\{ \frac{1}{2\pi} \int_0^{2\pi} d\theta'\; e^{i(r'\rho\cos(\theta'-\alpha)+m\theta')} \right\}$$

and $\hookrightarrow \{ \} = J_m(r'\rho)\, e^{im\alpha}$

and $\frac{1}{2\pi}\int_0^{2\pi} d\alpha\; e^{-ir\rho\cos(\theta-\alpha)+im\alpha}$

$= J_m(-r\rho)\, e^{im\theta}$ by this logic

So putting it all together we see that

$$f(r)e^{im\theta} = e^{im\theta}\int_0^\infty r'\,dr' \Big(\underbrace{\int_0^\infty \rho\, d\rho\, J_m(\rho r') J_m(\rho r)}_{\text{"}\frac{\delta(r-r')}{r'}\text{"}} \Big)$$

So we're done! $\Longrightarrow$

3.16 (b)

The Green's function with no conducting surfaces obeys

$$\nabla_x^2 G(\vec{x}, \vec{x}\,') = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')\,\delta(\phi-\phi')\,\delta(z-z')$$

To satisfy the δ-functions in ρ, ϕ, take G in the form (an ansatz):

$$G(\vec{x}, \vec{x}\,') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^\infty k\,dk\; J_m(k\rho)\, J_m(k\rho')\, Z_{km}(z,z')$$

Plugging this in gives an equation in z:

$$-k^2 Z(z,z') + Z''(z,z') = -4\pi\,\delta(z-z')$$

and the appropriate homogeneous solutions are

$u_1(z) = e^{kz}$ = regular @ $z\to-\infty$, if $k \geq 0$

and $u_2(z) = e^{-kz}$ = regular at $z \to +\infty$ if $k \geq 0$

so $Z(z,z') = -4\pi \dfrac{u_1(z_<)\, u_2(z_>)}{W(u_1,u_2)}$

$$\Rightarrow \quad Z(z,z') = \frac{4\pi}{2k} e^{k(z_< - z_>)} = \frac{2\pi}{k} e^{-|z-z'|}$$

and plugging this back in to our ansatz gives another identity:

$$G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k|z-z'|}$$

$$= \frac{1}{|\vec{x}-\vec{x}'|}$$

3.16(c) As a special case of this formula, set $\rho'=0 \Rightarrow$ all $J_m(k\rho') \longrightarrow 0$ unless $m=0$ and also set $z' \to 0$, which gives

$$\boxed{\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty e^{-k|z|} J_0(k\rho)\, dk} \qquad (1)$$

Next let $z' \to 0$ and $\phi' \to 0$ to see another special case:

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = (\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi)^{-1/2}$$
$$= \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im\phi} J_m(k\rho) J_m(k\rho') e^{-k|z|}$$

and one more identity is found by replacing ρ in Eq.(1) above by $(\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{1/2}$, namely

$$\int_0^\infty e^{-k|z|} J_0\left[k(\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{1/2}\right] dk$$

$$= \int_0^\infty dk\, e^{-k|z|} \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho')$$

and since this holds for ALL ρ and z, the integrands must be equal

$$\Rightarrow J_0\left[k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}\,\right] = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho')$$

Next, prove that $e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)$

Here I will use the generating function for Bessel functions:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} t^m J_m(x)$$

In this identity let $t \rightarrow e^{i(\phi+\frac{\pi}{2})}$

$$\Rightarrow e^{\frac{x}{2}\left[2i\sin(\phi+\frac{\pi}{2})\right]} = e^{ix\cos\phi} = \sum_{m=-\infty}^{\infty} e^{im\phi} \underbrace{e^{\frac{im\pi}{2}}}_{i^m} J_m(x)$$

or setting $x = k\rho \Rightarrow$ $\boxed{e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)}$

Alternative method to show orthonormality of continuous eigenfunctions more generally

[i.e. alternative solution of Jackson 3.16(a)]
consider p. 99 – 100 as an "appendix", not covered in class

Start from the Sturm – Liouville operator

$$\boxed{\hat{\mathcal{L}} \equiv \frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}}$$

whose eigenfunctions are Bessel functions:

$$\hat{\mathcal{L}}\, J_\nu(k\rho) = -k^2 J_\nu(k\rho)$$

and $\hat{\mathcal{L}}\, J_\nu(k'\rho) = -k'^2 J_\nu(k'\rho)$

Then consider $(k^2 - k'^2)\int_0^\infty \rho\, d\rho\, J_\nu(k\rho)\, J_\nu(k'\rho) \equiv I$

so

$$I = \int_0^\infty \rho\, d\rho \left\{ [\hat{\mathcal{L}} J_\nu(k'\rho)] J_\nu(k\rho) - J_\nu(k'\rho)[\hat{\mathcal{L}} J_\nu(k\rho)] \right\}$$

$$= \int_0^\infty d\rho \left\{ \left[\frac{d}{d\rho}\left(\rho \frac{dJ_\nu(k'\rho)}{d\rho}\right)\right] J_\nu(k\rho) - J_\nu(k'\rho)\left[\frac{d}{d\rho}\left(\rho\frac{dJ_\nu(k\rho)}{d\rho}\right)\right] \right.$$

$$\left. + \cancel{\rho\, J_\nu(k'\rho) J_\nu(k\rho)\left(\frac{\nu^2 - \nu^2}{\rho^2}\right)} \to 0 \right\}$$

but observe that we can rewrite the nonvanishing part of the integrand as:

$$I = \int_0^\infty d\rho \frac{d}{d\rho}\left\{ \rho \frac{dJ_\nu(k'\rho)}{d\rho} J_\nu(k\rho) - \rho\, J_\nu(k'\rho)\frac{dJ_\nu(k\rho)}{d\rho} \right\}$$

or $I = \lim\limits_{R\to\infty} \left\{ \rho\, W[J_\nu(k\rho), J_\nu(k'\rho)] \right\}\Big|_0^R$

Now, this vanishes at the lower limit, while the upper limit can be evaluated using the asymptotic expansion, namely

$$J_\nu(k\rho) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi k\rho}\right)^{1/2} \cos\Big(k\rho \underbrace{-\frac{\nu\pi}{2} - \frac{\pi}{4}}_{\equiv \beta}\Big)$$

So $W[J_\nu(kR), J_\nu(k'R)] \xrightarrow[R\to\infty]{} \left(\frac{2}{\pi R}\right)(kk')^{-1/2}$

$$\times \left\{ \cos(kR+\beta)(-k')\sin(k'R+\beta) + k\sin(kR+\beta)\cos(k'R+\beta) \right\}$$

now set $k'-k \equiv \Delta$, i.e. replace $k' \to k+\Delta$ everywhere expand in powers of small Δ:

$$\Rightarrow \lim_{R\to\infty} W = \frac{2}{\pi R}(-1)\underbrace{[k(k+\Delta)]^{-1/2}}_{\simeq \frac{1}{k}} \{ k\sin(\Delta R) - \Delta\cos(kR+\beta)\sin((k+\Delta)R+\beta) \}$$

only term relevant to the δ-function!

oscillates infinitely fast and vanishes at $R\to\infty$

$$\Rightarrow \int_0^\infty \rho J_\nu(k\rho) J_\nu(k'\rho)\, d\rho = \lim_{R\to\infty} \frac{2}{\pi} \frac{\sin R\Delta}{k^2-k'^2}, \text{ and } k+k' \simeq 2k$$

$$= \frac{1}{2k} \lim_{R\to\infty} \frac{2}{\pi} \frac{\sin R\Delta}{\Delta} = \frac{\delta(k-k')}{k} \checkmark$$

and we're done!

Chapter 4 – Multipole expansions + dielectrics

Next we consider:

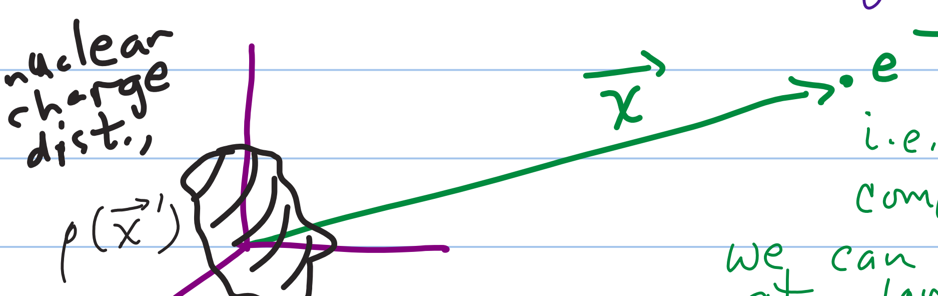
(i) Multipole expansion of $\Phi(\vec{x})$ produced by a localized $\rho(\vec{x})$

(ii) Electric fields and energy in macroscopic media, including charge rearrangement in response to an external field.

i.e. induced dipoles modify the electric field, as in $\vec{E} = \vec{E}_{ext} + \vec{E}_{induced}$

<u>Multipole expansion</u> First, our goal is to find properties like $\Phi(\vec{x})$, $\vec{E}(\vec{x})$, etc., produced by a localized $\rho(\vec{x}\,')$, e.g. when a nucleus produces a potential and then an e^- moves through it.

<u>Concept</u> Let's try to characterize effects of the charge distribution through a few parameters, = charge, electric dipole moment, electric quadrupole moment, etc.



i.e. the full $\rho(\vec{x}\,')$ could be complicated and detailed, but we can summarize its effects at large $|\vec{x}|$ more simply.

For now, develop the theory for only localized charges in free space, i.e. with no boundaries:

$$\Rightarrow \quad \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

or if we plug in our spherical harmonic expansion, for $r > r'$, this gives

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{1}{4\pi\epsilon_0}\left[\int d^3x' \, r'^l Y^*_{lm}(\hat{x}')\rho(\vec{x}')\right] \times \frac{1}{r^{l+1}} Y_{lm}(\hat{x})$$

or $$\Phi(\vec{x}) = \frac{1}{\epsilon_0}\sum_{lm} \frac{q_{lm}}{2l+1}\frac{Y_{lm}(\hat{x})}{r^{l+1}}$$

where the $$q_{lm} \equiv \int d^3x' \, r'^l Y^*_{lm}(\hat{x}')\rho(\vec{x}')$$ are called the multipole moments of $\rho(\vec{x}')$

Some basic properties

(1) Note that $q_{l,-m} = (-1)^m q^*_{lm}$, simply from the Y_{lm} properties

(2) $$q_{0,0} = \frac{1}{\sqrt{4\pi}}\int \rho(\vec{x}')\, d^3x' = \frac{q}{\sqrt{4\pi}}$$ ← electric monopole moment

where q = total charge

(3) ELECTRIC DIPOLE, $l=1$

$$q_{11} = -\int d^3x' \, r' \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta' e^{-i\phi'} \rho(\vec{x}\,')$$

It is instructive to rewrite this in terms of the "SPHERICAL UNIT VECTORS", defined as $\hat{\epsilon}_{\pm 1} = \dfrac{\mp \hat{x} - i\hat{y}}{\sqrt{2}}$, $\hat{\epsilon}_0 = \hat{z}$

which shows that the $l=1$ spherical harmonics are in fact just:

$$Y_{1,\mu}(\theta',\phi') = \left(\frac{3}{4\pi}\right)^{1/2} \frac{\vec{x}\,'}{r} \cdot \hat{\epsilon}_\mu$$

<u>Quick check</u>

does $Y_{1,1}(\theta',\phi') \stackrel{?}{=} \left(\frac{3}{4\pi}\right)^{1/2} \left(\frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}\right) \cdot \left(\frac{-\hat{x} - i\hat{y}}{\sqrt{2}}\right)$

$$= -\left(\frac{3}{8\pi}\right)^{1/2} (\sin\theta\cos\phi + i\sin\theta\sin\phi)$$

$$= -\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\phi}$$ ✓ it checks!

Note also that the spherical unit vectors obey:

$$\hat{\epsilon}_\mu^* \cdot \hat{\epsilon}_\nu = \delta_{\mu\nu}$$

<u>Addendum</u> ANY vector in 3D can be written in terms of its spherical (tensor) components as follows:

$$\vec{P} = \sum_{\mu=-1}^{1} \hat{\epsilon}_\mu \left(\hat{\epsilon}_\mu^* \cdot \vec{P}\right)$$

which is formally analogous to the Hilbert Space expansion in quantum mechanics,

$$|\psi\rangle = \sum_\mu |\mu\rangle \left(\langle\mu|\psi\rangle\right)$$

Using this notation, we have

$$q_{11} = \left(\frac{3}{4\pi}\right)^{1/2} \hat{\epsilon}^{\,*}_{+1} \cdot \int d^3x' \, \vec{x}' \rho(\vec{x}')$$

$$= \sqrt{\frac{3}{4\pi}} \, \hat{\epsilon}^{\,*}_{+1} \cdot \vec{p}$$

where $\vec{p} \equiv \int d^3x' \, \vec{x}' \rho(\vec{x}')$ is the usual ELECTRIC DIPOLE MOMENT of ρ

e.g. $q_{11} = -\frac{1}{\sqrt{2}}(p_x - ip_y)\left(\frac{3}{4\pi}\right)^{1/2}$

$$q_{10} = \left(\frac{3}{4\pi}\right)^{1/2} \hat{\epsilon}^{\,*}_{0} \cdot \vec{p} = \left(\frac{3}{4\pi}\right)^{1/2} p_z$$

$$q_{1,-1} = \left(\frac{3}{4\pi}\right)^{1/2} \hat{\epsilon}^{\,*}_{-1} \cdot \vec{p} = \left(\frac{3}{4\pi}\right)^{1/2} \frac{p_x + ip_y}{\sqrt{2}}$$

(4) Electric quadrupole moment ($l=2$)

$$q_{22} = \int d^3x' \, r'^2 \, Y^*_{22}(\hat{x}') \rho(\vec{x}')$$

and $Y^*_{22} = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \sin^2\theta' \, e^{-2i\phi'}$,

which we can also rewrite in terms of Cartesian components as

$$\frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \left(\frac{x' - iy'}{r}\right)^2$$

or $q_{22} = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \int (x' - iy')^2 \rho(\vec{x}') \, d^3x'$

It is conventional to write this in terms of a Cartesian quadrupole moment tensor, Q, i.e.

$$q_{22} = \frac{1}{12}\left(\frac{15}{2\pi}\right)^{1/2}\left(Q_{11} - 2iQ_{12} - Q_{22}\right)$$

and similarly $q_{21} = -\left(\frac{15}{8\pi}\right)^{1/2} \int z'(x-iy')\rho(\vec{x}\,')\,d^3x'$

$$\equiv -\frac{1}{3}\left(\frac{15}{8\pi}\right)^{1/2}\left(Q_{13} - iQ_{23}\right)$$

and

$$q_{20} = \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2} \int (3z'^2 - r'^2)\rho(\vec{x}\,')\,d^3x'$$

$$\equiv \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2} Q_{33}$$

where

makes Q_{ij} traceless

$$Q_{ij} \equiv \int (3x_i x_j - r'^2\delta_{ij})\rho(\vec{x}\,')\,d^3x'$$

called the electric quadrupole moment tensor

Then the external electric potential beyond the range of ρ can be expanded as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{ij}\frac{\overbrace{x_i Q_{ij} x_j}^{\equiv\, \vec{x}\cdot \underline{Q}\, \vec{x}}}{r^5} + \dots\right.$$

Next, let's work out the electric field of a single l,m multipole, using $\vec{E} = -\nabla\Phi$:

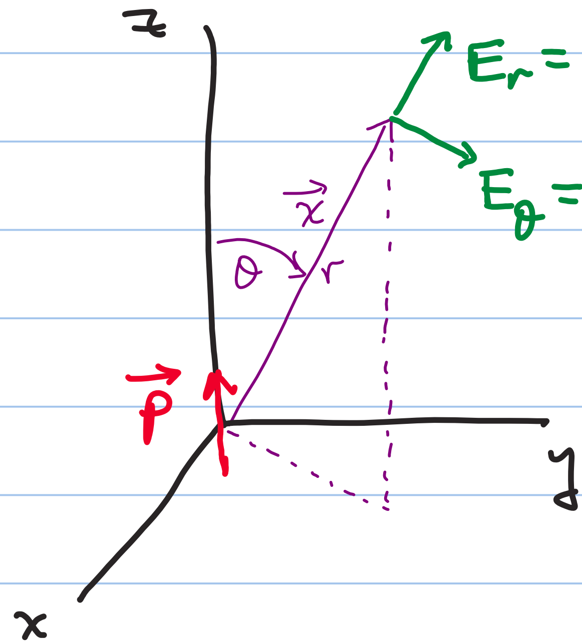
Here, use $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$

$$\Rightarrow \boxed{E_r = \frac{q_{lm}}{\epsilon_0}\frac{l+1}{2l+1}\frac{Y_{lm}(\hat{x})}{r^{l+2}}} \quad \boxed{E_\theta = \frac{-q_{lm}}{\epsilon_0(2l+1)}\frac{1}{r^{l+2}}\frac{\partial}{\partial\theta}Y_{lm}(\hat{x})}$$

$$\boxed{E_\phi = \frac{-q_{lm}}{\epsilon_0(2l+1)}\frac{im}{r^{l+2}\sin\theta}Y_{lm}(\hat{x})}$$

Specialize now to the case of an electric dipole and align it with the z-axis: $\vec{p} = p\hat{z}$

$\Rightarrow \ell = 1, m = 0$ only



$E_r = \dfrac{2p\cos\theta}{4\pi\epsilon_0 r^3}$

$E_\theta = \dfrac{p\sin\theta}{4\pi\epsilon_0 r^3}$ $\qquad E_\phi = 0$

We can recast these expressions into a more general form, involving $\vec{x}$ and $\vec{p}$ only.

First observe that $\vec{p} = \hat{r}(\vec{p}\cdot\hat{r}) + \hat{\theta}(\vec{p}\cdot\hat{\theta})$

hence $\hat{\theta} = \dfrac{\vec{p} - \hat{r}(\vec{p}\cdot\hat{r})}{\vec{p}\cdot\hat{\theta}}$

whereby

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left\{2p\cos\theta\,\hat{r} + p\sin\theta\left(\frac{\vec{p} - \hat{r}(\vec{p}\cdot\hat{r})}{\vec{p}\cdot\hat{\theta}}\right)\right\}$$

and by inspection,

$\vec{p}\cdot\hat{\theta} = p\cos\left(\frac{\pi}{2}+\theta\right) = -p\sin\theta$, $\vec{p}\cdot\hat{r} = p\cos\theta$

which simplifies $\vec{E}$ to the form:

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$

This form is no longer restricted to a dipole oriented along $\hat{z}$!

Or if $\vec{p}$ is located at $\vec{x}_0$, this reads,

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0 |\vec{x}-\vec{x}_0|^5}\left\{3\left[\vec{p}\cdot(\vec{x}-\vec{x}_0)\right](\vec{x}-\vec{x}_0) - \vec{p}\,|\vec{x}-\vec{x}_0|^2\right\}$$

It is worth noting that this expression gives 0 when $\vec{E}$ is integrated over a sphere centered on $\vec{p}$:

i.e. $$\int_{r<R} \frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{r^3}\, d^3r = 0$$

This is because the components of $\vec{E}$ are proportional to $Y_{2m}(\theta,\phi)$ and of course $\int Y_{lm}\, d\Omega = 0$ unless $l=m=0$

$\Rightarrow$ To see this explicitly, we use a few identities to work out the Cartesian components of $\vec{E}$.

First, observe that
$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

and
$$\hat{\phi} = \frac{\partial\hat{r}}{\partial\phi}\Big/\left|\frac{\partial\hat{r}}{\partial\phi}\right|$$
$$= -\hat{x}\sin\phi + \hat{y}\cos\phi$$

and $$\hat{\theta} = \hat{\phi}\times\hat{r} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

So for a dipole aligned with $\hat{z}$, i.e. $\vec{p} = p\hat{z}$

$$\Rightarrow 4\pi\epsilon_0 r^3\vec{E} = 2p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$$
$$= 2p\cos\theta(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)$$
$$+ p\sin\theta(\hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta)$$

or comparing to the $l=2$ spherical harmonics gives

$$\boxed{\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(\frac{8\pi}{15}\right)^{1/2}\left\{\hat{x}\,\frac{3}{2}\left(-Y_{2,1}+Y_{2,-1}\right)+\hat{y}\,\frac{3}{2i}\left(-Y_{21}-Y_{2,-1}\right) + \hat{z}\,\sqrt{2}\,Y_{2,0}\right\}}$$

Comments

(1) In general (see problem 4.4a), all multipoles of rank l HIGHER than the lowest nonzero one (l_{min}) depend on the coordinate origin.

illustration: a charge q at the origin has only one nonvanishing moment, namely

$$q_{0,0} = \frac{q}{\sqrt{4\pi}} = \text{the monopole moment}$$

However a charge q placed at $\vec{x}_0$ has the same monopole moment q_{00}, but also the other $q_{lm} = q\, r_0^l\, Y^*_{lm}(\theta_0, \phi_0) \neq 0$ for all l, m, in general.

(2) We have not yet investigated carefully the behavior near $r=0$, i.e. at the singularity. In fact a careful analysis shows that there is an additional δ-contribution, giving

an analogous term is important in atomic hyperfine structure, for magnetic moments

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{x}-\vec{x}_0)\right]$$

Here of course $\hat{n} = \dfrac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|}$

Proof Consider the volume integral of $\vec{E}(\vec{x})$ over a tiny spherical volume, $r<R$

$$\Rightarrow \int_{r<R} \vec{E}(\vec{x})\, d^3r = -\int_{r<R} \nabla \Phi(\vec{x})\, d^3x = -\underbrace{\int R^2 d\Omega\, \Phi(\vec{x})\, \hat{n}}_{\text{surface integral of radius } R}$$

this identity can be proved by applying the divergence theorem to $\vec{b}\,\Phi(\vec{x})$ where $\vec{b}$ = any constant vector

i.e. consider

$$\int_V \nabla \cdot (\vec{b}\,\Phi(\vec{x}))\, d^3x = \vec{b} \cdot \int \vec{\nabla}\Phi\, d^3x = \oint_S \vec{b} \cdot \hat{n}\, \Phi(\vec{x})\, d^3x$$

So since $\vec{b} \cdot \int \vec{\nabla}\Phi\, d^3x = \vec{b} \cdot \oint_S \hat{n}\, \Phi(\vec{x})\, da$ holds for arbitrary constant $\vec{b}$

$$\Rightarrow \boxed{\int \nabla \Phi(\vec{x})\, d^3x = \oint_S \Phi(\vec{x})\, \hat{n}\, da}$$

And returning to our above derivation, we have

$$\int_{r<R} \vec{E}(\vec{x})\, d^3x = \frac{-R^2}{4\pi\epsilon_0} \int d^3x' \rho(\vec{x}') \int_{r=R} \frac{d\Omega\, \hat{n}}{|\vec{x} - \vec{x}'|}$$

(i.e. $|\vec{x}| = R$ on surface)

and $\hat{n} = \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi + \hat{z} \cos\theta$

then recall $\hat{n} = \left(\frac{4\pi}{3}\right)^{1/2} \sum_\mu Y_{1\mu}^*(\hat{x})\, \hat{\epsilon}_\mu$

$$\Rightarrow \int \frac{\hat{n}\, d\Omega}{|\vec{x}-\vec{x}'|} = \int d\Omega \sum_{lm} \frac{4\pi}{2l+1}\left(\frac{4\pi}{3}\right)^{1/2} \sum_{\mu} \underbrace{Y^*_{1\mu}(\hat{x})\,\hat{\epsilon}_\mu \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\hat{x}')\,Y_{lm}(\hat{x})}_{\delta_{l,1}\,\delta_{m,\mu}}$$

$$= \underbrace{\frac{4\pi}{3}\left(\frac{4\pi}{3}\right)^{1/2} \sum_{\mu} Y^*_{1\mu}(\hat{x}')\,\hat{\epsilon}_\mu}_{=\hat{n}'} \frac{r_<}{r_>^2}$$

and so we see that

$$\int_{r<R} \vec{E}(\vec{x})\, d^3r = \left(-\frac{4}{3}\pi R^2\right)\frac{1}{4\pi\epsilon_0}\int \frac{r_<}{r_>^2}\,\rho(\vec{x}')\,\hat{n}'\, d^3x'$$

Thus if <u>all</u> $\rho(\vec{x}')$ is INSIDE the sphere, then $r_< = r'$, $r_> = R$, whereby

$$\int \vec{E}(\vec{x})\, d^3x = -\frac{\left(\frac{4}{3}\pi R^2\right)}{4\pi\epsilon_0 R^2}\underbrace{\left(\int \rho(\vec{x}')\,\vec{x}'\, d^3x'\right)}_{=\vec{p}}$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{4\pi}{3}\vec{p},$$

this is <u>independent</u> of the sphere radius R provided R is outside of the charges ρ

In conclusion, for a dipole $\vec{p}$ at $\vec{x}'=0$,

$$\boxed{\vec{E}_{dipole}(\vec{x}) = \frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{4\pi\epsilon_0 r^3} - \frac{\vec{p}}{3\epsilon_0}\,\delta(\vec{x})}$$

Corollary to the preceding derivation

Consider a different limit now, where all charge $\rho(\vec{x}')$ lies OUTSIDE the sphere,

$\Rightarrow$ the same math applies except now we have $r_< = R$, $r_> = r'$

$$\Rightarrow \int_{r<R} \vec{E}(\vec{x})\, d^3x = -\frac{4\pi}{3} R^3 \; \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')\, \hat{n}'\, d^3x'}{r'^2}$$

but this is precisely $\vec{E}(0)$!

$$\Rightarrow \boxed{\int_{r<R} \vec{E}(\vec{x})\, d^3x = \frac{4}{3}\pi R^3\, \vec{E}(0)}$$

this holds provided NO CHARGE lies inside the sphere

Multipole expansion of the interaction energy of $\rho(\vec{x})$ with an external field

The interaction energy with the external $\Phi(\vec{x})$ is:

$$W = \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x$$

Suppose now that the potential $\Phi(\vec{x})$ is slowly varying throughout the region where $\rho(\vec{x}) \neq 0$, we can Taylor expand $\Phi(\vec{x})$ about a suitable origin inside ρ:

$$\Phi(\vec{x}) = \Phi(0) + \vec{x}\cdot\vec{\nabla}\Phi(0) + \frac{1}{2}\sum_{ij} x_i \frac{\partial^2 \Phi(0)}{\partial x_i \partial x_j} x_j + \ldots$$

$$= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j(0)}{\partial x_i} + \ldots$$

$$= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j(0)}{\partial x_i} + \ldots$$

$\Rightarrow$ In that last term we have added a term

$$0 = \frac{r^2}{6}\nabla\cdot\vec{E},$$

in order to simplify the final expression.

Next, plugging this into the above energy expression gives the following:

$$W = q_{tot}\,\Phi(0) - \vec{E}(0)\cdot\int \vec{x}\,\rho(\vec{x})\,d^3x$$
$$-\frac{1}{6}\sum_{ij}\frac{\partial E_j(0)}{\partial x_i}\int\left(3x_i x_j - r^2\delta_{ij}\right)\rho(\vec{x})\,d^3x$$

or, more compactly,

$$W = q_{tot}\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\left[\nabla\cdot\left(\underline{Q}\,\vec{E}(\vec{x})\right)\right]_0 + \ldots$$

where $\underline{Q}$ = constant
= electric quadrupole Cartesian matrix.

e.g. 2 dipoles have:

$$W_{12} = \frac{\left[\vec{p}_1\cdot\vec{p}_2 - 3(\hat{n}\cdot\vec{p}_1)(\hat{n}\cdot\vec{p}_2)\right]}{4\pi\epsilon_0 r^3} + \frac{\vec{p}_1\cdot\vec{p}_2}{3\epsilon_0}\,\delta(\vec{x}_1-\vec{x}_2)$$

Electrostatics in macroscopic media:

- how matter affects or is affected by fields.

e.g. - why does light pass through a window, but not through wood?

some challenging aspects:

- Matter is mostly neutral, but it consists of charged particles

- Matter is rearranged, or polarized by external fields

- The polarization (usually expressed as induced dipole moments) produces "internal fields"

- The total field = sum of external + internal fields

- We must <u>average</u> over a "microsopically large but macroscopically small" volume, i.e. over a volume that might contain $\simeq 1000$ atoms

e.g., call $\vec{E}_{micro}(\vec{x})$ the primitive or microscopic E-field that obeys Maxwell's equations in free space and includes <u>ALL</u> charges in the universe

picture this as fluctuating wildly in space + time on the atomic scale, e.g. inside a solid.

$\Rightarrow$ We want to find equations obeyed by a coarse-grained or <u>averaged</u> <u>E-field</u>,

$$\Rightarrow \quad \boxed{\vec{E}(\vec{x}) = \frac{\int_{|\vec{x}-\vec{x}'|<R} d^3x' \, \vec{E}_{micro}(\vec{x}')}{\frac{4}{3}\pi R^3}}$$

Point (1) Since $\nabla \times \vec{E}_{micro} = 0$ everywhere in electrostatics, clearly $\underline{\nabla \times \vec{E} = 0}$ still after averaging

Hence $\vec{E}$ is still derivable from a potential

i.e. $\boxed{\vec{E} = -\nabla \Phi}$

Point (2) Next, determine the internal fields produced by electric dipole moments $\vec{p}$ induced in the medium. (Assume that higher multipoles only produce small corrections.)

$\Rightarrow$ Start from $\Phi(\vec{x}) = \dfrac{\vec{p} \cdot (\vec{x} - \vec{x}')}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|^3}$

for a dipole at $\vec{x}'$. We can rewrite this in differential form, for the potential $d\Phi$ produced by a dipole $d\vec{p}$ at $\vec{x}'$:

$$d\Phi(\vec{x}) = \frac{d\vec{p}}{4\pi\epsilon_0} \cdot \nabla' \frac{1}{|\vec{x} - \vec{x}'|} = \frac{-d\vec{p}}{4\pi\epsilon_0} \cdot \nabla \frac{1}{|\vec{x} - \vec{x}'|}$$

So if we have a continuous average distribution of dipole moment, we write $d\vec{p}(\vec{x}') = \vec{P}(\vec{x}')\, d^3x'$ where we have defined

$$\vec{P}(\vec{x}') \equiv \text{polarization} = \frac{\text{average dipole moment}}{\text{unit volume}}$$

$$= \sum_i N_i \langle \vec{p}_i \rangle$$

where N_i = number density of the ith type of molecule, having an average electric dipole moment, $\langle \vec{p}_i \rangle$

Of course, in addition to the dipoles that represent "bound charge", we could also have some "additional" or "free" charge present, ρ_{free}

But for now, consider only the dipoles.

$$\Rightarrow \Phi(\vec{x}) = \int_V \frac{\vec{P}(\vec{x}')}{4\pi\epsilon_0} \cdot \nabla' \frac{1}{|\vec{x}-\vec{x}'|} d^3x'$$

or recalling the identity:

$$\nabla' \cdot \left(\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) = \frac{1}{|\vec{x}-\vec{x}'|} \nabla' \cdot \vec{P}(\vec{x}') + \vec{P}(\vec{x}') \cdot \nabla' \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \nabla' \cdot \left(\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) d^3x' - \int_V \frac{d^3x'}{4\pi\epsilon_0} \frac{\nabla' \cdot \vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

and from the divergence theorem, of course, this becomes

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\left(-\nabla' \cdot \vec{P}(\vec{x}')\right) d^3x'}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi\epsilon_0} \oint_S \frac{\vec{P}(\vec{x}') \cdot \hat{n}' da'}{|\vec{x}-\vec{x}'|}}$$

Interpretation These 2 integrals now have the SAME FORM as for the potential produced by a volume "bound charge" $\rho_b \equiv -\nabla' \cdot \vec{P}(\vec{x}')$ and/or a surface bound charge $\sigma_b \equiv \vec{P}(\vec{x}') \cdot \hat{n}'$

where "b" stands for "bound charges"

Thus, interpreted this way, our old formulas for $\Phi(\vec{x})$ still work in the presence of media, namely

$$\Phi(\vec{x}) = \int_V \frac{\rho(\vec{x}')\,d^3x'}{4\pi\epsilon_0|\vec{x}-\vec{x}'|} + \oint_S \frac{\sigma(\vec{x}')\,da'}{4\pi\epsilon_0|\vec{x}-\vec{x}'|}$$

provided we use:

$$\rho(\vec{x}') = \rho_{free}(\vec{x}') + \rho_b(\vec{x}') \quad \leftarrow -\nabla'\cdot\vec{P}(\vec{x}')$$

and

$$\sigma(\vec{x}') = \sigma_{free}(\vec{x}') + \sigma_b(\vec{x}') \quad \leftarrow \vec{P}(\vec{x}')\cdot\hat{n}'$$

And since this equation has the same form for $\Phi(\vec{x})$ as before (without media), we know:

$$-\nabla^2\Phi(\vec{x}) = \nabla\cdot\vec{E} = \frac{4\pi}{4\pi\epsilon_0}(\rho_{free} + \rho_b)$$

or $$\nabla\cdot\vec{E} + \frac{1}{\epsilon_0}\nabla\cdot\vec{P} = \frac{\rho_{free}}{\epsilon_0}$$

Or defining the ELECTRIC DISPLACEMENT as

$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P},$$

we obtain the appropriate Maxwell equation in the presence of macrosopic media,

$$\boxed{\nabla\cdot\vec{D} = \rho_{free}} \Rightarrow \boxed{\oint_S \vec{D}\cdot d\vec{a} = Q^{free}_{enclosed}}$$

This, in addition to $\boxed{\nabla \times \vec{E} = 0,}$ defines the electrostatic field.

Generally, we cannot solve the Maxwell equations

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \times \vec{E} = 0$$

unless we know how $\vec{D}$ and $\vec{E}$ are related.

Often, it is a good approximation to assume, especially for weak fields, that the induced polarization is proportional to $\vec{E}$, i.e. $\vec{P} = \chi_e \epsilon_0 \vec{E}$

where χ_e is the electric susceptibility of the medium = CONSTANT for an isotropic medium.

$\Longrightarrow \quad \vec{D} = \epsilon \vec{E} \Rightarrow \epsilon = \epsilon_0 (1 + \chi_e)$ is called the DIELECTRIC CONSTANT

An Important Special Case

Often ϵ = constant within some region. When this is true, we simply have

$$\nabla \cdot \vec{E} = \rho / \epsilon \quad \text{WITHIN THAT REGION,}$$

while different regions are connected by BOUNDARY CONDITIONS, namely

$\boxed{(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{2 \leftarrow 1} = \sigma_F} \quad \boxed{(\vec{E}_2 - \vec{E}_1) \times \hat{n}_{2 \leftarrow 1} = 0}$

Note $\vec{P}$ is often NOT proportional to $\vec{E}$, notably:

(i) in ferroelectric materials, where $\vec{P}$ can be nonzero even when $\vec{E}=0$

or

(ii) for strong E-field where $\vec{P}$ can be a more complicated, nonlinear function of E, e.g. in nonlinear optics

There is another important class of problems where the relationship between $\vec{P}$ and $\vec{E}$ is LINEAR but ANISOTROPIC, as in a birefringent crystal like calcite. For such a case, we can write:

$$\vec{P} = \epsilon_0 \underline{\chi_e} \vec{E}$$

where $\underline{\chi_e}$ is a 3×3 matrix rather than a scalar.

Boundary Value Problems with Dielectrics

Category I: Problems solvable using image charge methods: e.g. a point charge q near a planar interface between 2 dielectrics.

$z > 0$

$\epsilon = \epsilon_1$

Φ_1

q at $(0,0,d) \equiv \vec{x}_0$

$z = 0$ plane

$z < 0$

$\epsilon = \epsilon_2$

Φ_2

try image q' at $(0,0,-d)$

″ $\vec{x}_0'$

when calculating Φ_1 at $z > 0$

Then a plausible hypothesis is that $\Phi(\vec{x})$ can be written as follows, for some q',

$$\Phi_1 = \frac{q}{4\pi\epsilon_1 |\vec{x} - \vec{x}_0|} + \frac{q'}{4\pi\epsilon_1 |\vec{x} - \vec{x}_0'|}$$

and another plausible guess/ansatz is that Φ_2 can be written in the form:

$$\Phi_2 = \frac{1}{4\pi\epsilon_2} \frac{q''}{|\vec{x} - \vec{x}_0|}$$

Recall that it is not allowed to put an image charge into the volume of interest!

Next, attempt to match the B.C.s for some choice of q', q'', using $\frac{1}{|\vec{x}-\vec{x}_0|} = [x^2+y^2+(z-d)^2]^{-1/2}$

$$\frac{1}{|\vec{x}-\vec{x}_0'|} = [x^2+y^2+(z+d)^2]^{-1/2}$$

Since there are no free surface charges, the BCs are

$$\epsilon_1 E_z\Big|_{z=0^+} = \epsilon_2 E_z\Big|_{z=0^-}$$

and

$$E_x\Big|_{z=0^+} = E_x\Big|_{z=0^-}$$

i.e., $-\epsilon_1 \frac{\partial \Phi_1}{\partial z}(x,y,0) = -\epsilon_2 \frac{\partial \Phi_2}{\partial z}(x,y,0)$

$$\Rightarrow \frac{-\epsilon_1}{4\pi\epsilon_1}\left\{\frac{+qd}{(\rho^2+d^2)^{3/2}} - \frac{q'd}{(\rho^2+d^2)^{3/2}}\right\} = \frac{-\epsilon_2}{4\pi\epsilon_2}\frac{q''d}{(\rho^2+d^2)^{3/2}}$$

and $-\frac{\partial \Phi_1}{\partial x}(x,y,0) = -\frac{\partial \Phi_2}{\partial x}(x,y,0)$

or

$$-\frac{1}{4\pi\epsilon_1}\left[\frac{-qx}{(\rho^2+d^2)^{3/2}} - \frac{q'x}{(\rho^2+d^2)^{3/2}}\right] = -\frac{1}{4\pi\epsilon_2}\frac{(-q''x)}{(\rho^2+d^2)^{3/2}}$$

and simplifying gives:

$$\begin{cases} q - q' = q'' \\ \dfrac{q+q'}{\epsilon_1} = \dfrac{q''}{\epsilon_2} \end{cases}$$

2 equations & 2 unknowns to solve

and the solution is $q' = -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right)q$, $q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}q$

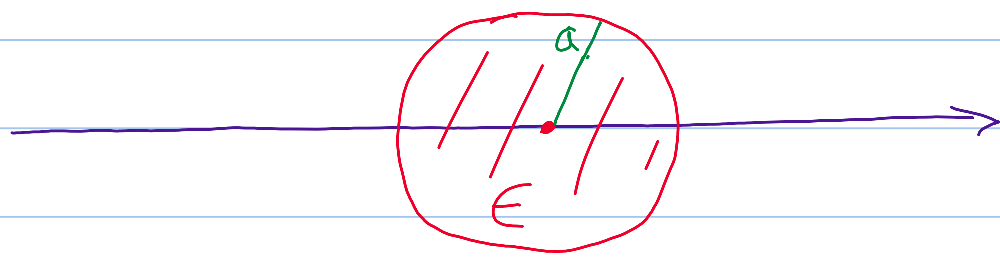
On your own, verify the following:

(1) The potential is <u>continuous</u> at the boundary $\Rightarrow$ we could have imposed this as a requirement

(2) Compute the polarization surface charge density, $\sigma_b = \vec{P} \cdot \hat{n}$ ← Jackson calls this σ_{pol}

Category II – Boundary value problem solved by separation of variables

e.g. a dielectric sphere in a uniform E-field



$\vec{E}_0 = E_0 \hat{z}$

$\Rightarrow$ Here we have AZIMUTHAL SYMMETRY and NO FREE CHARGES, $\rho_{free} = 0$

$$\Rightarrow \Phi_{in} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Phi_{out} = \sum_\ell \left(B_\ell r^\ell + \frac{C_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Boundary Conditions

(1) $\lim_{r\to\infty} \Phi_{out} = -E_0 z = -E_0 r P_1(\cos\theta)$

i.e. $\boxed{B_1 = -E_0}$ and all other $B_\ell = 0$ for $\ell \neq 1$

(2) $E_{\parallel}$ is continuous $\Rightarrow -\frac{1}{r}\frac{\partial \Phi_{in}}{\partial \theta}\Big|_{r=a} = -\frac{1}{r}\frac{\partial \Phi_{out}}{\partial \theta}\Big|_{r=a}$

$\Rightarrow A_\ell a^\ell = \frac{C_\ell}{a^{\ell+1}}$ if $\ell \neq 1$, and $A_1 a = B_1 a + \frac{C_1}{a^2}$

(3) $\epsilon E_\perp(in) = \epsilon_0 E_\perp(out)$

$\Rightarrow -\epsilon \frac{\partial \Phi_{in}}{\partial r}\Big|_{r=a} = -\epsilon_0 \frac{\partial \Phi_{out}}{\partial r}\Big|_{r=a}$

$\Rightarrow -\epsilon \ell A_\ell a^{\ell-1} = -\epsilon_0(-\ell-1) C_\ell a^{-\ell-2}$, $\ell \neq 1$

and $-\epsilon A_1 = -\epsilon_0(-E_0) + \frac{2C_1}{a^2}$, $\ell = 1$

These are 2 equations, 2 unknowns for each ℓ, whose solutions are:

$$A_\ell = C_\ell = 0, \quad \ell \neq 1$$

$$A_1 = \frac{-3E_0}{2 + \frac{\epsilon}{\epsilon_0}}, \quad C_1 = \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) a^3 E_0$$

$\Rightarrow$ the final solution is

$$\Phi_{in}(r,\theta) = -\frac{3}{2 + \frac{\epsilon}{\epsilon_0}} E_0 r \cos\theta, \quad r < a$$

$$\Phi_{out}(r,\theta) = -E_0 r\cos\theta + \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0 \frac{a^3}{r^2}\cos\theta, \quad r>a$$

Note that here again, Φ is continuous at a.

$$\Rightarrow \vec{E}_{inside} = \frac{3}{\frac{\epsilon}{\epsilon_0}+2} E_0 \hat{z}$$ (observe that E_{inside} is smaller than E_0, if $\epsilon > \epsilon_0$)

and $\vec{E}_{outside} = E_0\hat{z} + \vec{E}_{dipole}$

where

$\vec{E}_{dipole}$ is the field due to a dipole of magnitude $\vec{p} = \hat{z}\, 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{\frac{\epsilon}{\epsilon_0}+2}\right) a^3 E_0$

i.e. $\vec{p} = \int d^3x\, \vec{P}(x)$ where $\vec{P}(\vec{x}) = (\epsilon - \epsilon_0)\vec{E}$

$$= 3\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0}+2}\right)\vec{E}_0$$

or more directly using $\vec{E}_{out} = -\vec{\nabla}\Phi_{out}$,

$$\vec{E}_{out} = E_0\hat{z} - \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0\, a^3\, \nabla\left(\frac{\cos\theta}{r^2}\right)$$

$$= E_0\hat{z} - \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0}+2}\right) E_0\, a^3 \left(-\frac{\cos\theta\,\hat{r}}{r^3}(2) - \frac{\sin\theta\,\hat{\theta}}{r^3}\right)$$

Compare with Eq. 4.12

Observe that the bound (polarization) charge density is 0 throughout the volume $r<a$, since there $\vec{P}(\vec{x}) = \text{constant}$,

$\Rightarrow -\vec{\nabla}\cdot\vec{P}(\vec{x}) = 0$

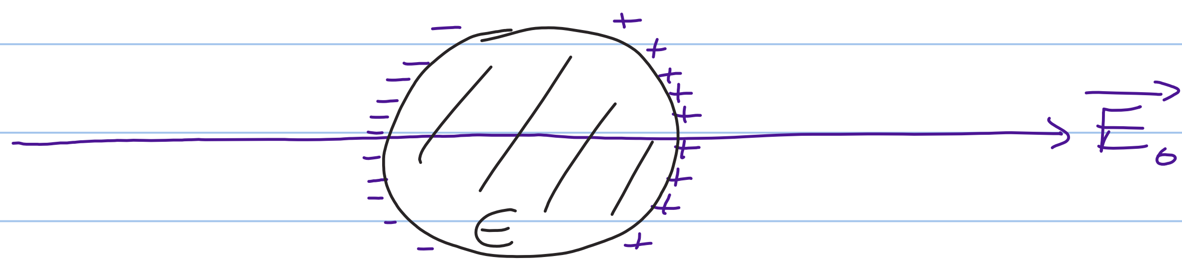
However, a surface bound charge remains,

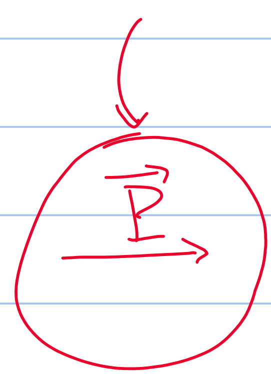
$$\sigma_b = \vec{P} \cdot \hat{r} = 3\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2} \right) E_0 \cos\theta$$

<u>Comment</u> The same math would apply to a spherical CAVITY in a dielectric medium, except that we must interchange $\epsilon \longleftrightarrow \epsilon_0$ everywhere!

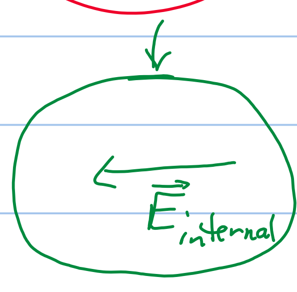
e.g. $\vec{E}_{in} = \dfrac{3\epsilon}{2\epsilon + \epsilon_0} \vec{E}_0$, which is $> E_0$ if $\epsilon > \epsilon_0$

Qualitative Picture for a dielectric sphere



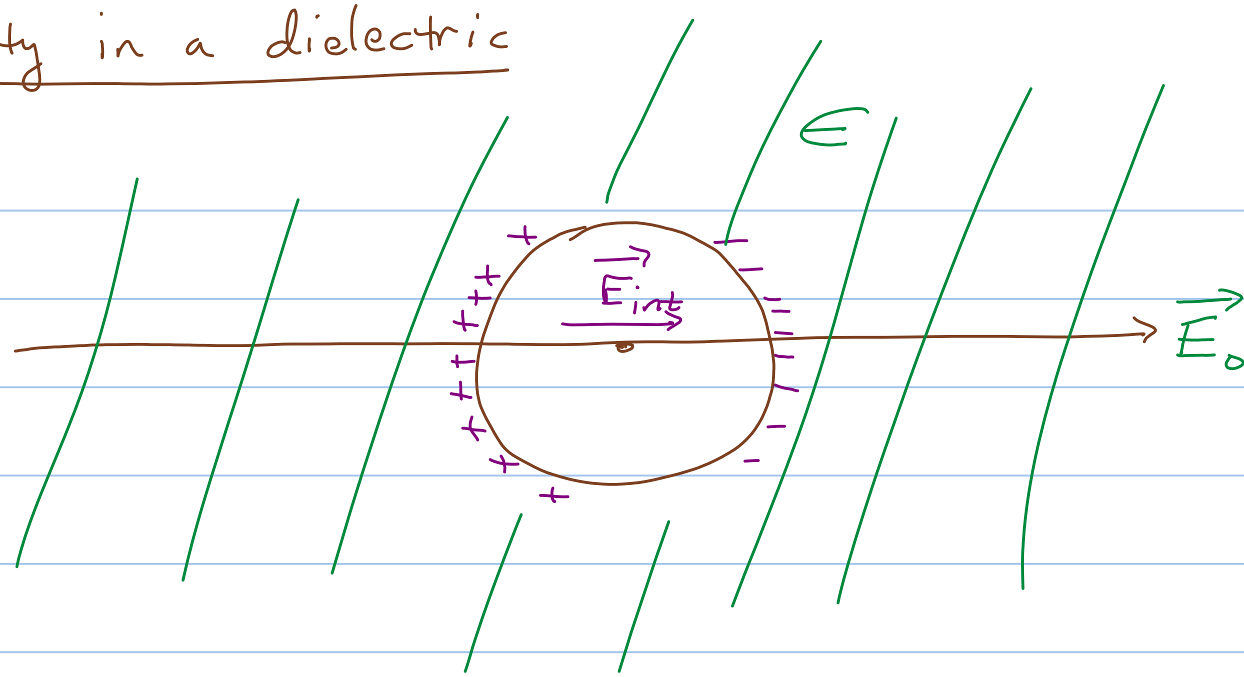


polarization induced by $\vec{E}_0$



$\vec{E}$-field induced by bound polarization charges σ_b REDUCES $\vec{E}$ when added to $\vec{E}_0$

Cavity in a dielectric



Now, for a cavity, the $\vec{E}$-field due to σ_b INCREASES the field higher than what it is in the dielectric far from the cavity.

IMPORTANT: When there are no free charges in a medium, we expect $\rho_b = -\nabla \cdot \vec{P} = 0$ since $\nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} \to 0$ and $\vec{P} = \epsilon_0 \chi_e \vec{E}$