Physics 630
Classical electrodynamics
Instructor: Chris Greene

Our goal this semester is to study the equations and phenomena of classical E+M, which follow from that tremendous synthesis into 4 vector equations, by James Clerk Maxwell. In vacuum, written in SI units, Maxwell's eqns are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (1)$$

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad (2)$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad (3)$$

$$\nabla \cdot \vec{B} = 0 \qquad (4)$$

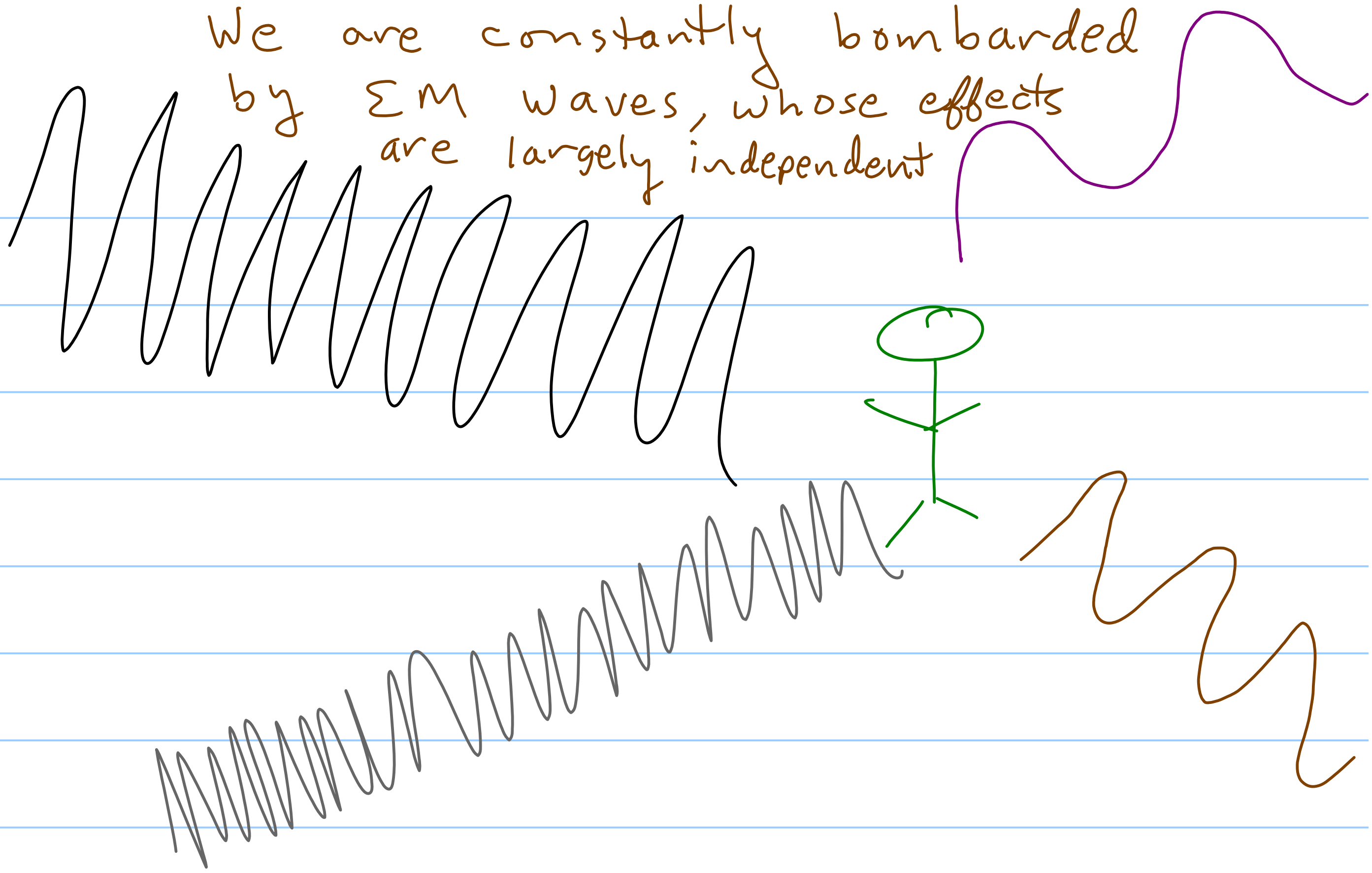
and these equations imply an auxiliary equation, derived by taking $\nabla\cdot(2)$ and plugging in (1):

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (5)$$

charge conservation law

IMPORTANT: Maxwell's equations are LINEAR, first-order differential equations

We are constantly bombarded by EM waves, whose effects are largely independent



We also must describe the effects of $\vec{E}, \vec{B}$ on the motion of a charged particle q. In classical mechanics this is the Lorentz force law,

$$\boxed{\vec{F} = q\vec{E}(\vec{x},t) + q\underbrace{\frac{d\vec{x}}{dt}}_{\vec{v}} \times \vec{B}(\vec{x},t)} \qquad (6)$$

$\vec{E}(\vec{x},t)$ = electric field at position $\vec{x}$, time t, $\left(\frac{V}{m}\right)$ or $\left(\frac{N}{C}\right)$

$\vec{B}(\vec{x},t)$ = magnetic field " " ", time t, (T)

$\rho(\vec{x},t)$ = charge density, $\left(\frac{\text{Coulombs}}{m^3}\right)$

$\vec{J}(\vec{x},t)$ = current density, $\left(\frac{\text{Coulombs}}{m^2 \cdot s}\right)$

And in a linear, homogeneous, dielectric medium, $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$

whereby (1) becomes $\nabla \cdot \vec{D} = \rho$ (1′)

(2) becomes $\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$ (2′)

c = vacuum light speed $\equiv 299792458 \frac{m}{s}$, EXACTLY

$= \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

The first step in development of the subject of E + M was the discovery of Coulomb's Law: For 2 point charges in a vacuum, the Coulomb force on q_1 due to q_2 is

$$\vec{F}_{1,2} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

Since the force on q_1 is proportional to q_1 it is useful to define the electric field at $\vec{x}_1$, independently of whether q_1 is actually there, i.e. as

$$\vec{E}(\vec{x}_1) = \lim_{q_1 \to 0} \frac{\vec{F}_{1,2}}{q_1} = \frac{q_2}{4\pi\epsilon_0} \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

If there are many charges, their net electric field at $\vec{x}$ is the sum

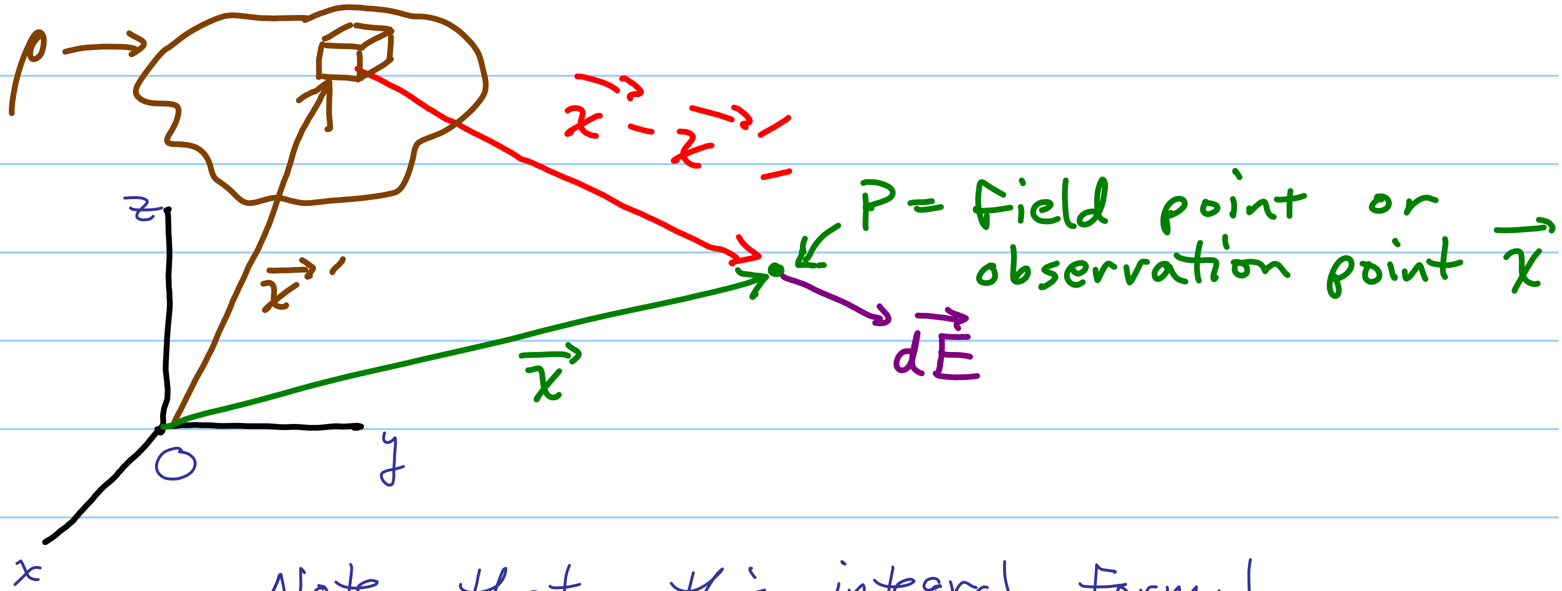
$$\vec{E}(\vec{x}) = \sum_i \frac{q_i}{4\pi\epsilon_0} \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}$$

which generalizes for a <u>continuous</u> distribution of charges, through the substitutions

$$\vec{x}_i \longrightarrow \vec{x}\,'$$
$$q_i \longrightarrow dq' \equiv \rho(\vec{x}\,')\, d^3x',$$

giving the integral

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x'\, \rho(\vec{x}\,') \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3}}$$



Note that this integral formula can also be used for DISCRETE charges, provided we identify

$$\rho(\vec{x}\,') = \sum_i q_i\, \delta(\vec{x}\,' - \vec{x}_i)$$

Idealization Actually, all charges are discrete, so the most accurate picture would use sums only, but since there are often Avogadro's number of charges this is impractical. So approximating the sums by integrals is more practical and usually quite accurate.

Also, in classical E+M it is frequently stated that Gauss's Law implies that any net charge on a conductor in electrostatics must reside on its outermost surface, i.e. within an infinitesimally thin layer.

More realistic models including the quantum wavelike nature of the electrons shows (see Fig I.5) that the surface layer actually has a finite thickness of approximately $\Delta r \simeq 2\,\mathring{A}$

while this is not infinitely thin, it is thin enough so that for most applications it can be treated that way, i.e.

$$\rho(\vec{x}') = \sigma\,\delta(x'-R)$$

Math Review: Dirac delta functions

(1) $$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \text{"}\infty\text{"}, & x = a \end{cases}$$

(2) $$\int_c^d \delta(x-a)\,dx = \begin{cases} 1, & \text{if} \quad c<a<d \\ -1, & \text{if} \quad c>a>d \\ 0, & \text{otherwise} \end{cases}$$

(Note: more careful consideration is needed if either $c=a$ or $d=a$.)

explicit representations

$$\frac{1}{2}\frac{d^2}{dx^2}|x| = \delta(x)$$

$$\delta(x) = \pi^{-\frac{1}{2}} \lim_{\epsilon \to 0^+} \epsilon^{-1/2} \exp\left(-x^2/\epsilon\right)$$

$$\delta(x) = \pi^{-1} \lim_{\epsilon \to 0^+} \frac{\epsilon}{x^2+\epsilon^2}$$

$$\delta(x) = \pi^{-1} \lim_{N \to \infty} \frac{\sin Nx}{x}$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

(3) $$\int_{-\infty}^{\infty} dx \; f(x)\,\delta(x-a)\,dx = f(a) \quad , a = \text{real}$$

proof $$\int_{-\infty}^{\infty} dx \; f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-a)}\,dk\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{-ika}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\; f(x)\,e^{ikx} = f(a),$$

since this is the inverse Fourier transform of a Fourier transform.

(4) $$\int_{-\infty}^{\infty} f(x)\,\delta'(x-a)\,dx = f(x)\,\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x-a)\,dx$$

$$= -f'(a)$$

This result is sometimes written as

$$f(x)\frac{d}{dx}\delta(x-a) = -\delta(x-a)\frac{d}{dx}f(x)$$

(5) $$\delta(ax) = \frac{1}{|a|}\delta(x)$$

corollary $$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$$

provided $f(x)$ has only simple zeroes at x_i

e.g. $$\delta[(x-a)(x-b)] = \frac{1}{|a-b|}\Big(\delta(x-a) + \delta(x-b)\Big)$$

(6) δ-functions in multiple dimensions

$$\delta(\vec{x}-\vec{x}\,') = \delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(x_3-x_3')$$

(in Cartesian coordinates)

This vanishes except at the point $\vec{x} = \vec{x}\,'$

i.e. $$\int_V \delta(\vec{x}-\vec{x}\,')\,d^3x' = \begin{cases} 1 & \text{if } \vec{x} \text{ is in } V \\ 0 & \text{otherwise} \end{cases}$$

3-dimensional representation

$$\delta(\vec{x}-\vec{x}') = (2\pi)^{-3} \int d^3k \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

If we consider general orthogonal, curvilinear coordinates, q_1, q_2, q_3
with differential arc length,

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$$

and volume element $d^3x = h_1 h_2 h_3 \, dq_1 dq_2 dq_3$,

then $$\delta(\vec{x}-\vec{x}') = \frac{1}{h_1 h_2 h_3} \delta(q_1-q_1')\,\delta(q_2-q_2')\,\delta(q_3-q_3')$$

e.g. in spherical coordinates,

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2 \sin\theta} \delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

$$= \frac{1}{r^2} \delta(r-r')\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')$$

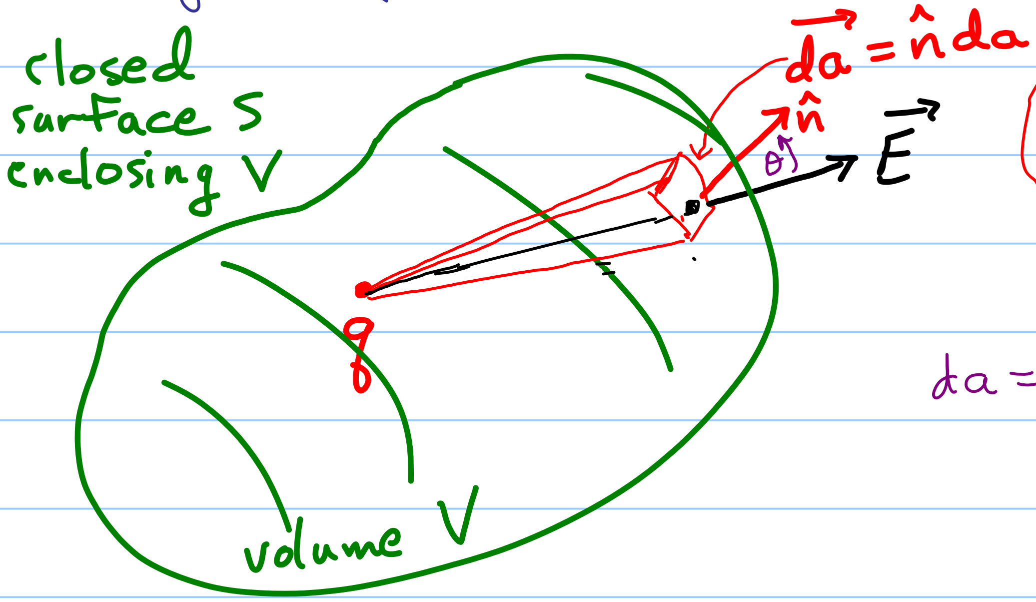
(7) **another identity:** $\nabla^2 \frac{1}{r} = -4\pi\,\delta(\vec{r})$

whereby we see that

$$\frac{1}{r} = \underbrace{\left[4\pi\,(2\pi)^{-3}\right]}_{(2\pi^2)^{-1}} \int d^3k \, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2}$$

Gauss's Law – This is often more convenient than evaluating a 3-dimensional Coulomb's Law integral. It also exemplifies the intuitive power and beauty of geometrically reinterpreting a math formula.

To demonstrate this, consider a point charge q at a point in space, and draw an imaginary closed surface S surrounding it:



$\hat{n}$ = outwardly directed unit vector normal to S

$da = |\vec{da}|$ = magnitude of differential area element

Now observe that $\vec{E} \cdot \vec{da} = \dfrac{q}{4\pi\epsilon_0}\left(\dfrac{\cos\theta\, da}{r^2}\right)$

But $d\Omega = \cos\theta \dfrac{da}{r^2}$,

where $d\Omega$ = differential solid angle subtended by da from charge q.

So we can write

$$\oint_S \vec{E} \cdot \vec{da} = \oint_S \frac{q}{4\pi\epsilon_0}\, d\Omega = \begin{cases} q/\epsilon_0, & \text{if } q \text{ is in } V \\ 0, & \text{if } q \text{ is outside } V \end{cases}$$

Of course, this argument generalizes if there are many charges, i.e. by superposition which is permitted by the linearity of the equations, giving the key result:

$$\boxed{\oint_S \vec{E}\cdot\vec{da} = \frac{q_{\text{enclosed}}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V d^3x'\,\rho(\vec{x}\,')}$$

where V = volume enclosed by S

Next, apply the divergence theorem:

$$\oint_S \vec{E}\cdot\vec{da} = \overbrace{\int_V \nabla\cdot\vec{E}\,d^3x = \frac{1}{\epsilon_0}\int_V d^3x\,\rho(\vec{x})}$$

and since the second equality holds for an ARBITRARY surface + volume, the integrands must themselves be equal everywhere, i.e.

$$\boxed{\nabla\cdot\vec{E} = \rho/\epsilon_0}$$

is equivalent to $\oint_S \vec{E}\cdot\vec{da} = \dfrac{q_{\text{enclosed}}}{\epsilon_0}$

= Gauss's Law in differential and integral form

Electrostatic Potential

Consider

$$\nabla_x \frac{1}{|\vec{x}-\vec{x}'|} = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= \frac{\hat{x}(-\frac{1}{2})(2)(x-x')+\hat{y}(-\frac{1}{2})(2)(y-y')+\hat{z}(-\frac{1}{2})(2)(z-z')}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}$$

$$= -\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

(Aside: sometimes we will abbreviate $\nabla_x \equiv \nabla$
$\nabla_{x'} \equiv \nabla'$)

Then Coulomb's Law can be rewritten as:

$$\vec{E}(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{4\pi\epsilon_0}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\int d^3x' \frac{\rho(\vec{x}')}{4\pi\epsilon_0}\nabla\frac{1}{|\vec{x}-\vec{x}'|}$$

or we can write $\boxed{\vec{E} = -\nabla\Phi(\vec{x})}$

where

$$\boxed{\Phi(\vec{x}) = \int \frac{d^3x'}{4\pi\epsilon_0}\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \equiv \text{Electrostatic Potential}}$$

Comments ① since $\nabla\times\nabla f = 0$ always $\Rightarrow \nabla\times\vec{E} = 0$ in electrostatics, always!

② This integral expression for Φ is not the most general, since it has implicitly assumed the B.C. that $\Phi(\vec{x}) \to 0$ as $r \to \infty$, whenever ρ has a finite extent.

③ If $\Phi(\vec{x})$ is one solution, then an equally good solution is $\Phi(\vec{x})$ + constant, since both give the same $\vec{E}(\vec{x})$

④ Don't forget that you need BOTH

$$\nabla\cdot\vec{E} = \rho/\epsilon_0$$

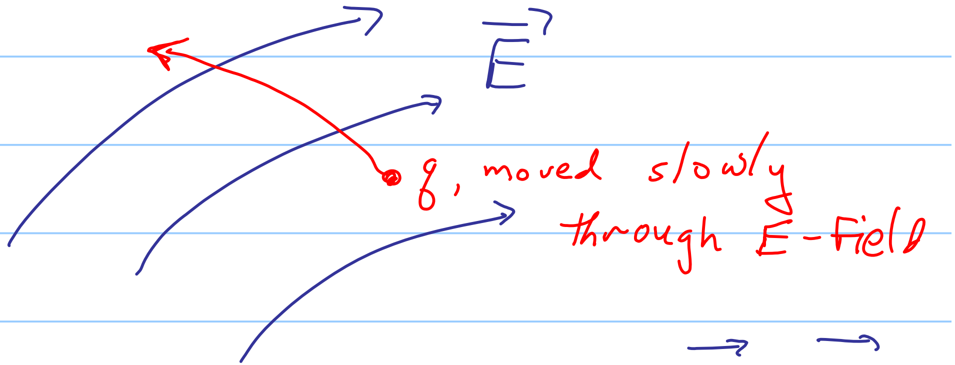
AND

$$\nabla\times\vec{E} = 0$$

to specify $\vec{E}$ completely, i.e. to within the gradient of a scalar function F that obeys Laplace's equation, $\nabla^2 F = 0$.

Relationship to energy

Let's try to discuss this carefully, since it is easy to make sign errors when discussing work and potential energy.



The electric force on a charge q is $\vec{F} = q\vec{E}$ so the force WE apply to hold q fixed, or to move it very slowly, is $\vec{F}_{us} = -q\vec{E}$

Thus the WORK that WE do in slowly moving q from $\vec{x}_1$ to $\vec{x}_2$ along some path $\mathcal{P}$ is

$$W_2 - W_1 = \int_{\vec{x}_1, \mathcal{P}}^{\vec{x}_2} \vec{F}_{us} \cdot \vec{dl} = - \int_{\vec{x}_1, \mathcal{P}}^{\vec{x}_2} \vec{F} \cdot \vec{dl}$$

or

$$W_2 - W_1 = -q \int_{\vec{x}_1}^{\vec{x}_2} \vec{E} \cdot \vec{dl} = +q \int_{\vec{x}_1}^{\vec{x}_2} \nabla \Phi \cdot \vec{dl}$$

$$= q \int_{\vec{x}_1}^{\vec{x}_2} d\Phi = q\, \Phi(\vec{x}_2) - q\, \Phi(\vec{x}_1)$$

Thus we can interpret $q\, \Phi(\vec{x})$ as the

ELECTRIC POTENTIAL ENERGY

i.e. if we do positive work, the potential energy must increase if kinetic energy = constant

observations

① The work done is path-independent

$$-q \int_{\vec{x}_1}^{\vec{x}_2} \vec{E} \cdot \vec{dl} = q\, \Phi(\vec{x}_2) - q\, \Phi(\vec{x}_1)$$

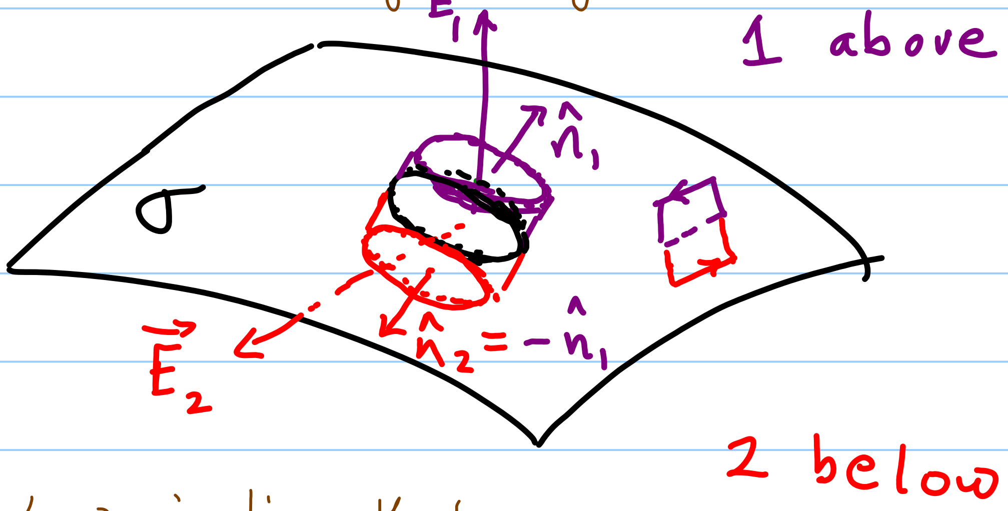
② $$\int_{\vec{x}_1}^{\vec{x}_1} \vec{E}(\vec{x}) \cdot \vec{dl} = \oint_C \vec{E}(\vec{x}) \cdot \vec{dl} = 0$$

for ANY closed loop

③ $$\oint_C \vec{E} \cdot \vec{dl} = \int_S \nabla \times \vec{E} \cdot \vec{da} = 0$$ is consistent with $\nabla \times \vec{E} = 0$

Implications of a surface charge distribution

Consider the electric field very close to a surface distribution of charge, σ



Now, Gauss's Law implies that

$$\oint \vec{E}\cdot\vec{da} = (\vec{E}_1\cdot\hat{n}_1 + \vec{E}_2\cdot\hat{n}_2)\,da = \frac{\sigma\,da}{\epsilon_0}$$

$$\text{or}\quad \boxed{(\vec{E}_1 - \vec{E}_2)\cdot\hat{n}_1 = \frac{\sigma}{\epsilon_0}}$$

and since $\oint \vec{E}\cdot\vec{dl} = 0$, $\Rightarrow$ $\boxed{\vec{E}_1\cdot\vec{l}_t = \vec{E}_2\cdot\vec{l}_t}$

where $\vec{l}_t$ is any vector lying in the plane tangent to the surface at that point.

Similarly, consider the effect of a dipole layer

i.e. consider a layer of oriented dipole molecules on a surface, consisting of surface charge densities $+\sigma(\vec{x})$ and $-\sigma(\vec{x})$ separated by a tiny distance $a(\vec{x})$.

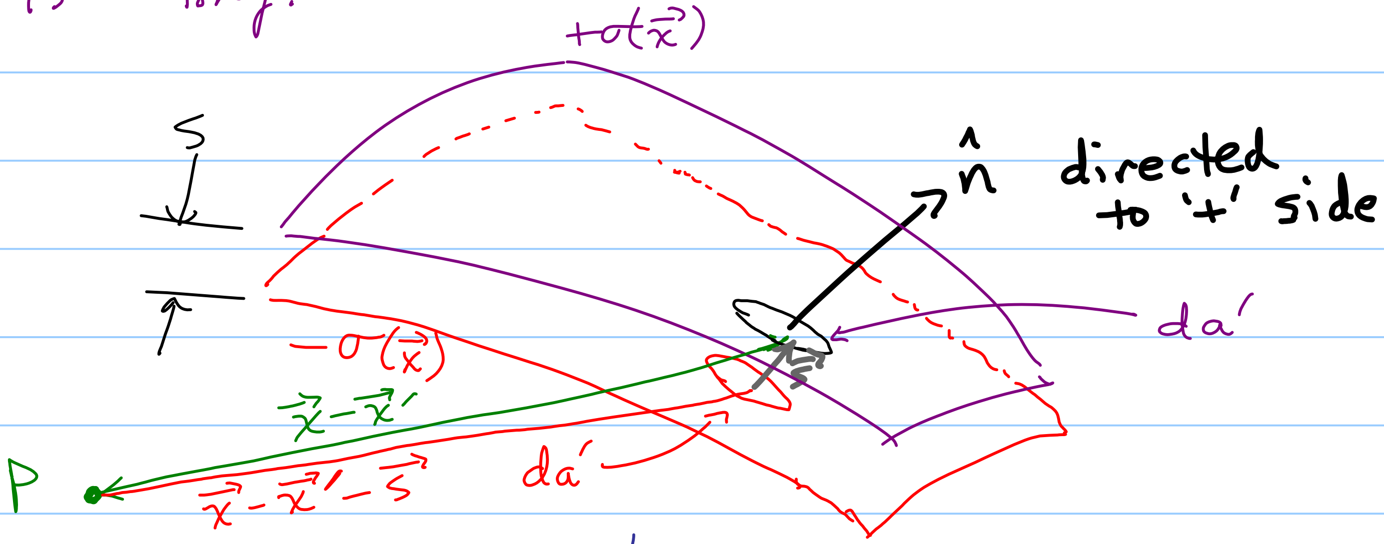
Then define $D(\vec{x}) = \sigma(\vec{x})\, s(\vec{x})$

Alternatively one can view a dipole layer as some $\frac{\#\text{ dipole molecules}}{\text{unit area}} \equiv n(\vec{x})$

each molecule having dipole moment $\vec{d} = e\vec{s}$

$$\Rightarrow \quad D(\vec{x}) = n(\vec{x})\,\vec{d} = n(\vec{x})\,e\vec{s}$$

Assume that the charge separation distance $\vec{s}$ is tiny.



Now, for a single layer of charge σ, the potential is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|}$$

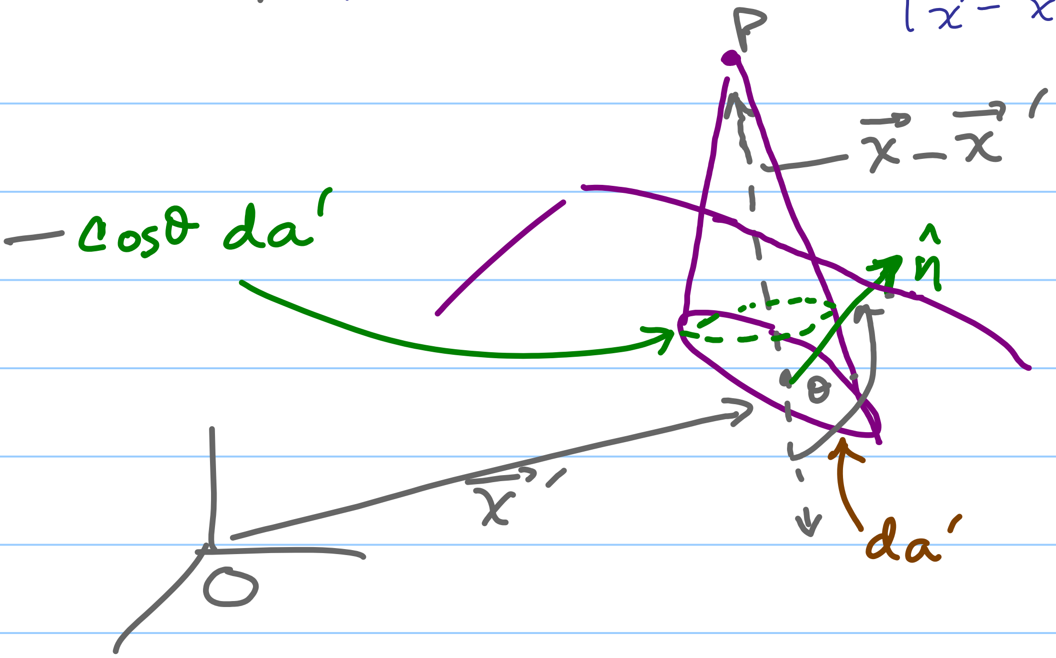
So for our dipole layer we have potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\int_{S'} \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|} - \int \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'+\vec{s}|}\right]$$

Now Taylor expand for small $\vec{s} \doteq \hat{n}s$

$$\frac{1}{|\vec{x}-\vec{x}'+\vec{s}|} \simeq \frac{1}{|\vec{x}-\vec{x}'|} + s\,\hat{n}\cdot\nabla\frac{1}{|\vec{x}-\vec{x}'|} + O(s^2)$$

$$\Rightarrow \Phi(\vec{x}) = \cancel{-}\frac{1}{4\pi\epsilon_0}\int \sigma(\vec{x}')\, s\, \hat{n}_0 \cdot \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\, da' \,\cancel{(-1)}$$



This can be re-expressed as $d\Omega'$, the solid angle subtended by da' from P, i.e.

$$-d\Omega' = -\frac{(\vec{x}'-\vec{x})\cdot\hat{n}\, da'}{|\vec{x}-\vec{x}'|^3} =$$

positive $(-d\Omega')$ is for $\vec{x}$ above plane

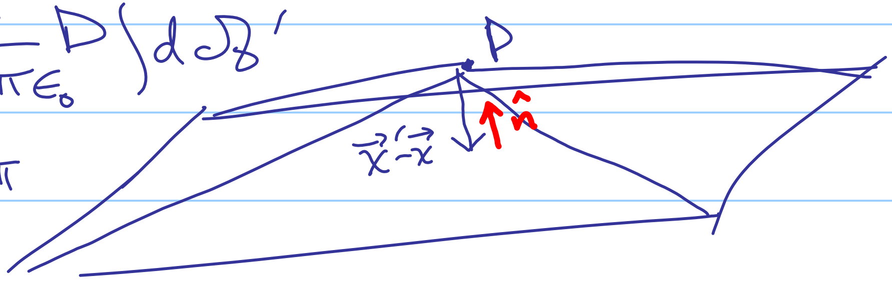
giving

$$\boxed{\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int D(\vec{x}')\, d\Omega'}$$

e.g. for a point just above a large dipolar plane, constant $D(\vec{x}) = D$,

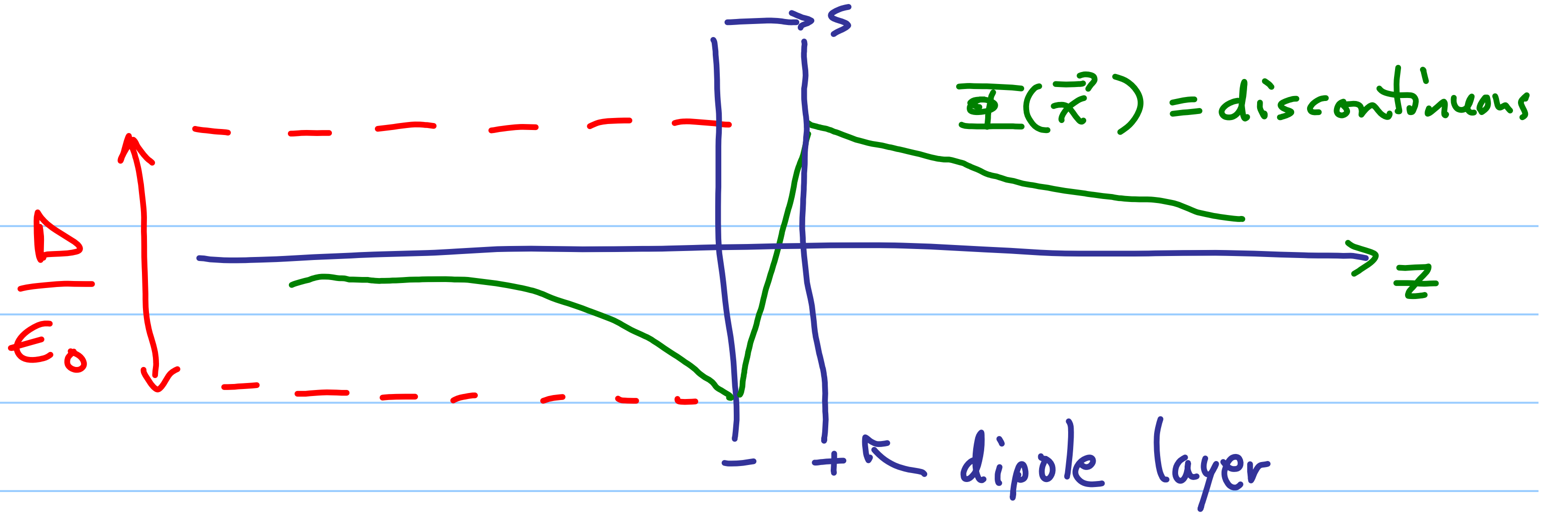
$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0} D \int d\Omega'$$

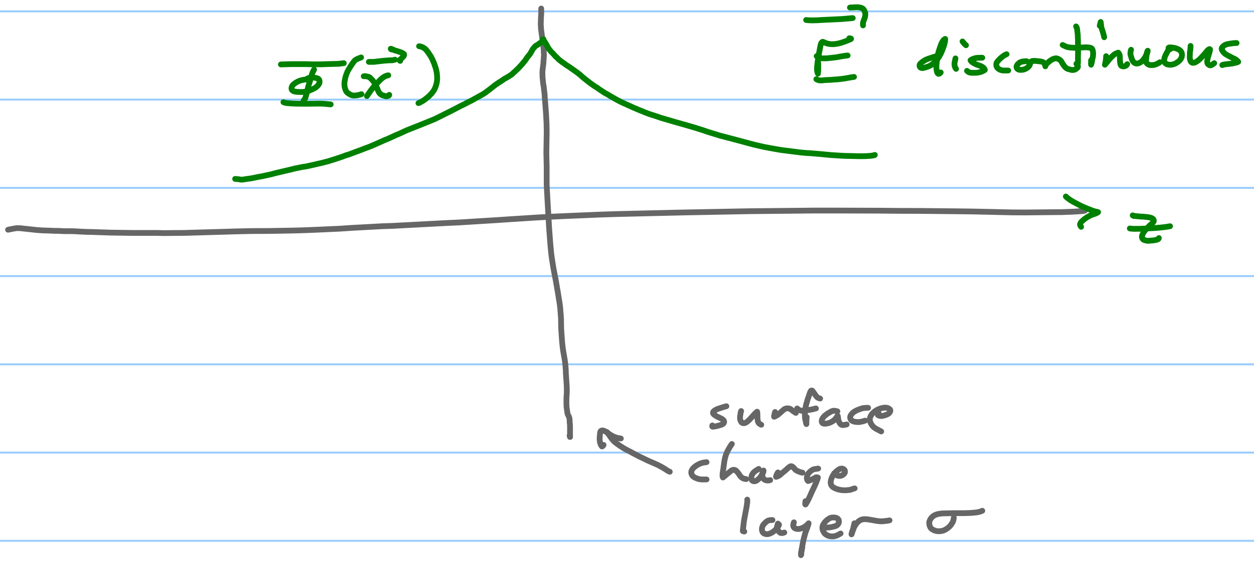
and $\int d\Omega' \to -2\pi$



so $\Phi^{\text{above}}(\vec{x}) \longrightarrow \frac{D}{2\epsilon_0}$, $\Phi^{\text{below}}(\vec{x}) \longrightarrow -\frac{D}{2\epsilon_0}$

and in general $\boxed{\Phi^{\text{above}}(\vec{x}) - \Phi^{\text{below}}(\vec{x}) = \frac{D}{\epsilon_0}}$





Poisson's and Laplace's equations

Again, a consequence of $\nabla\times\vec{E}=0$ and $\nabla\cdot\vec{E}=\rho/\epsilon_0$ is that $\vec{E}=-\nabla\Phi$ where Φ obeys Poisson's eqn,

$$\boxed{\nabla^2\Phi=-\rho/\epsilon_0}$$

A key special case is this equation in free space, with $\rho=0$, where

$$\boxed{\nabla^2\Phi=0}$$ = Laplace's equation

We will spend much time this year solving these kinds of equations.

An important check

Does $\Phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \frac{d^3x'}{|\vec{x}-\vec{x}'|}$

truly obey $\nabla^2 \Phi = -\rho/\epsilon_0$?

We can test this by taking derivatives, etc., everywhere except near the singularity at $\vec{x}=\vec{x}'$! Extra caution is always needed at singularities.

e.g., write $\nabla^2 \Phi = \left\{ \int_{V_1} + \int_{V_2} \right\} \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} d^3x'$

where V_2 = a tiny sphere of radius ϵ about $\vec{x}'=\vec{x}$

V_1 = everywhere else, outside of V_2,

where $\vec{x}' \neq \vec{x}$

So within V_1 we just differentiate normally

$$\Rightarrow \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -\nabla \cdot \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \right)$$

and with a variable change to $\vec{r} = \vec{x} - \vec{x}'$

$$\Rightarrow \nabla_x = \nabla_r \Rightarrow -\nabla_r \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3} \nabla_r \cdot \vec{r} + \vec{r} \cdot \nabla \frac{1}{r^3}$$

$$= \frac{3}{r^3} - 3\frac{\vec{r}\cdot\vec{r}}{r^5} = 0 \text{ if } \vec{r} \neq 0 \text{, true in } V_1$$

hence $\int_{V_1} d^3x' \, \rho(\vec{x}') \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = 0$!

Next, notice that $\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = \nabla'^2 \frac{1}{|\vec{x}-\vec{x}'|}$ ← prove to yourself!

and moreover, V_2 is so small that

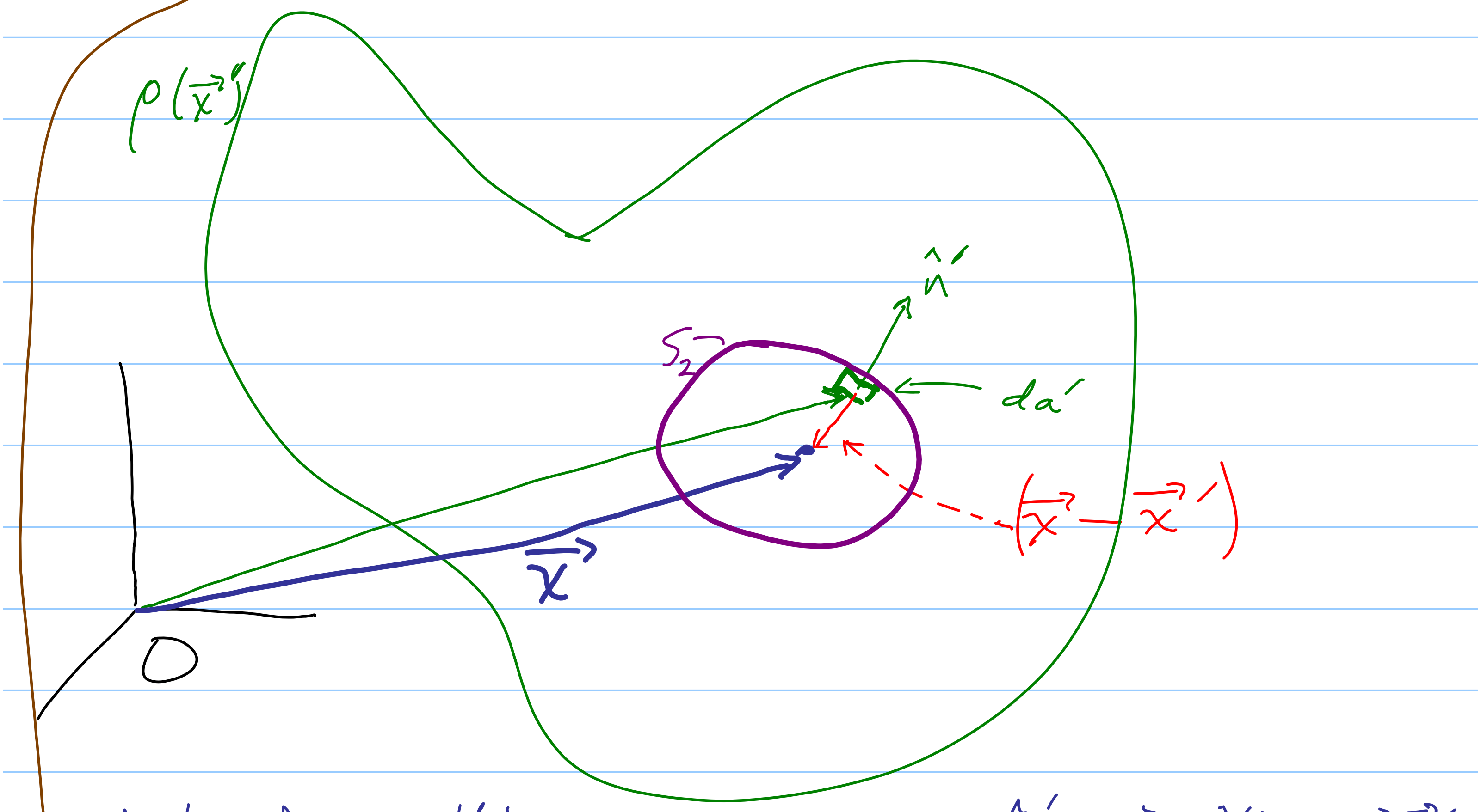
$\rho(\vec{x}\,') \simeq \rho(\vec{x})$ inside V_2,

i.e. approx. constant w.r.t. $\vec{x}\,'$

Thus $$\int_{V_2} \frac{\rho(\vec{x}\,')}{4\pi\epsilon_0} \nabla^2 \frac{1}{|\vec{x}-\vec{x}\,'|}\, d^3x'$$

$$\longrightarrow \frac{\rho(\vec{x})}{4\pi\epsilon_0} \int_{V_2} \nabla' \cdot \frac{(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\, d^3x'$$

$$= \frac{\rho(\vec{x})}{4\pi\epsilon_0} \oint_{S_2} \frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3} \cdot \hat{n}'\, da' \quad \left(\begin{array}{c}\text{divergence}\\ \text{theorem}\end{array}\right)$$



Note from this diagram that $\hat{n}' \cdot (\vec{x}-\vec{x}\,') = -|\vec{x}-\vec{x}\,'|$

$$= -\frac{\rho(\vec{x})}{4\pi\epsilon_0} \oint_{S_2} d\Omega' = -\frac{\rho(\vec{x})}{\epsilon_0}$$

Conclusions: We have thus proved that

$$\boxed{\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}}$$ as was desired

Moreover, we have demonstrated that $$\boxed{\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta(\vec{x}-\vec{x}')}$$

GREEN'S THEOREM

In some problems we don't know ρ or σ everywhere, but we can still determine Φ, $\vec{E}$. e.g. a conductor has $\vec{E}=0$ inside, whereby $\Phi(\vec{x})$ = constant everywhere on + through it. In such a problem, we might know Φ on the conductor, but not ρ.

Green's function methods are particularly useful for such problems.

I. 1st Green Identity

Consider two scalar functions ϕ, ψ. They obey the following identity

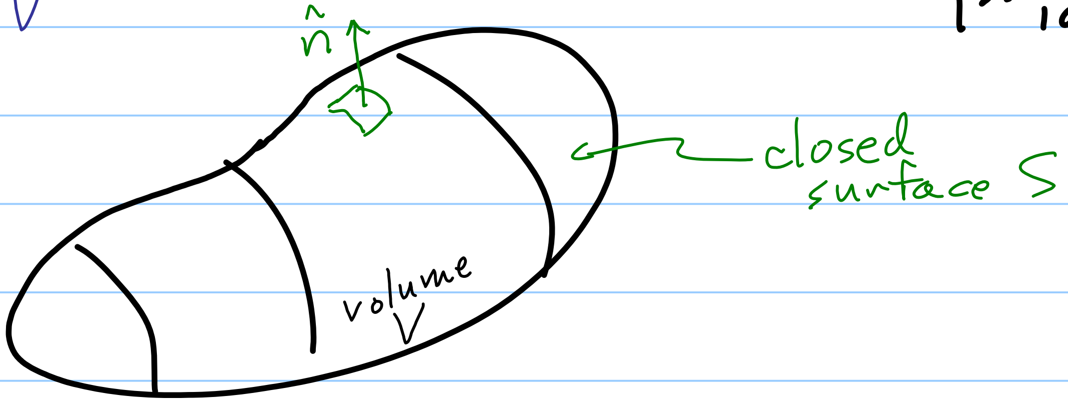
$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$$

Next, integrate this over an arbitrary volume V bounded by a closed surface S:

$$\int_V \nabla\cdot(\phi\nabla\psi)\,d^3x \overset{\text{divergence theorem}}{=} \oint_S \phi\nabla\psi\cdot\hat{n}\,da \equiv \oint_S \phi\,\overset{\text{outward normal partial derivative}}{\frac{\partial\psi}{\partial n}}\,da$$

$$= \int_V (\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi)\,d^3x$$

Green's 1st identity



II. Green's Theorem (2nd identity)

To derive it, just rewrite the 1st identity with ϕ, ψ interchanged, and subtract:

i.e. $$\left(\begin{array}{l} \int_V (\cancel{\nabla\phi\cdot\nabla\psi} + \phi\nabla^2\psi)\,d^3x \\ -\int_V (\cancel{\nabla\psi\cdot\nabla\phi} + \psi\nabla^2\phi)\,d^3x \end{array}\right) = \left(\begin{array}{l} \oint_S \phi\frac{\partial\psi}{\partial n}\,da \\ -\oint \psi\frac{\partial\phi}{\partial n}\,da \end{array}\right)$$

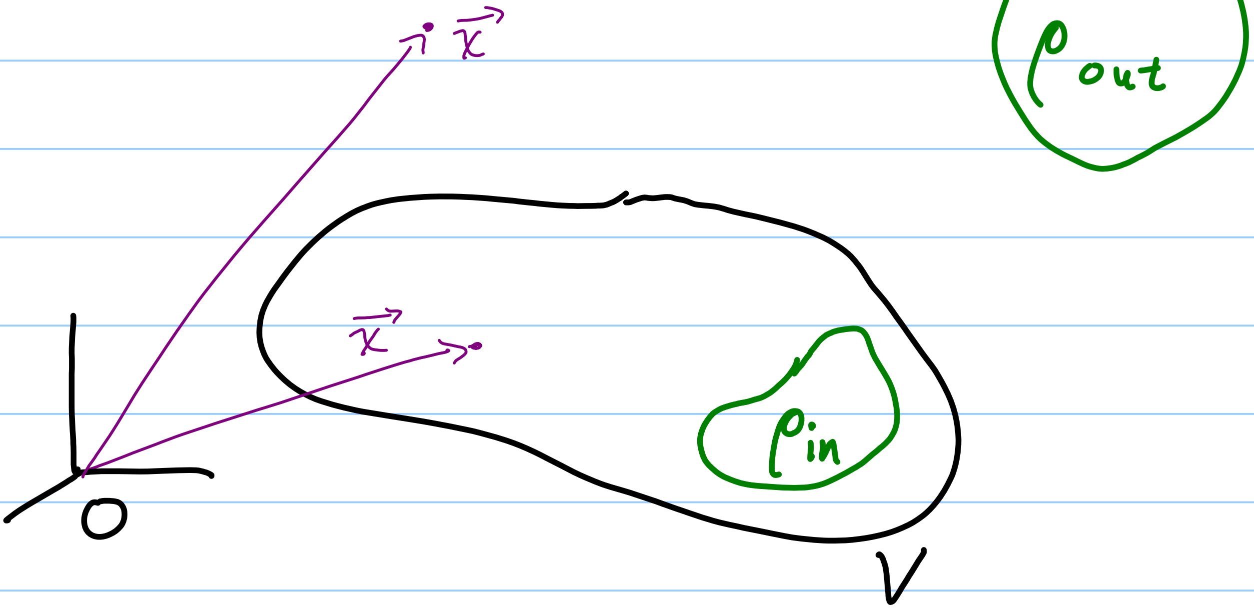
$$\Rightarrow \boxed{\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da}$$

Kewl!

Integral Equation Next we convert the Poisson PDE into an integral equation, starting from Poisson's eqn for $\Phi(\vec{x})$, written in terms of $\vec{x}'$ instead of $\vec{x}$:

$$\nabla'^2 \Phi(\vec{x}') = -\frac{\rho(\vec{x}')}{\epsilon_0}$$

We consider a volume V that contains a part ρ_{in} of this $\rho(\vec{x})$, while another part ρ_{out} is outside, i.e.



We will consider both cases, where the observation point is either inside or outside V.

Derivation In Green's Theorem let $\phi \to \Phi$, $\psi \to \frac{1}{|\vec{x}-\vec{x}'|}$

and let the integration variable be labelled as $\vec{x}'$

Also, use $\nabla'^2 \psi = -4\pi\delta(\vec{x}-\vec{x}')$

Then

$$\int_V \Big[\Phi(\vec{x}')\overbrace{\nabla'^2 \frac{1}{|\vec{x}-\vec{x}'|}}^{-4\pi\delta(\vec{x}-\vec{x}')} - \frac{1}{|\vec{x}-\vec{x}'|}\overbrace{\nabla'^2\Phi(\vec{x}')}^{-\frac{\rho(\vec{x}')}{\epsilon_0}}\Big] d^3x'$$

$$= \oint_S \Big[\Phi(\vec{x}')\underbrace{\frac{\partial}{\partial n'}\frac{1}{|\vec{x}-\vec{x}'|}}_{\hat{n}'\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}} - \frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial}{\partial n'}\Phi(\vec{x}')\Big]da'$$

To simplify the notation, define $R \equiv |\vec{x}-\vec{x}'|$.
Then if the point $\vec{x}$ lies within V, we have

$$-4\pi\int_V \Phi(\vec{x}')\delta(\vec{x}-\vec{x}')d^3x' = -4\pi\,\Phi(\vec{x})$$

$$= -\frac{1}{\epsilon_0}\int_V \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} + \oint_S\Big[\Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{R} - \frac{1}{R}\frac{\partial\Phi(\vec{x}')}{\partial n'}\Big]da'$$

OR

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi}\oint_S\Big[\frac{1}{R}\frac{\partial}{\partial n'}\Phi(\vec{x}') - \Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{R}\Big]da'}$$

notice that this integral only includes the charges INSIDE V!

observation If there are NO charges INSIDE V then $\Phi(\vec{x})$ is determined ENTIRELY by the values of $\Phi, \frac{\partial\Phi}{\partial n'}$ on S.

However this overdetermines the solution of the problem, since these are then Cauchy Boundary Conditions

(we discuss this issue later)

Next, what if $\vec{x}$ lies outside of the volume V?

Of course the δ-function integral now vanishes, so we get a different identity, namely

$$0 = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{R}\right]da',$$

which tells us nothing about $\Phi(\vec{x})$ outside!

Apparent error in Jackson, p.37, end of top paragraph, where he writes that this result gives:

"... zero field and zero potential outside the volume V."

Variational Formulation

Consider the following functional $I[\Phi]$,

$$I[\Phi] \equiv \frac{1}{2}\int_V \nabla\Phi\cdot\nabla\Phi\,d^3x - \int_V \frac{\rho}{\epsilon_0}\Phi\,d^3x$$

⇒ We can prove that this expression is in fact a variational principle that can be used to determine Φ. Consider this expression with

$$\Phi \longrightarrow \Phi + \delta\Phi \quad \leftarrow \text{a "small deviation"}$$

the exact solution

We can find δI by taking the 1st variation, i.e $\quad I[\Phi+\delta\Phi] \longrightarrow I[\Phi] + \delta I$

where $\quad \delta I = \int_V \nabla\Phi\cdot\nabla(\delta\Phi)\, d^3x - \int_V \frac{\rho}{\epsilon_0}\,\delta\Phi\, d^3x$

application of Green's first identity gives

$$\delta I = \int_V \underbrace{\left(-\nabla^2\Phi - \frac{\rho}{\epsilon_0}\right)}_{0}\delta\Phi\, d^3x + \oint_S \frac{\partial\Phi}{\partial n}\,\delta\Phi\, da$$

since Φ = exact solution

Hence $\delta I = 0$, i.e. the 1^{st} variation vanishes PROVIDED we only allow variations $\delta\Phi = 0$, that vanish on S

This variational principle can be used by choosing a trial potential $\Phi_t[a,b,c,\ldots]$ that depends on some set of parameters $\{a,b,c,\ldots\}$; and then choosing those parameters that give an extremum for I.

Uniqueness - What is the minimum amount of information needed to uniquely specify Φ within a volume V.

We saw already that

$$\Phi(\vec{x}) = \int_V \frac{\rho(\vec{x}')\, d^3x'}{4\pi\epsilon_0|\vec{x}-\vec{x}'|} + \frac{1}{4\pi}\oint_S \left[\Phi(\vec{x}')\frac{\partial}{\partial n'}\frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}'|}\frac{\partial}{\partial n'}\Phi(\vec{x}')\right] da'$$

But starting from BOTH Φ AND $\frac{\partial \Phi}{\partial n}$ on S overspecifies the solution,

$\Rightarrow$ it can lead to internal inconsistency in the solution

~~Ins~~tead, we consider 2 types of boundary conditions:

EITHER:

1) specify $\Phi(\vec{x})$ on S $\Leftarrow$ "Dirichlet BCs"

(if $\Phi(\vec{x}) = 0$ on S, call them homogeneous-Dirichlet BCs)

2) Specify $\frac{\partial \Phi}{\partial n}$ on S $\Leftarrow$ "Neumann BCs"

(if $\frac{\partial \Phi}{\partial n} = 0$ on S $\Rightarrow$ homogeneous Neumann BCs)

THEOREM knowledge of ρ inside of V, and either Dirichlet OR Neumann BCs, is enough to specify Φ in V uniquely.

Proof Suppose Φ_1 and Φ_2 are 2 solutions obeying:

(i) $\nabla^2 \Phi_i = -\rho/\epsilon_0$ in V

(ii) $\Phi_1 = \Phi_2$ on S (Dirichlet case) OR

$\frac{\partial \Phi_1}{\partial n'} = \frac{\partial \Phi_2}{\partial n'}$ on S (Neumann case)

Now define $U \equiv \Phi_1 - \Phi_2$

$\Rightarrow \nabla^2 U = 0$ in V, and $U = 0$ on S

<u>OR</u> $\frac{\partial U}{\partial n} = 0$ on S.

Start from the 1^{st} Green identity with $\phi \to U$, $\psi \to U$

$$\Rightarrow \int_V [U \cancelto{0}{\nabla^2 U} + \nabla U \cdot \nabla U] d^3x = \oint_S \cancelto{0}{U \frac{\partial U}{\partial n}} da$$

0 for either BC!

$$\Rightarrow \int_V |\nabla U|^2 d^3x = 0 \quad \text{for either B.C.}$$

But since $|\nabla U|^2 \geq 0$ everywhere, the only possibility is that $\nabla U = 0$ everywhere in V.

$\Rightarrow U = \Phi_1 - \Phi_2 =$ constant in V

meaning that Φ_1 and Φ_2 are equivalent, as the physics is unchanged by $\Phi \to \Phi +$ constant.

Moreover, since <u>either</u> Dirichlet <u>or</u> Neumann BCs gives a unique solution, one sees that arbitrarily specifying BOTH Φ AND $\left.\frac{\partial \Phi}{\partial n}\right|_S$ will not be consistent.

To learn more about this issue, e.g. that Cauchy B.C.'s are problematic in this type of PDE, see Morse + Feshbach, volume I, or Arfken + Weber,

Green's Function solution of Poisson's equation

This method is especially useful and practical for solution of the following two types of problems:

(i) Potentials are specified on a boundary S

OR

(ii) The charge distribution ρ is given in a region that is inside of a conducting boundary

The Green's function can be viewed as the electrostatic potential at an observation point $\vec{x}$ due to a unit point charge at a source point $\vec{x}'$, subject to Dirichlet or Neumann boundary conditions on the relevant surfaces.

Mathematically, the Green's function $G(\vec{x}, \vec{x}')$ obeys

$$\boxed{\nabla^2 G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')}$$

We already know a particular solution to be:

$$G(\text{particular}) = \frac{1}{|\vec{x} - \vec{x}'|}$$

since $\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi \delta(\vec{x} - \vec{x}')$

But this is not the most general solution. More generally we should write

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$$

where F obeys Laplace's equation in V, i.e.

$$\nabla^2 F(\vec{x}, \vec{x}') = 0$$

By choosing F appropriately, we can impose specific BCs on G, using tricks or more sophisticated math techniques.

One example of such a "trick" is the METHOD OF IMAGES, which determines F by placing "FAKE CHARGES" external to the volume V of interest.

Question How can we use $G(\vec{x}, \vec{x}')$ to obtain the desired unique solution for $\Phi(\vec{x})$?

Solution Choose G (i.e. specify F) to eliminate one of the two surface integrals in the general formula for $\Phi(\vec{x})$.

e.g. return to Green's theorem, setting now

$$\phi \rightarrow \Phi(\vec{x}') \text{ and } \psi \rightarrow G(\vec{x}, \vec{x}')$$

when we want to specify particular BCs on G, we use the abbreviations:

$G_D(\vec{x},\vec{x}')$ = Dirichlet Green's Function (vanishes on surface(s) S)

$G_N(\vec{x},\vec{x}')$ = Neumann BC Green's Function ($\frac{\partial G_N}{\partial n'}$ specified on S)

$$\Rightarrow \int_V \Big[\Phi(\vec{x}')\, \underbrace{\nabla'^2 G(\vec{x}',\vec{x})}_{\overset{''}{=} -4\pi\,\delta(\vec{x}-\vec{x}')} - G(\vec{x}',\vec{x})\, \underbrace{\nabla'^2 \Phi(\vec{x}')}_{\overset{''}{=} -\frac{\rho(\vec{x}')}{\epsilon_0}} \Big] d^3x'$$

$$= \oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x}',\vec{x})}{\partial n'} - G(\vec{x}',\vec{x}) \frac{\partial \Phi(\vec{x}')}{\partial n'} \right] da'$$

$$= -4\pi\,\Phi(\vec{x}) + \int_V \frac{\rho(\vec{x}')}{\epsilon_0}\, G(\vec{x}',\vec{x})\, d^3x'$$

Aside - We will see later that $G(\vec{x},\vec{x}') = G(\vec{x}',\vec{x})$

$\Rightarrow$ also $\frac{\partial G}{\partial n'}(\vec{x}',\vec{x}) = \frac{\partial}{\partial n'} G(\vec{x},\vec{x}')$

Thus we arrive at the key equation for Φ:

$$\boxed{\begin{aligned} \Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0} \int_V G(\vec{x},\vec{x}')\,\rho(\vec{x}')\, d^3x' \\ &\quad + \frac{1}{4\pi} \oint_S \left[G(\vec{x},\vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} - \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} \Phi(\vec{x}') \right] da' \end{aligned}}$$

Apply this formula to the 2 standard cases:

Dirichlet BCs – $\Phi(\vec{x})$ given on S

$\Rightarrow$ Choose $G_D(\vec{x}, \vec{x}\,') = 0$ for $\vec{x}$ on S
or for $\vec{x}\,'$ on S

$$\Rightarrow \boxed{\Phi(\vec{x}) = \int_V G_D(\vec{x}, \vec{x}\,') \frac{\rho(\vec{x}\,')}{4\pi\epsilon_0} d^3x' - \frac{1}{4\pi} \oint_S \frac{\partial G_D(\vec{x}, \vec{x}\,')}{\partial n'} \Phi(\vec{x}\,') da'}$$

For this Green's function choice, $\frac{\partial \Phi}{\partial n'}$ is irrelevant and not used!

Neumann BCs – $\frac{\partial \Phi}{\partial n'}$ is given on S

$\Rightarrow$ Specify $\frac{\partial G(\vec{x}, \vec{x}\,')}{\partial n'} = \frac{-4\pi}{S}$, a constant over S,

where $S = \oint da' =$ total area of surface S

Note If we had chosen $\left.\frac{\partial G}{\partial n'}\right|_S = 0$, this would cause inconsistencies,

since $\oint_S \frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x}\,') da' = \int_V \nabla'^2 G_N(\vec{x}, \vec{x}\,') d^3x'$

$= -4\pi$

$\Rightarrow$ our BC for G_N is consistent with this!

So in the Neumann BC problem,

$$\Phi(\vec{x}) = \int_V G_N(\vec{x}, \vec{x}\,') \frac{\rho(\vec{x}\,')}{4\pi\epsilon_0} d^3x'$$

$$+ \frac{1}{4\pi} \oint_S G_N(\vec{x}, \vec{x}\,') \frac{\partial \Phi}{\partial n'}(\vec{x}\,') da' - \frac{1}{4\pi} \oint_S \Phi(\vec{x}\,') \frac{\partial G}{\partial n'}(\vec{x}, \vec{x}\,') da'$$

and the last term here is

$$-\frac{1}{4\pi}\left(\frac{4\pi}{S}\right)\oint_S \Phi(\vec{x}\,')\,da' = \langle \Phi \rangle_S$$

= mean value of Φ on S

= a constant that is usually dropped

whereby we obtain

$$\boxed{\begin{aligned}\Phi(\vec{x}) = {} & \int_V G_N(\vec{x},\vec{x}\,')\frac{\rho(\vec{x}\,')}{4\pi\epsilon_0}\,d^3x' \\ & + \frac{1}{4\pi}\oint_S G_N(\vec{x},\vec{x}\,')\frac{\partial\Phi(\vec{x}\,')}{\partial n'}\,da'\end{aligned}}$$

A third possible choice of Green's function, not considered by Jackson but sometimes useful, (instead of Dirichlet or Neumann BCs), is to simply apply the same BCs to G that Φ must obey on all boundaries S. For this choice, all surface integrals vanish and we obtain simply

$$\boxed{\Phi(\vec{x}) = \int_V G(\vec{x},\vec{x}\,')\frac{\rho(\vec{x}\,')}{4\pi\epsilon_0}\,d^3x'}$$

GF symmetry (1) can show that $G_D(\vec{x},\vec{x}\,') = G_D(\vec{x}\,',\vec{x})$ is automatically true, => complete symmetry under interchange of source and observation points

(2) However, $G_N(\vec{x}, \vec{x}')$ need not automatically be equal to $G_N(\vec{x}', \vec{x})$, but one can demand this symmetry as a separate requirement, without loss of generality.

We will see examples of using these formulas in Chapters 2,3.

Electrostatic Potential Energy

** Reading Assignment: Read through Jackson Sec. 2.6

The energy needed to take N charges,

$$q_1, q_2 \ldots q_N$$

from infinity, and assemble them at respective positions $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_N$, is called the POTENTIAL ENERGY of the system, U.

⇒ This energy can be calculated by bringing in each charge, one-by-one:

e.g. $U_1 = 0$ $\qquad U_2 = q_2 \left(\dfrac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_2|} \right)$

$$U_3 = q_3 \left(\frac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_3|} + \frac{q_2}{4\pi\epsilon_0 |\vec{x}_2 - \vec{x}_3|} \right)$$

$$\vdots$$

$$U_N = q_N \left(\frac{q_1}{4\pi\epsilon_0 |\vec{x}_1 - \vec{x}_N|} + \cdots + \frac{q_{N-1}}{4\pi\epsilon_0 |\vec{x}_{N-1} - \vec{x}_N|} \right)$$

and

$$U = \sum_{i=1}^{N} U_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\vec{x}_i - \vec{x}_j|}$$

This result can be rewritten as

$$\boxed{U = \frac{1}{8\pi\epsilon_0} \sum_{\underbrace{i \neq j}} \frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \qquad (1)}$$

implies that "self-energy" terms are omitted w/ i=j

The most logical generalization of this expression (1) to a continuous distribution of charge $\rho(\vec{x})$ is evidently

$$U = \frac{1}{8\pi\epsilon_0}\int \frac{\rho(\vec{x})\ \rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\, d^3x\, d^3x'$$

or alternatively, using $\Phi(\vec{x}) = \int \frac{\rho(\vec{x}\,')\, d^3x'}{4\pi\epsilon_0|\vec{x}-\vec{x}\,'|}$

$$\Rightarrow U = \frac{1}{2}\int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x$$

Yet another form is $U = -\frac{\epsilon_0}{2}\int_{\text{all space}} \Phi(\vec{x})\nabla^2\Phi(\vec{x})\, d^3x$

and after applying Green's 1st identity with all surface terms at ∞ neglected,

$$\Rightarrow U = \frac{\epsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} |\nabla u|^2 d^3x$$

or finally $\boxed{U = \frac{\epsilon_0}{2}\int_{\substack{\text{all}\\ \text{space}}} |\vec{E}|^2 d^3x}$ (2)

This result is usually interpreted as implying that the $\vec{E}$-field at any point $\vec{x}$ in space stores an energy density equal to $u(\vec{x}) = \frac{\epsilon_0}{2}|\vec{E}|^2$ (i.e. J/m³)

Observe: We have an apparent inconsistency in our derivation!

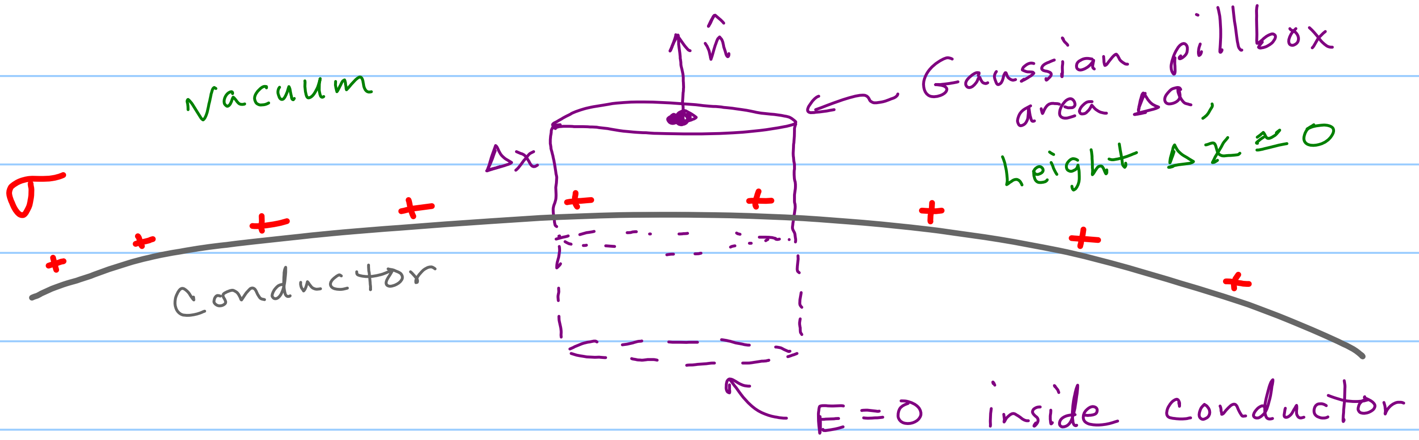
Namely, U in Eq. (1) can be positive OR negative, but U in Eq. (2) is positive definite!

Resolution This is because Eq. 2 includes the "self-energy" terms, i.e. when $\vec{x} = \vec{x}'$, whereas Eq. (1) excludes them!

Electrostatic Pressure

Consider the situation where charge density σ is on the surface of a conductor.

$\Rightarrow$ Electrostatic forces cause an outward pressure, as we now explore in detail:



Gauss's Law

$$\Rightarrow \oint \vec{E} \cdot \vec{da} = E\,\Delta a = \frac{\sigma\,\Delta a}{\epsilon_0}$$

$\Rightarrow \vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$ $\Rightarrow$ at the surface there is an energy density

$$u(\vec{x}) = \frac{\epsilon_0}{2}\frac{\sigma^2}{\epsilon_0^2} = \frac{\text{energy}}{\text{unit volume}}$$

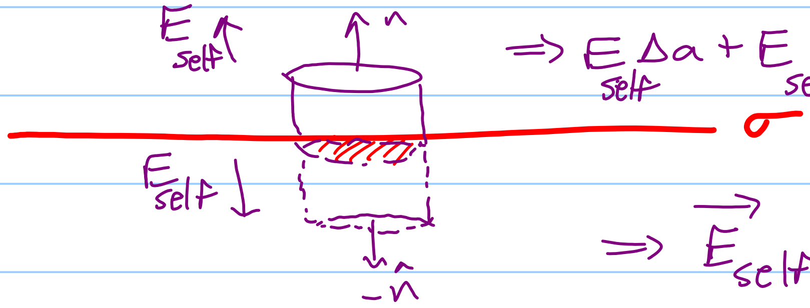
Suppose we now move a differential surface area element outward, in the direction $\hat{n}$, by a small distance Δx

$\Rightarrow$ the E-field does work

$$\Delta W = F_{other}\,\Delta x = (\sigma\,\Delta a)\,E_{other}\,\Delta x$$

In order to deduce what "E_{other}" is for this problem, consider what the electric field produced by this σ would be if this thin layer of charge were <u>not</u> on a conductor

$\Rightarrow$ Then



$$\Rightarrow E_{self}\,\Delta a + E_{self}\,\Delta a = \frac{\sigma}{\epsilon_0}\,\Delta a$$

$$\Rightarrow \vec{E}_{self} = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\,\hat{n}, & \text{above} \\ -\dfrac{\sigma}{2\epsilon_0}\,\hat{n}, & \text{below} \end{cases}$$

The <u>other</u> charges in the universe must therefore produce an electric field $\vec{E}_{other} = \frac{\sigma}{2\epsilon_0}\hat{n}$, which cancels $\vec{E}_{self}$ below the surface (inside the conductor), and doubles the total field outside. We are interested only in the force due to those "other" charges, since charge cannot exert a net force on itself.

$\Rightarrow \vec{F}_{other} = (\sigma\,\Delta a)\,\frac{\sigma}{2\epsilon_0}\,\hat{n}$, or the $\frac{\text{force}}{\text{unit area}}$ = PRESSURE

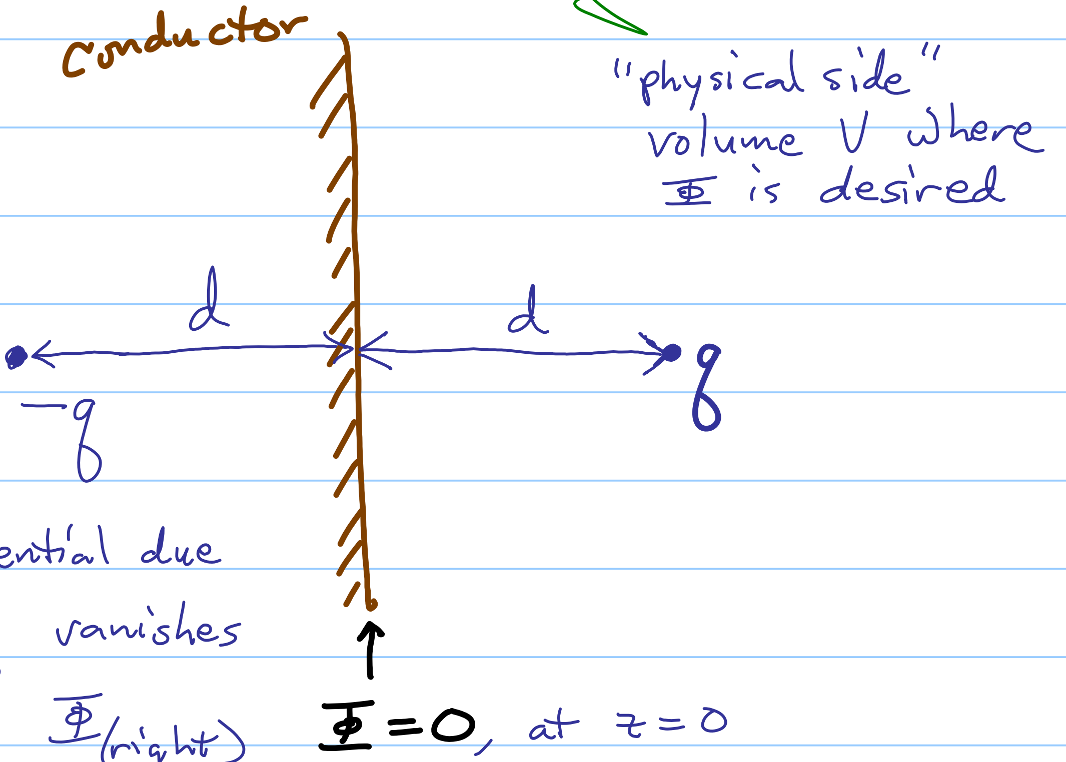
is $$\boxed{\frac{\Delta \vec{F}}{\Delta a} = \frac{\sigma^2}{2\epsilon_0}\,\hat{n}}$$

Chapter 2 Boundary value problems in electrostatics, part I

Sec. 2.1 For problems with charges near (a) boundary surface(s), we may use appropriately-placed "image charges" outside the volume V of interest, to simulate the boundary conditions.

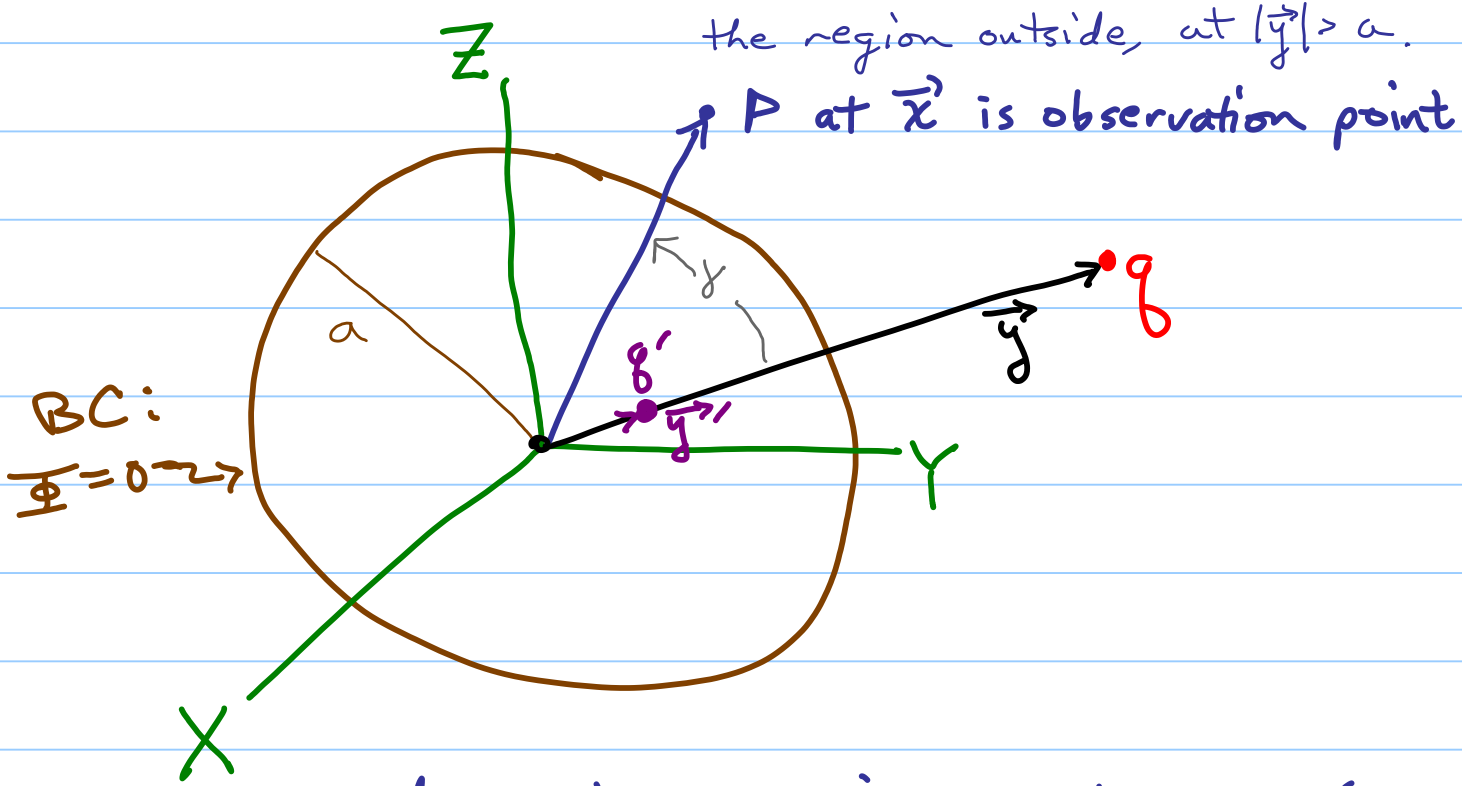
Note: a charge placed outside of V will obey Laplace's equation inside V, since $\nabla^2 \frac{1}{|\vec{x}-\vec{x}_0|} = 0$ except near $\vec{x} \simeq \vec{x}_0$

e.g. the grounded, conducting plane



Clearly the potential due to q and $-q$ vanishes at $z=0$, so $\Phi_{(\text{right})}$ can be calculated using q and $(-q)$ as point charges, only!

Sec. 2.2 Consider a point charge q at position $\vec{y}$ outside a grounded, conducting sphere of radius a. The volume V of interest is the region outside, at $|\vec{y}| > a$.



Idea: place an image charge q' INSIDE the sphere at $\vec{y}'$, and choose q', $\vec{y}'$ such that $\Phi = 0$ on the sphere surface.

By symmetry, of course, $\vec{y}'$ must be along $\vec{y}$, i.e. $\vec{y}' \propto \vec{y}$, pointing towards q. So if we set $\vec{x} = x\hat{n}$, $\vec{y} = y\hat{n}'$, $\vec{y}' = y'\hat{n}'$, with $\hat{n}$ and $\hat{n}'$ unit vectors, the total potential in V is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|x\hat{n} - y\hat{n}'|} + \frac{q'}{|x\hat{n} - y'\hat{n}'|}\right)$$

Now demand that

$$4\pi\epsilon_0 \Phi(x=a) = 0 = \frac{q}{|a\hat{n} - y\hat{n}'|} + \frac{q'}{|a\hat{n} - y'\hat{n}'|}$$

or

$$0 = \frac{q}{(a^2+y^2-2ay\cos\gamma)^{1/2}} + \frac{q'}{(a^2+y'^2-2ay'\cos\gamma)^{1/2}}$$

where $\hat{n}\cdot\hat{n}' \equiv \cos\gamma$

$$\Rightarrow \quad \frac{q'}{q} = -\frac{(a^2+y'^2-2ay'\cos\gamma)^{1/2}}{(a^2+y^2-2ay\cos\gamma)^{1/2}} \quad \Rightarrow \frac{q'}{q} < 0$$

square this and collect terms

$$\Rightarrow \left[\left(\frac{q'}{q}\right)^2(a^2+y^2) - (a^2+y'^2)\right] + \cos\gamma\left[-\left(\frac{q'}{q}\right)^2(2ay) + 2ay'\right] = 0$$

Since this equation holds for ALL γ, both terms in [ ] must separately vanish, giving 2 eqns and 2 unknowns (q'/q) and y':

SOLUTION:

$$\boxed{\frac{q'}{q} = -\frac{a}{y}, \quad y' = \frac{a^2}{y}}$$

Note that $y' < y$, so $\vec{y}'$ is outside of V, a necessary condition for an image charge.

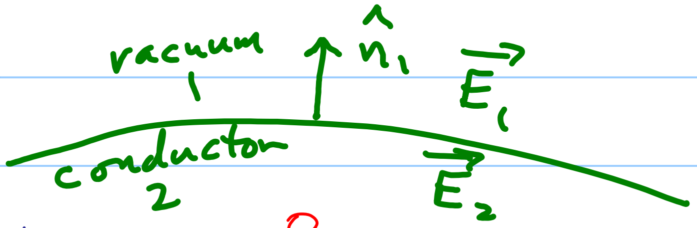
We can now determine any desired observable, e.g.

$$\Phi(x,\gamma) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{(x^2+y^2-2xy\cos\gamma)^{1/2}} - \frac{qa/y}{\left(x^2+\left(\frac{a^2}{y}\right)^2-2x\frac{a^2}{y}\cos\gamma\right)^{1/2}}\right\}$$

which is valid at all $x>a$, $y>a$.

Charge Density One quantity often of interest is the charge density on any conducting surface.

For the sphere problem



we saw previously that

$$(\vec{E}_1 - \cancelto{0}{\vec{E}_2})\cdot\hat{n}_1 = \frac{\sigma}{\epsilon_0}$$

So here,

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial r}\bigg|_{r=a^+}$$

or

$$\boxed{\sigma = -\frac{q}{4\pi a^2}\,\frac{a}{y}\,\frac{(1-a^2/y^2)}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma\right)^{3/2}}}$$

Consider the limit as $y \gg a$

$$\Rightarrow \sigma \xrightarrow[y\gg a]{} \frac{-q}{4\pi a^2}\,\frac{a}{y}\left(1+\left(-\frac{3}{2}\right)\left(-2\frac{a}{y}\right)\cos\gamma + O\left(\frac{a^2}{y^2}\right)\right)$$

$$\hookrightarrow \frac{-q}{4\pi a y} - \frac{3q}{4\pi y^2}\cos\gamma + O\left(\frac{a}{y^3}\right),$$

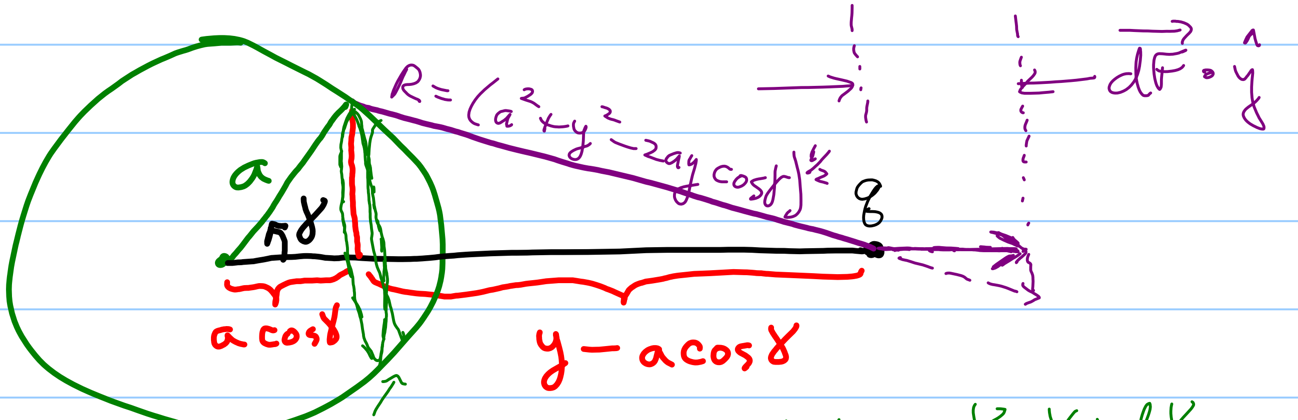
which looks approximately like a small net charge and a small electric dipole moment as $y\to\infty$.

Observe that $\oint \sigma da = q' = -q\frac{a}{y}$ at all y, which makes sense that the magnitude of charge on the surface equals the image charge magnitude!

Force on the charge q

Let's calculate directly the force on q, using the "real charges" we have just derived on the surface of the sphere. This is worth studying from different viewpoints, because with force and energy there can be factors of 2 or $\frac{1}{2}$ lurking.

method 1 integrate force from each $\sigma da'$:



da swept out between γ, $\gamma + d\gamma$

$$\text{so } \vec{dF}\cdot\hat{y} = \left(\frac{y-a\cos\gamma}{R}\right)\frac{q}{4\pi\epsilon_0 R^2}\sigma(\gamma)\,2\pi a^2\sin\gamma\, d\gamma$$

$$\equiv dF_y$$

$$\Rightarrow dF_y = \frac{2\pi a^2 q}{4\pi\epsilon_0}\underbrace{\left\{\frac{-q}{4\pi a y}\left(1-\frac{a^2}{y^2}\right)\left(1+\frac{a^2}{y^2}-\frac{2a}{y}\cos\gamma\right)^{-3/2}\right\}}_{\sigma(\gamma)}$$

$$\times \frac{\sin\gamma\, d\gamma}{(y^2+a^2-2ay\cos\gamma)^{3/2}}(y-a\cos\gamma)\cdot\frac{y^3}{(y^2)^{3/2}}$$

$$= \frac{-q^2}{8\pi\epsilon_0}\,\frac{a}{y}\left(1-\frac{a^2}{y^2}\right)y^3\,\frac{\sin\gamma\,(y-a\cos\gamma)}{(a^2+y^2-2ay\cos\gamma)^3}$$

Now integrate this from $\gamma=0$ to π, giving

$$\vec{F} = \frac{-q^2}{8\pi\epsilon_0}\,a\,(y^2-a^2)\left[\frac{2y}{(y^2-a^2)^3}\right]\hat{y}$$

Mathematica

or finally $\vec{F} = -\hat{y}\,\frac{q^2}{4\pi\epsilon_0}\,\frac{ay}{(y^2-a^2)^2}$

method 2 interestingly, the above result agrees exactly with the trivial method of just computing the force on q due to the point change q' at y'!

$\Rightarrow$ much simpler to compute that way!

method 3 Use Newton's 3rd law, compute the force on the sphere due to q, then multiply by (-1) to get the force on q. This gives the same answer also!

Section 2.3 Next treat a point charge q outside a charged, insulated, conducting sphere carrying total charge Q

Concept First solve for the potential in the case where q is at distance $y > a$ from a grounded sphere.

⇒ We saw previously that this induces a total charge $q' = -q\frac{a}{y}$ on the sphere, in order for the sphere to be an equipotential.

⇒ Start from that solution and add a small additional amount of charge.

CLAIM: The additional charge must spread over the sphere surface symmetrically, because the sphere is an equipotential. If we keep adding charge this way until the total amount of charge is Q, this means we must add a total amount of extra charge equal to

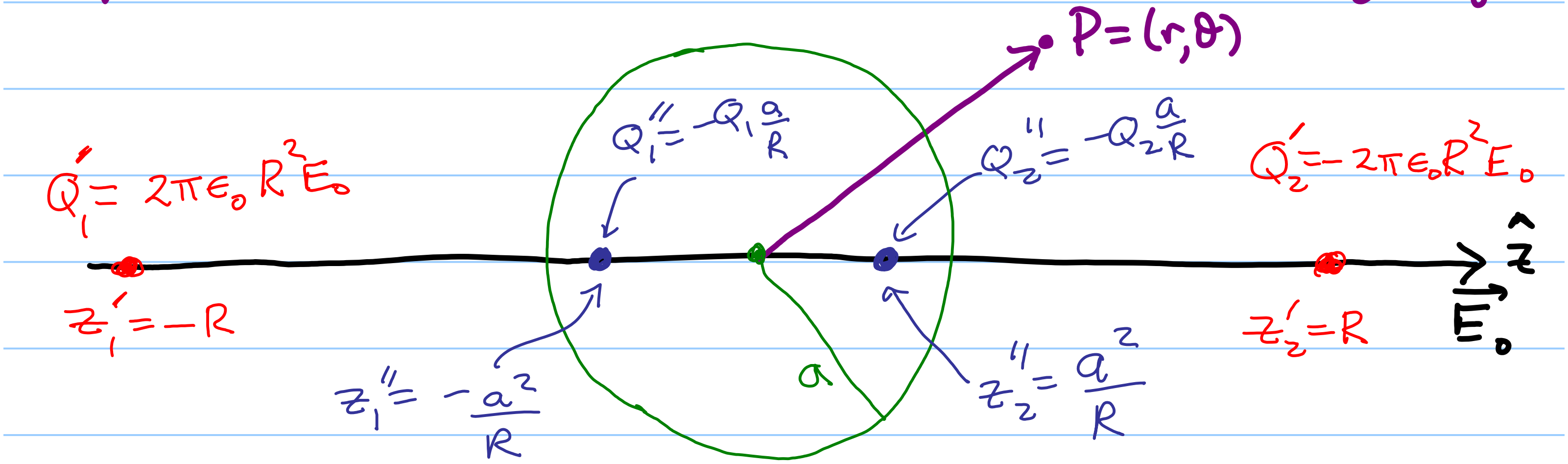
$$Q_{NEW} = Q - q' = Q + \frac{a}{y}q$$

We also know that the potential outside a spherically-symmetric distribution of charge Q_{NEW} is the same as if it were all at the center!

Therefore the total potential outside is equal to

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{y}\frac{1}{|\vec{x}-\frac{a^2}{y^2}\vec{y}|} + \frac{Q+\frac{a}{y}q}{|\vec{x}|} \right\}}$$

Section 2.5 Consider a grounded conducting sphere in a uniform external E-field, $\vec{E}_0 = E_0\hat{z}$



We solve this as a limiting case, with two charges far away $(R\to\infty)$ that set up the uniform electric field. The solution is again carried out using image charges.

$\Rightarrow$ The potential at all points outside the sphere is then

$$\Phi(r,\theta) = \lim_{R\to\infty} \frac{1}{4\pi\epsilon_0}\frac{Q_1'}{[R^2+r^2-2rR\cos(\pi-\theta)]^{1/2}}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{(-aQ_1'/R)}{[(a^2/R)^2+r^2-2r(a^2/R)\cos(\pi-\theta)]^{1/2}} + \frac{1}{4\pi\epsilon_0}\frac{Q_2'}{[R^2+r^2-2rR\cos\theta]^{1/2}}$$

$$+\frac{1}{4\pi\epsilon_0}\frac{(-aQ_2'/R)}{[(a^2/R)^2+r^2-2r(a^2/R)\cos\theta]^{1/2}}$$

It is easiest to find the large $R \to \infty$ limit if we first make a binomial expansion

$$\text{i.e.} \quad (1+\epsilon)^{\gamma} = 1 + \gamma\epsilon + \frac{\gamma(\gamma-1)}{2!}\epsilon^2 + \ldots$$

... skipping a few lines of algebra ...

$$\Rightarrow \Phi(r,\theta) = \lim_{R\to\infty} \frac{E_0 R}{2}\left[1 - \frac{r}{R}\cos\theta - 1 - \frac{r}{R}\cos\theta\right]$$

$$- \frac{E_0 R}{2}\frac{a}{r}\left[1 - \frac{a^2}{rR}\cos\theta - 1 - \frac{a^2}{rR}\cos\theta\right]$$

$$+ \text{ terms that vanish at } R \to \infty$$

$$\text{or} \quad \boxed{\Phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta}$$

Aside: observing that this Φ vanishes at $\theta = \frac{\pi}{2}$, we notice that we have solved another problem too, namely a grounded sphere + a grounded conducting plane at $z=0$.

Induced surface charge on the sphere:

$$\sigma(a,\theta) = -\epsilon_0 \frac{\partial\Phi}{\partial r}\bigg|_{r=a} = \epsilon_0 E_0\cos\theta + \epsilon_0 E_0 \frac{2a^3}{a}\cos\theta$$

$$\text{or} \quad \boxed{\sigma(a,\theta) = 3\epsilon_0 E_0 \cos\theta}$$

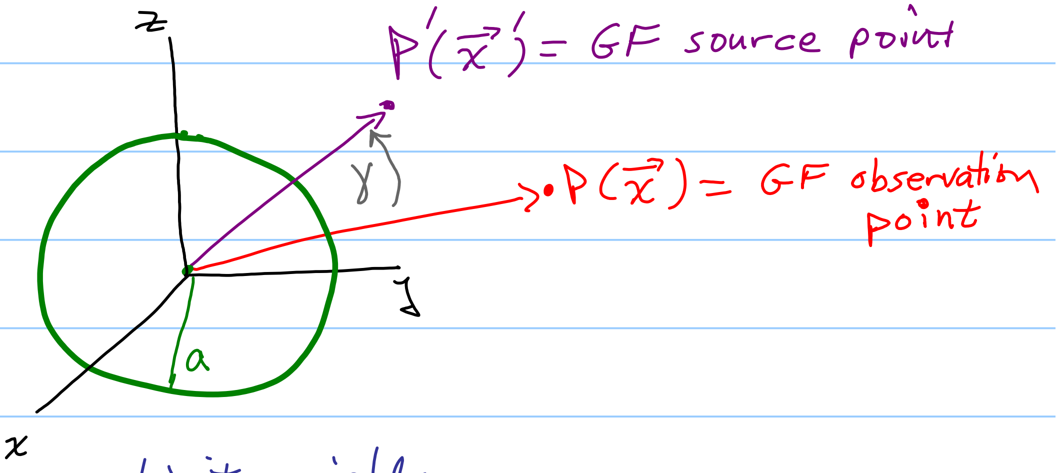
and since the integral vanishes, this means there is NO NET CHARGE on the sphere.

Dirichlet Green's Function for a sphere

Using the Green's function, we could solve many different problems. Recall that the Dirichlet GF for the exterior of a sphere must vanish at $r=a$, and it must obey $\nabla^2 G(\vec{x},\vec{x}\,') = -4\pi\delta(\vec{x}-\vec{x}\,')$ where $\vec{x}\,', \vec{x}$ are arbitrary positions outside the sphere, i.e. $G(|\vec{x}|=a, \vec{x}\,') = 0$.

This GF can be found using the method of images now, simply by inspection:

$$G(\vec{x},\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} - \frac{a}{x'\left|\vec{x}-\frac{a^2}{x'^2}\vec{x}\,'\right|}$$



Making this more explicit yields

$$\boxed{G(\vec{x},\vec{x}\,') = (x^2+x'^2-2xx'\cos\gamma)^{-1/2} - \left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{-1/2}}$$

To solve a general Dirichlet problem we will sometimes need the normal derivatives of G, i.e.

$$\boxed{\begin{aligned}\left.\frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\right|_{x'=a} &= -\left.\frac{\partial G(\vec{x},\vec{x}')}{\partial x'}\right|_{x'=a}\\ &= \frac{a^2-x^2}{a(a^2+x^2-2ax\cos\gamma)^{1/2}}\end{aligned}}$$

Next – A typical application: Find Φ outside a sphere whose two conducting hemispheres are held at equal + opposite potentials $\pm V$.

e.g. top is at $+V$ $(0 \le \theta \le \pi/2)$, (sphere radius a)

bottom (insulated from top) is at $-V$ $(\frac{\pi}{2} < \theta \le \pi)$

Solution Here $\rho = 0$ outside so there are only surface integrals in this Dirichlet BC problem.

$$\Rightarrow \Phi(r,\theta,\phi) = \frac{1}{4\pi}\underbrace{\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \, a^2}_{\int da'} \left.\frac{\partial G}{\partial n'}(\vec{x},\vec{x}')\right|_{x'=a} V(\theta',\phi')$$

$$= \frac{a^2 V}{4\pi}\int_0^{2\pi} d\phi' \left[\int_0^1 d(\cos\theta') - \int_{-1}^0 d(\cos\theta')\right] \times$$

$$\times \left\{\frac{a^2-x^2}{a(a^2+x^2-2ax\cos\gamma)^{3/2}}\right\}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

which has a complicated dependence on $\theta, \phi, \theta', \phi'$.

Jackson works this out for a special case, $\theta = 0 = \phi$.

Some mathematical preliminaries or review

⇒ Orthogonal Functions and expansions

- Expansions can provide an excellent method for electrostatic potential problems, and in other physics subfields, especially QM.

The simplest and usual case works with an orthonormal set of functions, say,

$$\{u_n(\xi)\}, \quad n = 1, 2, \ldots \infty$$

with the independent variable ξ having a finite range $\quad a \le \xi \le b$

with orthogonality relation written as:

$$\int_a^b u_n^*(\xi)\, u_m(\xi)\, d\xi = \delta_{nm}$$

Assume for now that the $u_n(\xi)$ are square-integrable, i.e.

$$N^2 \equiv \int_a^b |\bar{u}_n(\xi)|^2 d\xi < \infty$$

whereby one can construct normalized $u_n(\xi)$ as

$$u_n(\xi) = \frac{1}{N} \bar{u}_n(\xi)$$

Usually the set $\{u_n(\xi)\}$ is also <u>complete</u>, in which case any square-integrable function $f(\xi)$ over $a \le \xi \le b$ can be expanded as a linear combination, $f(\xi) = \sum_{n=1}^{\infty} a_n u_n(\xi)$

When $f(\xi)$ is known, the a_n can be determined by what Griffiths calls "Fourier's trick"

$$a_n = \int_a^b u_n^*(\xi) f(\xi) d\xi$$
$$= \int_a^b u_n^*(\xi') f(\xi') d\xi'$$

Plugging this last expression into the $f(\xi)$ expansion gives

$$f(\xi) = \int_a^b d\xi' \left[\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) \right] f(\xi')$$

This implies the <u>completeness</u> or <u>closure</u> relation,

$$\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) = \delta(\xi' - \xi)$$

Using notation slightly different from Jackson, here is one example of a Fourier series over a symmetric interval $-\frac{a}{2} \le x \le \frac{a}{2}$

$$U_m^e(x) = \sqrt{\frac{2-\delta_{m,0}}{a}} \cos\left(\frac{2\pi m x}{a}\right), \; m = 0, 1, 2, \ldots$$

$$U_m^o(x) = \sqrt{\frac{2}{a}} \, \sin\left(\frac{2\pi m x}{a}\right), \; m = 1, 2, 3, \ldots$$

where the combination of both the even (e) and odd (o) sets gives a complete, orthonormal set, $\{ U_m^e(x), U_m^o(x) \}$, can write for any $f(x)$,

$$\Rightarrow f(x) = \sum_{m=0}^{\infty} A_m U_m^e(x) + \sum_{m=1}^{\infty} B_m U_m^o(x)$$

where $$A_m = \left(\frac{2-\delta_{m,0}}{a}\right)^{1/2} \int_{-a/2}^{a/2} \cos\frac{2\pi m x}{a} f(x)\,dx$$

$$B_m = \left(\frac{2}{a}\right)^{1/2} \int_{-a/2}^{a/2} \sin\frac{2\pi m x}{a} f(x)\,dx$$

and these relations generalize to 2 or more dimensions in the natural way, e.g.

$$F(\xi,\eta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, u_m(\xi)\, v_n(\eta),$$

and, e.g. $$A_{mn} = \int_a^b d\xi \int_c^d d\eta\, u_m^*(\xi) v_n^*(\eta) F(\xi,\eta)$$

etc.

Generalization to an INFINITE INTERVAL:

We can rewrite the above sin, cos example in terms of exponentials, e.g.

let $$U_n(x) = a^{-1/2} e^{\frac{2i\pi n x}{a}}, \quad n=0,\pm1,\pm2,\ldots$$

$$\Rightarrow \quad F(x) = \sum_{n=-\infty}^{\infty} A_n \frac{1}{\sqrt{a}} e^{2i\pi n x/a}$$

where $$A_n = \int_{-\frac{a}{2}}^{a/2} \frac{1}{\sqrt{a}} e^{-2i\pi \frac{n x}{a}} F(x)\,dx$$

We take the $a \longrightarrow \infty$ limit, defining:

$$k \equiv \frac{2\pi n}{a}$$

$$\sum_m \longrightarrow \int dm = \frac{a}{2\pi}\int dk$$

$$A_m \longrightarrow \left(\frac{2\pi}{a}\right)^{1/2} A(k)$$

Then (a) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$

(b) $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x')\, dx'$

and

(c) $f(x) = \int_{-\infty}^{\infty} dx' f(x') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}\right]$

which again implies a completeness relation,

(d) $\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x')$

and we will need the "orthonormality relation",

(e) $\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik'x}\, e^{ikx}\, dx = \delta(k-k')$

These are the basic equations of Fourier Analysis

Solution of partial differential equations by separation of variables

IDEA when possible, there is great simplification if a PDE can be solved by the separable product form

e.g. $\Phi(x,y,z) = X(x)\, Y(y)\, Z(z)$

For any specific problem the solution may not have such a simple form, but it is still often useful to solve the PDE in this form as an initial step in the solution process.

Let's use this ansatz to solve

$$\nabla^2 \Phi(x,y,z) = 0 = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Plug in the above separable form, divide the entire equation by $X(x)Y(y)Z(z)$

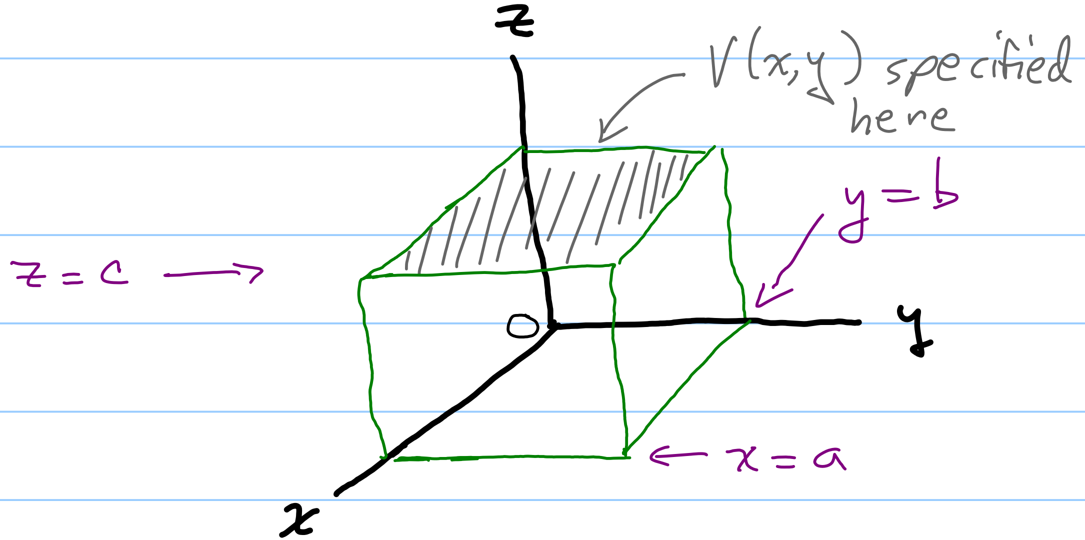
$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

and since this must hold for <u>all</u> x, y, z, each term must separately be a constant, i.e.

$$\frac{X''(x)}{X(x)} = -\alpha^2, \quad \frac{Y''(y)}{Y(y)} = -\beta^2$$

$$\frac{Z''(z)}{Z(z)} = \alpha^2 + \beta^2$$

separation constants



A useful rule of thumb for problems like this: Choose the sign of the separation constants to produce <u>oscillatory</u> solutions in any dimension having a

$\Phi = 0$ boundary condition at both limits.

e.g. suppose we are given the BCs to be

$$\Phi(0,y,z) = \Phi(a,y,z) = 0$$
$$\Phi(x,0,z) = \Phi(x,b,z) = 0$$
$$\Phi(x,y,0) = 0; \quad \Phi(x,y,c) = V(x,y)$$

$\Rightarrow$ the above choice of signs of α, β works well for this PDE. Assume initially that α, β = real, but keep open the possibility that they could later be found to be imaginary or complex.

$\Rightarrow$ $X(x)$ = linear combinations of $e^{\pm i\alpha x}$

$Y(y)$ = " " " $e^{\pm i\beta y}$

$Z(z)$ = " " $e^{\pm\sqrt{\alpha^2+\beta^2}\,z}$

By inspection the linear combinations needed in x, y are:

$$X_m(x) = \left(\frac{2}{a}\right)^{1/2} \sin\frac{m\pi x}{a}, \quad 0 \le x \le a,$$
$$\left(\text{i.e. } \alpha = \frac{m\pi}{a}, \; m = 1, 2, 3, \ldots\right)$$

and

$$Y_n(y) = \left(\frac{2}{b}\right)^{1/2} \sin\frac{n\pi y}{b}, \quad 0 \le y \le b$$
$$\left(\beta = \frac{n\pi}{b}, \; n = 1, 2, \ldots\right)$$

And the B.C. $Z(0) = 0$ implies that the only solution in z must be proportional to

$$Z_{m,n}(z) = \sinh\left[\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\; z\right]$$

for $0 \le z \le c$

Hence an infinite set of product solutions that obey all BC's in the problem $X_m(x)\,Y_n(y)\,Z_{mn}(z)$ EXCEPT at $z = c$, where we must still find a way to satisfy the B.C.:

$$\Phi(x,y,c) = V(x,y) \leftarrow \text{specified + assumed to be known}$$

Using the linearity of the PDE, we can superpose this set with coefficients A_{mn} still to be determined to match the BCs:

$$\Rightarrow \Phi(x,y,z) = \sum_{m,n=1}^{\infty} A_{mn} \left(\frac{2}{a}\right)^{1/2} \left(\frac{2}{b}\right)^{1/2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{z} \sinh(\gamma_{mn} z)$$

where $\gamma_{mn} \equiv \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\;\pi$

DEMAND $\xrightarrow[z\to c]{} V(x,y)$

Use the orthonormality of the $\left(\frac{2}{a}\right)^{1/2} \sin\frac{m\pi x}{a}$ and $\left(\frac{2}{b}\right)^{1/2} \sin\frac{n\pi y}{b}$

$$\left[\text{i.e. } \int_0^a dx\, \frac{2}{a} \sin\frac{m\pi x}{a} \sin\frac{m'\pi x}{a} = \delta_{mm'}\right]$$

Now apply "Fourier's Trick", mutiplying both sides by $\left(\frac{2}{a}\right)^{1/2}\sin\frac{m'\pi x}{a}\left(\frac{2}{b}\right)^{1/2}\sin\frac{n'\pi y}{b}$ and integrate, giving

$$\boxed{A_{mn} = \frac{\int_0^a dx \int_0^b dy \left(\frac{4}{ab}\right)^{1/2} V(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\sinh\gamma_{mn} c}, \quad \begin{array}{l} m=1,2,\ldots\infty \\ n=1,2,\ldots\infty \\ \gamma_{mn}=\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\,\pi \end{array}}$$

Now when these constant coefficients are used in the infinite expansion above, then the above expression for $\Phi(x,y,z)$ will obey all the BCs and the PDE $\nabla^2\Phi = 0$.

Other little tricks:

- If $\Phi \neq 0$ is specified on any of the other sides, simply solve for one nonzero side at a time, but then use linearity and add them all up to make the final solution.

- If $\rho \neq 0$ inside as well, then we must use an appropriate Green's Function inside, as discussed in Chapter 3.

Chapter 3 Boundary value problems in electrostatics II

IMPORTANT: it is crucial whenever possible to solve a PDE in a coordinate system adapted to the shapes of the boundary surfaces.

Laplace's equation in spherical coordinates

$$\nabla^2 \Phi(r,\theta,\phi) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$= 0$$

separable spherical solutions are found by setting $\Phi(r,\theta,\phi) = \frac{u(r)}{r} P(\theta) Q(\phi)$

$$\Rightarrow r^2\sin^2\theta\left\{\frac{u''(r)}{u(r)} + \frac{1}{r^2\sin\theta}\frac{(\sin\theta\, P'(\theta))'}{P(\theta)}\right\} + \frac{Q''(\phi)}{Q(\phi)} = 0 \qquad (*)$$

$\Rightarrow \frac{Q''(\phi)}{Q(\phi)} = -m^2 =$ constant since $(*)$ must hold for all r, θ, ϕ

$$\Rightarrow Q(\phi) = e^{\pm i m\phi}$$

and if the full angular range $0 \le \phi < 2\pi$ is allowed, $Q(\phi)$ must be single-valued,

$\Rightarrow m =$ any integer

Returning to the r, θ equations,

$$\Rightarrow \underbrace{r^2\frac{u''(r)}{u(r)}}_{\text{all } r\text{-dependence}} + \underbrace{\frac{1}{\sin\theta}\frac{(\sin\theta\, P'(\theta))'}{P(\theta)} - \frac{m^2}{\sin^2\theta}}_{\text{all } \theta\text{-dependence}} = 0$$

⇒ The two circled terms must equal constants that sum to 0, call them $\pm \ell(\ell+1)$

$$\Rightarrow \quad -\frac{1}{\sin\theta}\left(\sin\theta\, P'(\theta)\right)' + \frac{m^2}{\sin^2\theta} P(\theta) = \ell(\ell+1) P(\theta) \qquad (**)$$

and then $$u''(r) - \frac{\ell(\ell+1)}{r^2} u(r) = 0 \qquad (***)$$

For any given (ℓ, m) there are 2 linearly independent solutions called <u>Associated Legendre Functions</u>

$$P_{\ell m}(\cos\theta), \quad Q_{\ell m}(\cos\theta)$$

LegendreP[ℓ, m, Cos[θ]], in Mathematica
or Q

- If $m =$ integer, only the solutions $P_{\ell m}(\cos\theta)$ are regular at $\theta = 0$, and these will not be well-behaved at $\theta = \pi$ unless $\ell =$ integer $\geq |m|$ i.e. $\ell = |m|, |m|+1, \ldots$ etc.

- If the electrostatic potential is known to be azimuthally symmetric about the z-axis, (meaning independent of ϕ), then only $m = 0$ can be present. The associated Legendre functions for $m=0$ are simpler + are called Legendre polynomials, $P_\ell(x)$ or LegendreP[ℓ, x] ($x = \cos\theta$ here)

Legendre Polynomial properties:

- Differential equation:

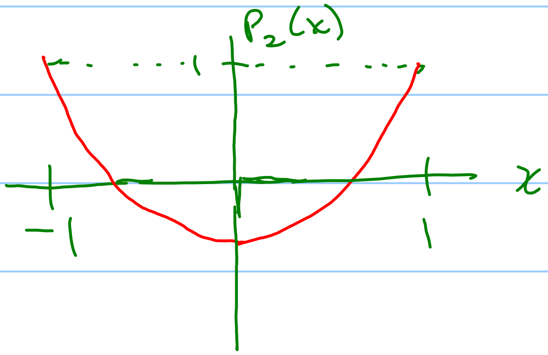
$$\frac{d}{dx}(1-x^2)\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell(x) = 0$$

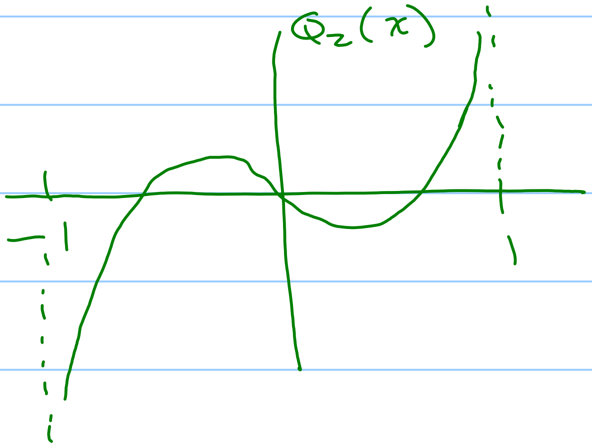
to solve it, this is simple via power series, or it can be coaxed out of Mathematica, e.g.

```
DSolve[(1-x^2)P''[x]-2xP'[x]+l(l+1)P[x]==0, P[x],x]
```

returns

```
P[x] -> C[1] LegendreP[l,x] + C[2] LegendreQ[l,x]
```





Things you should know about Legendre polynomials:

- First few:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$
$$\vdots$$

- Highest power of x in $P_\ell(x)$ is x^ℓ
- Parity under $x \to -x$ is $(-1)^\ell$, i.e. $P_\ell(-x) = (-1)^\ell P_\ell(x)$

- Rodriguez' formula $P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

- Orthogonality $\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$

(can prove this using Rodriguez' formula, then integrate by parts)

- Completeness: over $-1 \le x \le 1$, can expand "any" function, as

$$F(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$

where $$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} P_\ell(x) f(x)\, dx$$

- Recurrence relations, e.g.

(a) $P'_{\ell+1} - P'_{\ell-1} = (2\ell+1) P_\ell$

(b) $(\ell+1) P_{\ell+1} - (2\ell+1) x P_\ell + \ell P_{\ell-1} = 0$

(c) $P'_{\ell+1} - x P'_\ell - (\ell+1) P_\ell = 0$

(d) $(x^2-1) P'_\ell - x\ell P_\ell + \ell P_{\ell-1} = 0$

↖ for others, see Jackson Eq. 3.29

use such relations to evaluate integrals like

$$\int_{-1}^{1} x^k P_\ell(x) P_{\ell'}(x)\, dx, \text{ etc.}$$

Finally, the radial equation is solved as:

$$r^2 \frac{u''(r)}{u(r)} = \ell(\ell+1) \Rightarrow u(r) = r^{\ell+1} \text{ or } r^{-\ell}$$

$$\Rightarrow \frac{u(r)}{r} = A_\ell r^\ell + B_\ell r^{-\ell-1}$$ is the general solution.

Azimuthally-symmetric boundary value problems

For problems having no ϕ-dependence, the general solution to $\nabla^2 \Phi = 0$ takes the form:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

where the constants A_ℓ, B_ℓ are found by imposing boundary conditions.

EXAMPLE Suppose a sphere of radius a has a specified surface potential equal to

$$V(\theta) = V_0 \cos^2\theta$$

BCs: Require $\Phi \xrightarrow[r\to 0]{}$ finite

and $\Phi \xrightarrow[r\to\infty]{} 0$ ← can demand this only for finite charge distributions

$\Rightarrow B_\ell \xrightarrow[\text{inside}]{} 0$, $A_\ell \xrightarrow[\text{outside}]{} 0$

i.e. $\Phi_{\text{inside}}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

$$\Phi_{\text{outside}}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Since there is no dipole layer, Φ = continuous,

$\Rightarrow$ $$\Phi_{\text{inside}}(a,\theta) = V(\theta) = \Phi_{\text{outside}}(a,\theta)$$

SOLUTION use these and orthonormality of $P_l(\cos\theta)$ to find A_l, B_l:

$$\text{e.g. } \sum_l A_l a^l P_l(\cos\theta) = V_0 \cos^2\theta$$

mutiply by $P_{l'}(\cos\theta) \equiv P_{l'}(x)$ and integrate:

$$\int_{-1}^{1} dx\, P_{l'}(x) \sum_l A_l a^l P_l(x) = V_0 \int_{-1}^{1} P_{l'}(x)\, x^2 dx$$

or $$\frac{2}{2l+1} A_l a^l = V_0 \int_{-1}^{1} P_l(x)\, x^2 dx$$

But notice that $x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x)$

$$\Rightarrow \int_{-1}^{1} P_l(x)\, x^2 dx = \frac{2}{5}\left(\frac{2}{3}\right)\delta_{l,2} + \frac{1}{3}(2)\,\delta_{l,0}$$

$$\Rightarrow A_2 = \frac{2}{3}\frac{V_0}{a^2}, \quad A_0 = \frac{1}{3}V_0$$

and similarly $\frac{2}{5} B_2 a^{-3} = V_0 \frac{4}{15}$ or $B_2 = \frac{2}{3} a^3 V_0$

and $B_0 = \frac{1}{3} a V_0$

So finally $$\boxed{\begin{aligned} \Phi_{\text{inside}}(r,\theta) &= \tfrac{1}{3}V_0 + \tfrac{2}{3}V_0\left(\tfrac{r}{a}\right)^2 P_2(\cos\theta) \\ \Phi_{\text{outside}}(r,\theta) &= \tfrac{1}{3}V_0\frac{a}{r} + \frac{2V_0}{3}\frac{a^3}{r^3} P_2(\cos\theta) \end{aligned}}$$

EXAMPLE Construct the potential everywhere in space for an azimuthally-symmetric problem, starting from the solution known only on the symmetry axis.

e.g. since we know that in a charge-free region of space, the general ϕ-independent potential is

$$\Phi(r,\theta) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

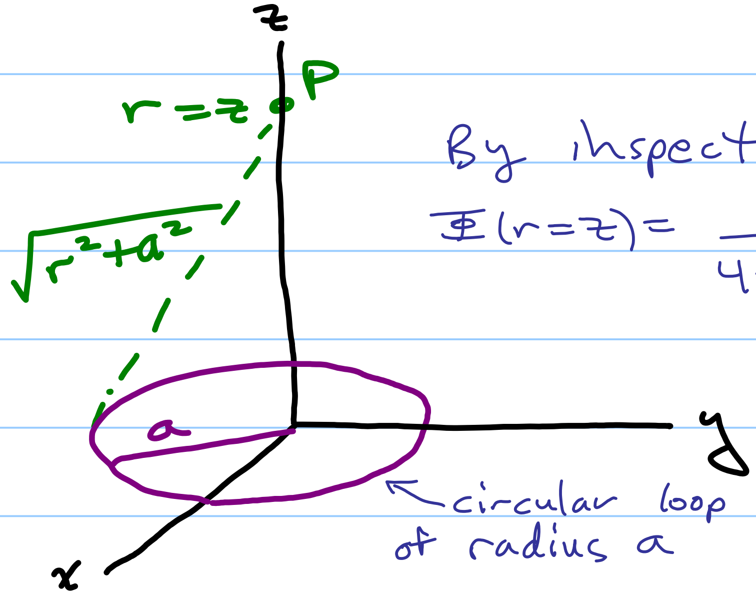
we can evaluate this on axis where $\theta = 0$, $P_\ell(1) = 1$,

$$\Rightarrow \quad \Phi(r,0) = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)$$

We can turn this logic around, as follows:

Suppose we know $\Phi(r=z)$ along the z-axis for an azimuthally-symm. problem. If we expand $\Phi(r=z)$ in a series, we can deduce $\Phi(r,\theta)$ everwhere.

To be concrete, consider a ring of total charge Q in the xy-plane,



By inspection

$$\Phi(r=z) = \frac{Q}{4\pi\epsilon_0\sqrt{r^2+a^2}}$$

Next, expand Φ at $r>a$ as a binomial expansion:

$$\Phi(r=z) = \frac{Q}{4\pi\epsilon_0 r}\left(1+\frac{a^2}{r^2}\right)^{-1/2}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1-\frac{1}{2}\frac{a^2}{r^2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{a^2}{r^2}\right)^2+\ldots\right\}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left\{1-\frac{1}{2}\frac{a^2}{r^2}+\frac{3}{8}\frac{a^4}{r^4}-\frac{15}{48}\frac{a^6}{r^6}+\ldots\right\}$$

$$= \sum_\ell \frac{B_\ell}{r^{\ell+1}} \quad \text{with } B_0 = \frac{Q}{4\pi\epsilon_0},\; B_1 = 0$$

$$B_2 = \frac{-Qa^2}{8\pi\epsilon_0}, \text{ etc...}$$

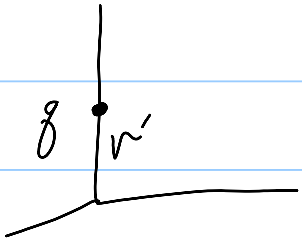
Then by the above logic, the potential off-axis at $r>a$ must be

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{r}-\frac{a^2}{2r^3}P_2(\cos\theta)+\frac{3}{8}\frac{a^4}{r^5}P_4(\cos\theta)+\ldots\right\}$$

Another application of this logic:

The potential at $\vec{x}$ due to a point change q at $\vec{x}\,'$ is $\Phi(\vec{x},\vec{x}\,') = \dfrac{q}{4\pi\epsilon_0 |\vec{x}-\vec{x}\,'|}$

If we choose $\vec{x}\,'$ to lie <u>on</u> <u>the</u> <u>z-axis</u> then we have axial symmetry, i.e. $r'=z'$



Now, the potential at any point $\vec{x} = r\hat{z}$ on the z-axis is of course

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0 |r-r'|} = \frac{q}{4\pi\epsilon_0}\begin{cases} \frac{1}{r}\left(1-\frac{r'}{r}\right)^{-1}, & r'<r \\ \frac{1}{r'}\left(1-\frac{r}{r'}\right), & r'>r \end{cases}$$

or in a more succinct notation, <u>define</u>

$$r_< = \min(r, r')$$

$$r_> = \max(r, r')$$

So in general, on axis,

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0}\frac{1}{r_>}\left(1-\frac{r_<}{r_>}\right)^{-1}$$

and performing a binomial expansion gives

$$\Phi(r=z, r'=z') = \frac{q}{4\pi\epsilon_0}\frac{1}{r_>}\left\{1 + \frac{r_<}{r_>} + \frac{(-1)(-2)}{2!}\left(\frac{r_<}{r_>}\right)^2 + \dots\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}$$

and as before, this result generalizes off-axis to

$$\Phi(r,\theta; r', 0) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}} P_l(\cos\theta)$$

Finally, noting that there was nothing special about choosing $\vec{x}\,'$ on the z-axis, we have generally

$$\boxed{\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}} P_l(\cos\gamma)}$$

where γ = angle between $\vec{x}$ and $\vec{x}\,'$

This last result is also equal to $G(\vec{x}, \vec{x}')$, the free space Green's function with no finite boundaries.

Note that we can write $\cos\gamma$ in terms of the separate spherical coordinates of $\vec{x}$ and $\vec{x}'$, as follows:

Start from $\hat{x} \cdot \hat{x}' = \cos\gamma$

but $\vec{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\vec{x}' = r'(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$

and $\hat{x} \cdot \hat{x}' = \sin\theta\sin\theta'(\cos\phi\cos\phi' + \sin\phi\sin\phi') + \cos\theta\cos\theta'$

or $$\boxed{\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')}$$

↑useful!

Read sec. 3.4 on your own.

Next, suppose the system has <u>no azimuthal symmetry</u> ⇒ NEED SPHERICAL HARMONICS!

The angular diff. equation is now

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP}{d\theta} + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)P = 0$$

+ solutions are associated Legendre functions,

$$P_l^m(x), \quad x = \cos\theta, \quad -1 \leq x \leq 1$$

and for $l, m =$ integers, an explicit formula is

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

Other relations: $P_\ell^m(1) = \delta_{m,0}$

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell, \text{ for } -\ell \le m \le \ell$$

and also, $$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

Important! The $P_\ell^m(x)$ with the SAME m but different ℓ are orthogonal and complete, i.e.

$$\int_{-1}^{1} P_\ell^m(x) P_{\ell'}^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell,\ell'}$$

Recalling that the ϕ-dependence for the Laplace equation solutions must be $e^{im\phi}$, the full solution in any charge-free region can be written (assuming no angular boundaries) as:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}\right) P_\ell^m(\cos\theta)\, e^{im\phi}$$

But it is usually more convenient if we work instead with spherical harmonics, having unit normalization, $Y_{\ell m}(\theta,\phi)$, obeying orthonormality,

$$\int d\Omega\, Y^*_{\ell' m'}(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\Rightarrow Y_{lm}(\vartheta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\vartheta)\, e^{im\phi}$$

whose completeness relation reads:

$$\sum_{lm} Y^*_{lm}(\vartheta',\phi')\, Y_{lm}(\vartheta,\phi) = \delta(\phi-\phi')\,\frac{\delta(\vartheta-\vartheta')}{\sin\vartheta}$$

$$= \delta(\phi-\phi')\,\delta(\cos\vartheta-\cos\vartheta')$$

or with the useful abbreviations in spherical coordinates, $\hat{x} \equiv (\vartheta,\phi)$, $\hat{x}' \equiv (\vartheta',\phi')$

$$\Rightarrow \sum_{l,m} Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x}) = \delta^{(2)}(\hat{x}-\hat{x}')$$

Miscellaneous useful relations:

$$Y^*_{lm}(\vartheta,\phi) = (-1)^m\, Y_{l,-m}(\vartheta,\phi)$$

and $$Y_{l0}(\vartheta,\phi) = \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\vartheta)$$

and in Mathematica,

SphericalHarmonicY[L, M, θ, ϕ]

Spherical Harmonic Addition Theorem:

important! $$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\vartheta',\phi')\, Y_{lm}(\vartheta,\phi)$$

where $\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\phi-\phi')$,

i.e. γ = angle between $\hat{x}$ and $\hat{x}'$

or $$P_l(\hat{x}\cdot\hat{x}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x})$$

Read Jackson's Proof in Sec. 3.6 and/or test it, e.g using Mathematica.

$$\Rightarrow \boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x})}$$

memorize this!

Next: Green's function construction

Again, for the Dirichlet problem, we pick $G \to 0$ on boundary surfaces, and then

$$\Phi(\vec{x}) = \int_V G(\vec{x},\vec{x}') \frac{\rho(\vec{x}')}{4\pi\epsilon_0} d^3x' - \frac{1}{4\pi} \oint_S \frac{\partial G}{\partial n'}(\vec{x},\vec{x}') \Phi(\vec{x}') da'$$

when $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

For many problems with spherical boundary surfaces, it is convenient to expand G into spherical harmonics, starting from:

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')$$

$$= \frac{1}{r^2}\delta(r-r') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x})$$

Strategies to construct multidimensional G:

Ⓘ Eigenfunction in ALL coordinates

e.g., to solve

$$\nabla^2 G(\vec{x}, \vec{x}\,') = -4\pi\,\delta(\vec{x} - \vec{x}\,'), \qquad (1)$$

subject to some BCs, such as homogeneous Dirichlet BCs $\Rightarrow$ $G(\vec{x}, \vec{x}\,') = 0$ for $\vec{x}$ on S,

we can consider a companion problem, the homogeneous eigenvalue problem for eigenfunctions $\psi_n(\vec{x})$ obeying the SAME BCs as $G(\vec{x}, \vec{x}\,')$, i.e. consider

$$-\nabla^2 \psi_n(\vec{x}) = \lambda_n \psi_n(\vec{x}) \qquad (2)$$

Now, one can show that these $\psi_n(\vec{x})$ are eigenfunctions of a Hermitian operator and they can be chosen to be orthonormal and complete, i.e.

$$\int_V \psi_n^*(\vec{x})\, \psi_{n'}(\vec{x})\, d^3x = \delta_{nn'} \qquad (3)$$

and

$$\sum_n \psi_n^*(\vec{x}\,')\, \psi_n(\vec{x}) = \delta(\vec{x} - \vec{x}\,') \qquad (4)$$

So expanding G into this complete set looks like

$$G(\vec{x}, \vec{x}\,') = \sum_n C_n(\vec{x}\,')\, \psi_n(\vec{x}) \qquad (5)$$

and plug this into (1) to find the expansion coefficients $C_n(\vec{x}\,')$:

i.e. $$-\nabla^2 \sum_n C_n(\vec{x}\,')\,\Psi_n(\vec{x}) = +4\pi\,\delta(\vec{x}-\vec{x}\,')$$

$$\hookrightarrow = \sum_n C_n(\vec{x}\,')\,\lambda_n \Psi_n(\vec{x}) = 4\pi \sum_n \Psi_n^*(\vec{x}\,')\,\Psi_n(\vec{x})$$

$$\Rightarrow \quad \lambda_n C_n(\vec{x}\,') = 4\pi\,\Psi_n^*(\vec{x}\,')$$

Hence finally

$$\boxed{G(\vec{x},\vec{x}\,') = 4\pi \sum_n \frac{\Psi_n^*(\vec{x}\,')\,\Psi_n(\vec{x})}{\lambda_n}} \quad (6)$$

The case of a continuous eigenvalue spectrum

$\Rightarrow$ if λ_n are continuous, we must generalize the $\sum_n$ to an integral, e.g.

$$-\nabla^2 \Psi_{\vec{k}}(\vec{x}) = k^2\,\Psi_{\vec{k}}(\vec{x})$$

If V = infinite free space, the solutions are

$$\Psi_{\vec{k}}(\vec{x}) = (2\pi)^{-3/2}\, e^{i\vec{k}\cdot\vec{x}}$$

And when the eigenvalue spectrum is continuous, must generalize the normalization condition to

$$\delta(\vec{k}-\vec{k}\,') = \int_V \Psi_{\vec{k}\,'}^*(\vec{x})\,\Psi_{\vec{k}}(\vec{x})\,d^3x$$

and the above derivation for G generalizes to

$$G(\vec{x},\vec{x}\,') = 4\pi \int d^3k\, \frac{\Psi_{\vec{k}}^*(\vec{x}\,')\,\Psi_{\vec{k}}(\vec{x})}{k^2}$$

It will be instructive, especially later in the course, to evaluate this integral. We can use a "trick", namely to choose the z-axis in k-space to lie along the direction of $\vec{x} - \vec{x}'$. We may do this because we're integrating over all k-space.

$$G(\vec{x}, \vec{x}') = \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{k^2 dk}{k^2} \int_{-1}^{1} d(\cos\theta_k) \int_0^{2\pi} d\phi_k \, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

where θ_k = angle between $\vec{k}$ and $(\vec{x}-\vec{x}')$

$$= \frac{1}{2\pi^2}(2\pi) \int_0^\infty dk \int_{-1}^{1} d(\cos\theta_k) \, e^{ikR\cos\theta_k}$$

where $R = |\vec{x}-\vec{x}'|$

$$\Rightarrow G(\vec{x}, \vec{x}') = \frac{1}{\pi} \int_0^\infty dk \, \frac{e^{ikR} - e^{-ikR}}{ikR}$$

$$= \frac{2}{\pi} \int_0^\infty dk \, \frac{\sin kR}{kR} = \frac{2}{\pi R} \overbrace{\int_0^\infty du \, \frac{\sin u}{u}}^{``\frac{\pi}{2}"}$$

So finally $\boxed{G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}}$ = an alternative way to find this!

Method II Eigenfunction expansion in all coordinates EXCEPT one.

e.g. for a spherical problem, write an ansatz that is motivated by the spherical harmonic completeness relation, namely

$$G(\vec{x}, \vec{x}') = \sum_{l,m} g_l(r,r') \, Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)$$

Now use ∇^2 in the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\vec{L}^2(\theta,\phi)}{r^2}, \quad \text{where}$$

$$\vec{L}^2(\theta,\phi) = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

and recall that $\vec{L}^2 Y_{lm}(\theta,\phi) = l(l+1) Y_{lm}(\theta,\phi)$

Thus we can find the equation $g_l(r,r')$ obeys,

$$\nabla^2 G(\vec{x},\vec{x}') = \sum_{lm} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x}) \left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] g_l(r,r')$$

$$= -4\pi \frac{\delta(r-r')}{r^2} \delta(\cos\theta - \cos\theta')\delta(\phi-\phi')$$

and this equation will be satisfied if

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] g_l(r,r') = -4\pi \frac{\delta(r-r')}{r^2}$$

$\Rightarrow$ We must solve this equation, subject to the appropriate BCs, over $a \le r \le b$, and then we will have $G(\vec{x},\vec{x}')$

solution of the radial equation = 2nd-order linear diff. eqn.

Think of $g_l(r,r')$ = function of r, for fixed r'.

$\Rightarrow$ At any $r \neq r'$, the most general solution in r must be a linear combination of 2 independent solutions of the HOMOGENEOUS DIFF. EQN.

i.e. $$g_\ell(r,r') = \begin{cases} A\, r^\ell + B r^{-\ell-1}, & r<r' \\ C r^\ell + D r^{-\ell-1}, & r>r' \end{cases}$$

where A, B, C, D all depend on ℓ and r' but not r.

<u>Dirichlet case</u> Must satisfy the following 4 equations

① $g_\ell(a,r') = 0 = A\, a^\ell + B\, a^{-\ell-1}$

② $g_\ell(b,r') = 0 = C\, b^\ell + D\, b^{-\ell-1}$

③ Continuity at $r=r'$

$\Rightarrow \quad A r'^\ell + B r'^{-\ell-1} = C r'^\ell + D r'^{-\ell-1}$

④ Derivative <u>discontinuity</u> at $r=r'$, caused by $\delta(r-r')$

To derive the precise form of this deriv. discontinuity integrate the diff. eqn. for $g_\ell(r,r')$ over r from $r=r'-\epsilon$ to $r=r'+\epsilon$, then take $\lim_{\epsilon\to 0^+}$

i.e.

$$\int_{r=r'-\epsilon}^{r=r'+\epsilon} r^2 dr \left\{ \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial g(r,r')}{\partial r} - \underbrace{\cancel{\frac{\ell(\ell+1)}{r^2}\, g_\ell(r,r')}}_{\to\, 0} \right\}$$

0 since continuous

$$= -4\pi \int_{r'-\epsilon}^{r'+\epsilon} r^2 dr\, \frac{\delta(r-r')}{r^2}$$

$$\Rightarrow r^2 \frac{\partial g_\ell(r,r')}{\partial r}\bigg|_{r=r'-\epsilon}^{r=r'+\epsilon} = -4\pi$$

or $$\boxed{\frac{\partial g_\ell(r,r')}{\partial r}\bigg|_{r=r'+\epsilon} - \frac{\partial g_\ell(r,r')}{\partial r}\bigg|_{r=r'-\epsilon} = \frac{-4\pi}{r'^2}}$$

This gives the 4th equation needed to find the 4 constants A, B, C, D, namely:

$$l C r'^{l-1} + (-l-1) D r'^{-l-2}$$
$$- [l A r'^{l-1} - (l+1) B r'^{-l-2} = -\frac{4\pi}{r'^2}$$

=> Solution of these 4 equations and 4 unknowns gives the solution in the form written as Jackson Eq. 3.125, namely for a Dirichlet BC problem with spherical conducting boundaries at r = a and r = b, the GF at $a \le r \le b$ is

$$G(\vec{x}, \vec{x}\,') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\hat{x}')Y_{lm}(\hat{x})\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]}$$

and it is readily seen that the free space GF is recovered upon setting $a \to 0$, $b \to \infty$.

1-dimensional Green Functions – general treatment for the Sturm-Liouville operator.

(see, e.g. Arfken + Weber, Chap. 9)

This 2nd-order differential operator looks in general like

$$\hat{\mathcal{L}}_x = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

Let's find the Dirichlet GF obeying

$$\hat{\mathcal{L}}_x g(x, x') = -\frac{\delta(x - x')}{\beta(x')}$$

over $a \le x \le b$, subject to $g(a,x')=0=g(b,x')$

Again, there are two linearly-independent solutions to the homogeneous differential equation, $\boxed{\hat{\mathcal{L}}_x u(x)=0}$ (*)

Call $u_1(x)$ = a solution of (*) obeying the LEFT boundary condition, $u_1(a)=0$

and $u_2(x)$ = a solution obeying $u_2(b)=0$

Then the GF must have the form

$$g(x,x') = \begin{cases} c_1 u_1(x), & a \le x \le x' \\ c_2 u_2(x), & x' \le x \le b \end{cases}$$

where c_1, c_2 depend on x'.

Next impose continuity at $x=x'$

$$\Rightarrow \quad \boxed{c_1 u_1(x') = c_2 u_2(x')} \quad (**)$$

Next find the derivative discontinuity at $x=x'$, by integrating:

$$\lim_{\epsilon \to 0^+} \int_{x=x'-\epsilon}^{x=x'+\epsilon} \hat{\mathcal{L}}_x g(x,x')\,dx = \int_{x'-\epsilon}^{x'+\epsilon} \left(-\frac{\delta(x-x')}{\beta(x')}\right) dx$$

$$\Rightarrow \boxed{p(x')\frac{dg(x,x')}{dx}\bigg|_{x=x'+\epsilon} - p(x')\frac{dg(x,x')}{dx}\bigg|_{x=x'-\epsilon} = \frac{-1}{\beta(x')}} \quad (***)$$

Solve this along with (**) to get $c_1(x'), c_2(x')$

$$\Rightarrow \text{Find} \quad c_1(x') = \frac{-u_2(x')}{\beta(x')p(x')[u_1(x')u_2'(x') - u_1'(x')u_2(x')]}$$

$$c_2(x') = \frac{-u_1(x')}{\beta(x')p(x')[u_1(x')u_2'(x') - u_1'(x')u_2(x')]}$$

Notational aside: an important quantity in the theory of differential equations is the Wronskian: $W(u_1, u_2) \equiv u_1(x) u_2'(x) - u_1'(x) u_2(x)$

For our Sturm–Liouville operator one can prove that, for any 2 solutions $u_1(x), u_2(x)$ of the homogeneous diff. equation, $\underline{p W(u_1, u_2) = \text{constant}}$

Thus in concise form, we have

$$\boxed{\begin{array}{l} g(x,x') = \dfrac{-1}{\beta(x') \, p W(u_1, u_2)} u_1(x_<) u_2(x_>) \\ \text{is the GF for } \dfrac{d}{dx} p(x) \dfrac{dg}{dx} + q(x) g = -\dfrac{\delta(x-x')}{\beta(x')} \end{array}}$$

extremely useful!

Application to the radial GF for Laplace's equation in spherical coordinates

The diff. equation is:

$$\frac{d}{dr} r^2 \frac{d}{dr} g_\ell(r,r') - \ell(\ell+1) g_\ell(r,r') = -4\pi \delta(r-r')$$

which matches our Sturm–Liouville eqn if we identify

$$p(r) = r^2$$
$$q = -\ell(\ell+1)$$
$$\beta(r) = \frac{1}{4\pi}$$

$a \leq r \leq b$

$$\Rightarrow \quad g_\ell(r,r') = \frac{-4\pi \; u_1(r_<) u_2(r_>)}{r'^2 W[u(r'), u_2(r')]}$$

where $u_i(r)$ obey the homogeneous diff eqn,

$$\frac{d}{dr} r^2 \frac{du_i(r)}{dr} - \ell(\ell+1) u_i(r) = 0$$

$$\text{and} \quad u_1(a) = 0 \;, \quad u_2(b) = 0$$

Aside: if you forget what these solutions are, you can always TRY a power law, e.g. try $u(r) = r^\gamma$ and see whether it works, e.g. here

$$\frac{d}{dr} r^2 (\gamma r^{\ell-1}) - \ell(\ell+1) r^\gamma = 0 = [\gamma(\gamma+1) - \ell(\ell+1)] r^\gamma = 0$$

$\Rightarrow$ this solution works provided

$\gamma(\gamma+1) - \ell(\ell+1) = 0$, i.e. $\gamma = \ell$ or $-\ell-1$

$\Rightarrow$ the general solution is $u(r) = A r^\ell + B r^{-\ell-1}$

So the linear combinations that obey homogeneous Dirichlet BCs are

$$u_1(r) = \left(\frac{r}{a}\right)^\ell - \left(\frac{a}{r}\right)^{\ell+1}, \quad u_2(r) = \left(\frac{r}{b}\right)^\ell - \left(\frac{b}{r}\right)^{\ell+1}$$

whose p×Wronskian works out to be

$$r'^2 W(u_1, u_2) = (2\ell+1)\left(\frac{b^{\ell+1}}{a^\ell} - \frac{a^{\ell+1}}{b^\ell}\right)$$

giving the radial GF,

$$g_\ell(r, r') = \frac{-4\pi}{(2\ell+1)\left(\frac{b^{\ell+1}}{a^\ell} - \frac{a^{\ell+1}}{b^\ell}\right)} \left[\left(\frac{r_<}{a}\right)^\ell - \left(\frac{a}{r_<}\right)^{\ell+1}\right]\left[\left(\frac{r_>}{b}\right)^\ell - \left(\frac{b}{r_>}\right)^{\ell+1}\right]$$

and $G(\vec{x}, \vec{x}') = \sum_{\ell m} g_\ell(r, r') Y^*_{\ell m}(\hat{x}') Y_{\ell m}(\hat{x})$

Application Consider a uniform thin ring of total charge Q, of radius R, inside a grounded, conducting sphere of radius b. Find $\Phi(\vec{r})$ inside.

We can write this in terms of the Dirichlet GF that was just found, as

$$\Phi(\vec{x}) = \int_V \frac{G(\vec{x},\vec{x}\,')\rho(\vec{x}\,')}{4\pi\epsilon_0}\, d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial G(\vec{x},\vec{x}\,')}{\partial n'}\cancelto{0}{\Phi(\vec{x}\,')}\, da'$$

since grounded

Observe that we could use $G(\vec{x},\vec{x}\,')$ from our preceding derivation, and set $a \to 0$, or else we could use G from our method of images derivation in Chapter 2. Their equality implies an identity,

$$G(\vec{x},\vec{x}\,') = \sum_{l,m}\left(\frac{-4\pi}{2l+1}\right)\frac{r_<^l}{b^{l+1}}\left[\left(\frac{r_>}{b}\right)^l - \left(\frac{b}{r_>}\right)^{l+1}\right] Y^*_{lm}(\hat{x}')\, Y_{lm}(\hat{x})$$

$$= \frac{1}{|\vec{x}-\vec{x}\,'|} - \frac{b}{r'\left|\vec{x} - \frac{b^2}{r'^2}\vec{x}\,'\right|}$$

The charged ring has charge density

$$\rho(\vec{x}\,') = \frac{Q}{2\pi R^2}\,\delta(r'-R)\,\delta(\cos\theta)$$

For the solution, let's write separate formulas for $r<R$ and for $r>R$

e.g., for $r<R$, $\Rightarrow r_< = r$, $r_> = R = r'$ (owing to $\delta(r'-R)$)

$$\Phi(r,\theta,\phi) = \sum_{lm} \frac{Q}{2\pi R^2}\frac{1}{4\pi\epsilon_0}\left(-\frac{4\pi}{2l+1}\right)\frac{r^l}{b^{l+1}}\, Y_{lm}(\theta,\phi)$$

$$\times \int_0^{2\pi} d\phi' \int_{-1}^{1} d(\cos\theta')\, Y^*_{lm}(\theta',\phi')\,\delta(\cos\theta')$$

$$\times \int_0^{b} r'^2 dr'\, \delta(r'-R)\left[\left(\frac{r'}{b}\right)^l - \left(\frac{b}{r'}\right)^{l+1}\right]$$

$$= 2\pi\,\delta_{m,0}\, Y_{lm}\left(\frac{\pi}{2},0\right) = \delta_{m,0}\left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(0)$$

$$R^2\left[\left(\frac{R}{b}\right)^l - \left(\frac{b}{R}\right)^{l+1}\right]$$

And the final answer, for $r<R$, is:

$$\Phi(r,\theta,\phi) = \frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty} r^l\left(\frac{1}{R^{l+1}} - \frac{R^l}{b^{2l+1}}\right)P_l(0)\,P_l(\cos\theta)$$

and one could simplify this further if it is desired, using, e.g.

$$P_{2n+1}(0) = 0, \quad P_{2n}(0) = \frac{(-1)^n}{2^n n!}(2n-1)!!$$

Laplace's equation in cylindrical coordinates

Now $(\rho, \phi, z) \equiv (q_1, q_2, q_3)$

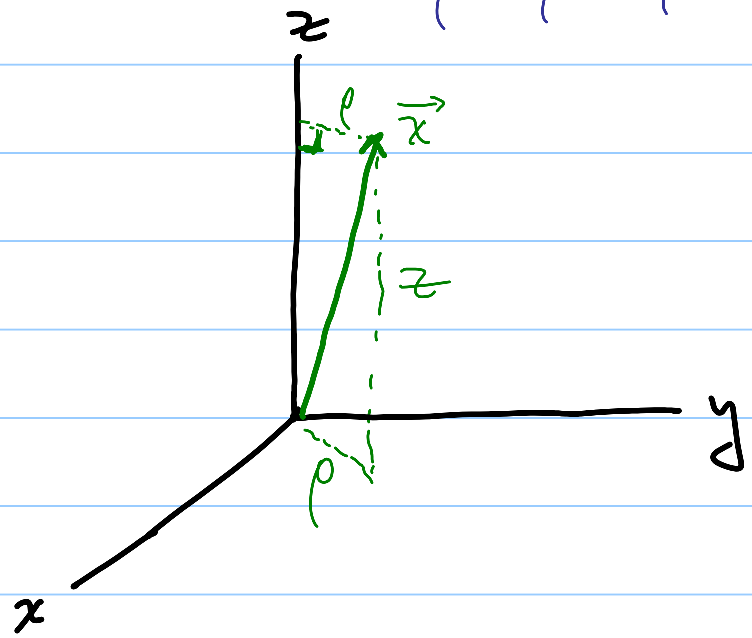
and $h_\rho = 1, \ h_\phi = \rho, \ h_z = 1$

So recalling that

$$\nabla^2 \Phi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\right) + \text{,, ...}$$

cyclic permutations

$$\Rightarrow \nabla^2 \Phi(\rho, \phi, z) = 0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right. + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$



Again solve this PDE by separation of variables. That is, we initially look for separable solutions, of the form

$$\Phi(\rho, \phi, z) = R(\rho)\, Q(\phi)\, Z(z)$$

which gives the separated equations,

$$\frac{d^2 Q(\phi)}{d\phi^2} + \nu^2 Q(\phi) = 0 \Rightarrow e^{\pm i\nu\phi} \text{ are the 2 solns}$$

when $\nu \neq 0$

and $$e^{i\nu\phi}\Big|_{\nu\to0} \longrightarrow A + B\phi$$

Next,

$$\frac{d^2 Z(z)}{dz^2} - k^2 Z(z) = 0$$

$$\Rightarrow Z(z) = e^{\pm kz}$$

or sinh kz, cosh kz...

and finally in ρ,

$$\Rightarrow \frac{d^2R(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dR(\rho)}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R(\rho) = 0$$

or dividing this last equation by k^2 and calling $x = k\rho$, gives

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$$

This can be solved by power series (Fröbenius method), be sure you know how to do this! The solution is

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j\,(x/2)^{2j}}{j!\,\Gamma(\nu+j+1)}$$

Observation, since the differential equation depends only on ν^2, we know immediately that

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(-\nu+j+1)}\left(\frac{x}{2}\right)^{2j}$$

is ALSO a solution to the diff. equation!

Theory of the Neumann Bessel Function.

These two functions $J_\nu(x)$, $J_{-\nu}(x)$ could serve as our two linearly-independent solutions, provided they are in fact independent.

We can assess their independence by looking at their asymptotic forms (see Arfken + Weber, or Morse + Feshbach):

$$J_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

and

$$J_{-\nu}(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

This is proportional to $J_\nu(x)$ when ν=integer

i.e. this shows that $J_{-m}(x) = (-1)^m J_m(x)$ when m=integer.

However, if we could define a solution $N_\nu(x)$ that behaves asymptotically like

$$N_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$

then it would be linearly-independent of $J_\nu(x)$ at all values of ν, x! Actually, we can find such a solution by starting from the identity,

$$\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\cos\pi\nu - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$
$$= \cos\left(x + \frac{\nu\pi}{2} - \pi/4\right)$$

i.e. $J_\nu(x)\cos\pi\nu - N_\nu(x)\sin\pi\nu = J_{-\nu}(x)$

and this in turn suggests that we should define

$$\boxed{N_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}}$$

which is called the "Neumann function", or Bessel function of the 2nd kind, sometimes written instead as $Y_\nu(x)$

Through this construction $J_\nu(x)$ and $N_\nu(x)$ are now linearly-independent for ALL values of ν, including $\nu = m =$ integer!

For electrostatics problems, these Bessel functions are most useful when $\Phi \to 0$ boundary conditions are imposed on a surface in ρ, because they are oscillatory in ρ for k=real, i.e. when $k^2>0$. It is also crucial to recognize that for small distances,

$$J_\nu(k\rho) \xrightarrow[k\rho\to 0]{} \text{regular} \quad \text{for} \quad \nu \geq 0$$

$$N_\nu(k\rho) \xrightarrow[k\rho\to 0]{} \text{IRREGULAR at all } \nu \geq 0$$

See, e.g., Jackson Eqs. 3.89, 3.90

Also, in some problems we will the values of k at which $J_\nu(k\rho_0)$ vanishes, i.e. the roots of

$$J_\nu(x_{\nu n}) = 0$$

with $x_{\nu n} = n^{th}$ zero of $J_\nu(x)$

Tables of these $x_{\nu n}$ can be found, e.g., in the NIST Tables, the old Abramowitz + Stegun, but also Mathematica can provide them with the statement BesselJZeros[ν, n]

first n zeroes of $J_\nu(x_{\nu n}) = 0$.

Example application Find the potential Φ inside a right-circular cylinder of radius a, height L, all of whose walls are held at $\Phi = 0$ (think of a grounded conductor), Except for the top cap at $z = L$, which is instead held at $V(\rho, \phi) = V_0 \frac{\rho}{a} w(\phi)$ and $w(\phi)$ is some known function of ϕ.

Solution Start from a sum over the separable Bessel solutions, regular at $\rho = 0$, which vanish at $\rho = a$:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left(\frac{A_{m,n} e^{im\phi} + A_{-m,n} e^{-im\phi}}{1 + \delta_{m0}} \right) \times J_m\left(x_{mn} \frac{\rho}{a}\right) \sinh\left(x_{mn} \frac{z}{a}\right)$$

i.e. write $k = \frac{x_{mn}}{a}$ in our earlier separable solution.

This has been designed to satisfy the PDE

$$\nabla^2 \Phi = 0$$

and all BCs in the problem except one, the one at $z = L$. To find the A_{mn} such that this last BC is obeyed, use the orthogonality relation obeyed by

$$e^{im\phi} J_m\left(x_{mn} \frac{\rho}{a}\right)$$

For a derivation, see Jackson pp. 114-115, namely,

$$\int_0^a J_m\left(x_{mn'} \frac{\rho}{a}\right) J_m\left(x_{mn} \frac{\rho}{a}\right) \rho \, d\rho = \delta_{nn'} \frac{a^2}{2} \left[J_{m+1}(x_{mn})\right]^2$$

Another useful aside: from the last result, notice that the functions

$$\boxed{u_n^{(m)}(\rho) = \frac{\sqrt{2}}{a J_{m+1}(x_{mn})} J_m\left(x_{mn} \frac{\rho}{a}\right)}$$

form a complete, orthonormal set, with weight function ρ, over $0 \le \rho \le a$, for each $m = 0, 1, 2, \ldots$

i.e. $\int_0^a u_n^{(m)}(\rho) \, u_{n'}^{(m)}(\rho) \, \rho \, d\rho = \delta_{nn'}$

and $\sum_{n=1}^{\infty} u_n^{(m)}(\rho) \, u_n^{(m)}(\rho') = \underline{\delta(\rho - \rho')} \over \rho$

for $0 \le \rho \le a$ and $0 \le \rho' \le a$

To complete this solution, use "Fourier's Trick" again, using orthogonality in ρ, ϕ, i.e.

$$A_{\pm m, n} = \frac{1}{2\pi} \frac{1}{\sinh\left(\frac{x_{mn} L}{a}\right)} \frac{2}{a^2 \left[J_{m+1}(x_{mn})\right]^2}$$

$$\times \int_0^{2\pi} e^{\mp i m \phi} d\phi \int_0^a \rho \, d\rho \, J_m\left(x_{mn} \frac{\rho}{a}\right) V(\rho, \phi)$$

$$= \frac{V_0}{\pi a^2 \left(J_{m+1}(x_{mn})\right)^2} \frac{\int_0^{2\pi} e^{\mp i m \phi} w(\phi) \, d\phi}{\sinh\left(x_{mn} \frac{L}{a}\right)}$$

$$\times \left(\int_0^a \frac{\rho^2}{a} J_m\left(x_{mn} \frac{\rho}{a}\right) d\rho \right)$$

An aside on orthogonality + completeness

$\Rightarrow$ Read this on your own!

We have already commented that eigenfunctions $u_n(x)$ of the Sturm-Liouville operator are orthogonal in general, if the $u_n(x)$ all obey the same BCs on $\frac{u'}{u}$ at the boundaries.

Careful statement: Let $\{u_n(x)\}$ be the set of all eigenfunctions of

can prove that the λ_n are real

$$\hat{\mathcal{L}} u_n(x) + \lambda_n w(x) u_n(x) = 0,$$

over $a \leq x \leq b$, where $w(x)$ is a nonnegative weight function, and where each $u_n(x)$ obeys the BCs $\frac{u_n'(a)}{u_n(a)} = \alpha$, $\frac{u_n'(b)}{u_n(b)} = \beta$

Then one can show (see, e.g. Arfken + Weber, Chap. 9):

ORTHOGONALITY $\int_a^b u_n^*(x)\, u_m(x)\, w(x)\, dx = 0$

if $\lambda_n \neq \lambda_m$. Moreover, if the eigenvalue spectrum is discrete (countable), then it is possible to normalize the eigenfunctions to unity, as

$$\int_a^b |u_n(x)|^2 w(x)\, dx = 1$$

and we assume hereafter that this has been done $\Rightarrow \int_a^b u_n^*(x)\, u_m(x)\, w(x) dx = \delta_{nm}$

Completeness One can also show that the $\{u_n(x)\}$, $n = 1, 2, \ldots \infty$ are complete. To derive the actual completeness relation, start from the following expansion of an arbitrary function $F(x)$, over $a \le x \le b$: $F(x) = \sum_n c_n u_n(x)$ (I)

The c_n obtained from "Fourier's Trick" are found the usual way (multiply by $u_m^*(x)\, w(x)$ and integrate)

$$\Rightarrow c_m = \int_a^b u_m^*(x')\, F(x')\, w(x')\, dx'$$

↖ use x' as dummy integration variable.

Now plug this back into (I)

$$\Rightarrow F(x) = \int_a^b \Big(\sum_n u_n^*(x')\, u_n(x)\Big) F(x')\, w(x') dx'$$

$\Rightarrow$ For this to hold at all x, we MUST have:

$$\sum_n u_n^*(x')\, u_n(x) = \delta(x - x')/w(x')$$

Back to cylindrical electrostatics problems

For some problems we will need the <u>other</u> family of Laplace equation solutions, which are oscillatory in z rather than ρ.

i.e. where we chose $\frac{Z''(z)}{Z(z)} = k^2$, we <u>could</u> have chosen $-\kappa^2$

$\Rightarrow$ This new family of solutions can be obtained by analytical continuation, which means setting $k \to i\kappa$ with $\kappa = \text{real}, \geq 0$ now.

$\Rightarrow$ solutions for $Z(z)$ become $\sinh(i\kappa z) = i \sin \kappa z$

$$\cosh(i\kappa z) = \cos \kappa z$$

whereas the Bessel solutions become

$$J_\nu(i\kappa\rho), \; N_\nu(i\kappa\rho),$$

and both of these <u>diverge</u> <u>exponentially</u> at $\kappa\rho \to \infty$ analogous to $\sinh \kappa\rho$, $\cosh \kappa\rho$.

Because of this divergence, it is often convenient to work with a different 2nd solution called $K_\nu(\kappa\rho)$ instead of with $N_\nu(i\kappa\rho)$, where $K_\nu(\kappa\rho)$ is defined to be regular at $\kappa\rho \to \infty$. (<u>Analogy</u>: this is similar to the fact that <u>one</u> linear combination of $\sinh \kappa z$, $\cosh \kappa z$, namely $e^{-\kappa z}$, is regular at ∞.)

Similarly, it is conventional to introduce a new solution $I_\nu(k\rho)$ proportional to $J_\nu(ik\rho)$ such that $I_\nu(k\rho)$ is real for real $k\rho > 0$, and still regular at $k\rho \to 0$

Definitions of non-oscillatory Bessel functions:

$$I_\nu(x) \equiv i^{-\nu} J_\nu(ix) = \text{real for real } x$$

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1}\left[J_\nu(ix) + i N_\nu(ix)\right]$$

$$= \frac{\pi}{2} i^{\nu+1} \underbrace{H_\nu^{(1)}(ix)}$$

↑ Hankel function of the 1st kind.

Terminology and limiting forms

$I_\nu(x)$ and $K_\nu(x)$ are often called

MODIFIED BESSEL FUNCTIONS

Small x $\quad I_\nu(x) \xrightarrow[|x|\to 0]{} \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu$ = regular at 0 if $\nu \geq 0$

$$K_\nu(x) \xrightarrow[|x|\to 0]{} \begin{cases} -\ln x - \gamma + \ln 2 + \cdots, & \nu = 0 \\ 2^{-\nu-1}(\nu-1)!\, x^{-\nu}, & \nu > 0 \end{cases}$$

= IRREGULAR at 0

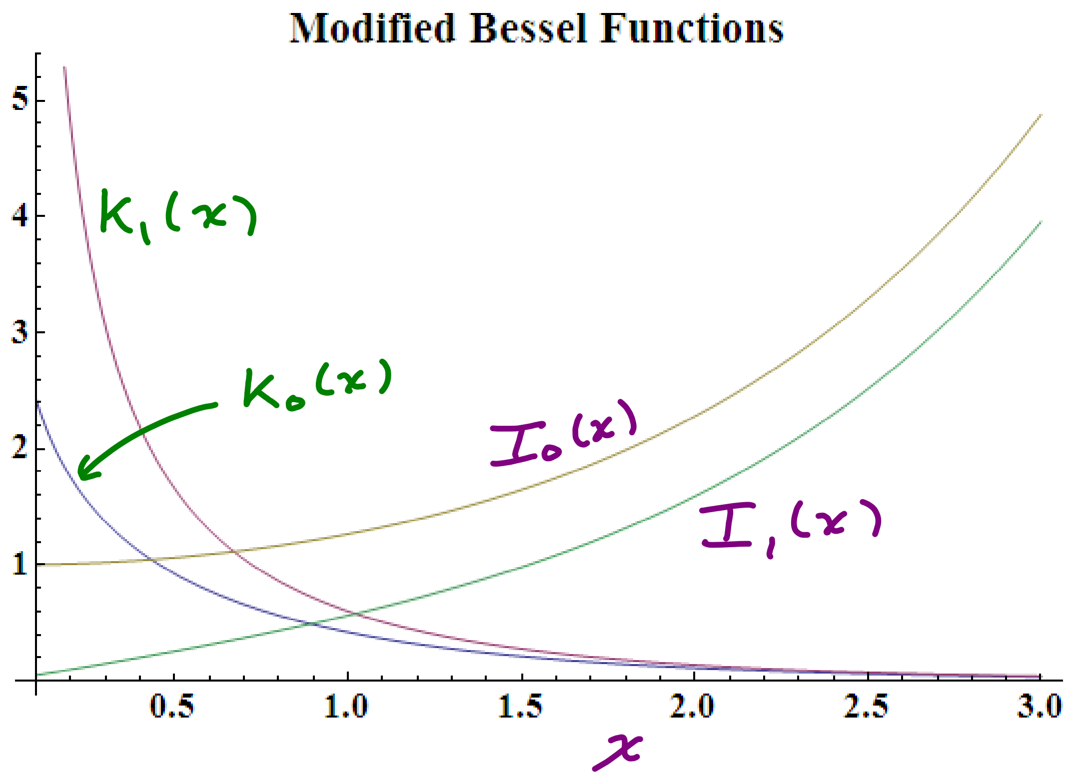
Large $|x|$ $\quad I_\nu(x) \xrightarrow[|x|\to\infty]{} \text{const} \times \frac{e^x}{\sqrt{x}}$, ⇒ IRREGULAR at ∞

$$K_\nu(x) \xrightarrow[|x|\to\infty]{} \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \Rightarrow \text{REGULAR at } \infty$$

```
In[3]:= $DefaultFont = {"HelveticaBold", 16};

In[6]:= Plot[{BesselK[0, x], BesselK[1, x], BesselI[0, x], BesselI[1, x]}, {x, 0.1, 3},
         BaseStyle → {FontWeight → "Bold", FontSize → 18},
         PlotLabel → "Modified Bessel Functions"]
```

Out[6]=



Example This class of solutions is useful for problems, especially if the top $^{z=L}$ and bottom $^{z=0}$ caps of a cylinder are held at $\Phi = 0$. Then we can utilize a complete, orthonormal set of solutions in z that vanish at $z=0, L$, with wavenumbers $k = \frac{n\pi}{L}$: $\left(\frac{2}{L}\right)^{1/2} \sin\frac{n\pi z}{L}$,

$n = 1, 2, 3, \ldots$

The solutions inside this cylinder are then

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{im\phi} \left(\frac{2}{L}\right)^{1/2} \sin\frac{n\pi z}{L} \, I_m\left(\frac{n\pi\rho}{L}\right)$$

- K_m cannot be present if $\rho = 0$ is within the volume V since K_m = irregular at $\rho = 0$
- Note that $I_{-m}(x) = I_m(x)$ for integer m.

Green's Function in Cylindrical Coordinates

Problem Find $G(\vec{x},\vec{x}')$ in free space, using oscillatory solutions in ϕ and z, i.e. e^{ikz}

In this infinite space volume V, we want to solve

$$\nabla_x^2 G(\vec{x},\vec{x}') = -4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi')\delta(z-z')$$

IDEA Start from the completeness relations in ϕ and z, namely

$$\text{(i)}\quad \delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(z-z')}\,dk$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \cos k(z-z')\,dk$$

$$\text{(ii)}\quad \delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

Then building the Green's function out of these forms, with a still-unknown ρ,ρ'-dependence,

$$\Rightarrow G(\vec{x},\vec{x}') = \frac{1}{2\pi^2}\sum_m \int_0^{\infty} dk\,\cos k(z-z')\,e^{im(\phi-\phi')}\,g_m(\rho,\rho';k)$$

Plug this into the PDE to find the equation obeyed by $g_m(\rho,\rho';k)$:

i.e.

$$\nabla^2 G(\vec{x},\vec{x}') = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) G(\vec{x},\vec{x}')$$

$$= \frac{1}{2\pi^2}\sum_m \int_0^\infty dk \cos k(z-z') e^{im(\phi-\phi')} \left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - k^2 - \frac{m^2}{\rho^2}\right\} g_m(\rho,\rho';k)$$

$$= -4\pi\frac{\delta(\rho-\rho')}{\rho}\left\{\frac{1}{2\pi^2}\sum_m e^{im(\phi-\phi')}\int_0^\infty \cos k(z-z')\,dk\right\}$$

Clearly, this last equation will be satisfied iff g_m obeys:

$$\cancel{\frac{1}{\rho}}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} g_m(\rho,\rho';k) - \left(k^2+\frac{m^2}{\rho^2}\right) g_m(\rho,\rho';k) = -4\pi\frac{\delta(\rho-\rho')}{\cancel{\rho}}$$

and observe that this fits the Sturm-Liouville equation discussed previously, namely

$$\frac{d}{dx}p(x)\frac{d}{dx}g(x,x') + q(x)\,g(x,x') = -\frac{\delta(x-x')}{\beta(x')}$$

where we derived $g(x,x') = \dfrac{-u_1(x_<)u_2(x_>)}{\beta(x')p(x')W[u_1,u_2]\big|_{x'}}$

So to apply this here, we must let

$x\equiv\rho$ $\qquad p\longrightarrow\rho$, $\beta\to\frac{1}{4\pi}$

And the appropriate solutions of the homogeneous differential equation are:

$u_1(\rho) = I_m(k\rho)$, obeys the B.C. at $\rho\to 0$

and $u_2(\rho) = K_m(k\rho)$, obeys the B.C. at $\rho\to\infty$

-and $\rho' W(I_m,K_m)\big|_{\rho'}$ can be evaluated at ANY ρ' since it is constant.

I will evaluate it at $\rho \to 0$ where

$$I_m(k\rho) \xrightarrow[\rho\to 0]{} \frac{1}{|m|!}\left(\frac{k\rho}{2}\right)^{|m|}$$

$$K_m(k\rho) \xrightarrow[\rho\to 0]{} 2^{|m|-1}\,(|m|-1)!\,(k\rho)^{-|m|}, \quad \text{for } m\neq 0$$

and thus $\rho' W(I_m, K_m)_{\rho'} = \lim\limits_{\rho'\to 0} \frac{\rho'}{2|m|}\left(\frac{-|m|}{\rho'} - \frac{|m|}{\rho'}\right) = -1$

and this result TURNS OUT to hold also for $m=0$. (Prove this on your own!)

Therefore, finally, $g_m(\rho,\rho';k) = 4\pi I_m(k\rho_<)K_m(k\rho_>)$ and another expression for the Green's function with NO BOUNDARIES PRESENT is:

$$G(\vec{x},\vec{x}\,') = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk \cos k(z-z')\, e^{im(\phi-\phi')}\, I_m(k\rho_<)K_m(k\rho_>)$$

$$= \frac{1}{|\vec{x}-\vec{x}\,'|}$$

Example problem 3.16

(a) δ-Function orthogonality proof

Consider the 2-dim^{l} Fourier transform of a function f(x,y):

$$F(s,\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\, f(x,y)\, e^{i(sx+\sigma y)}$$

and now specialize to a particular f(x,y) written in polar coordinates,

$f(x,y) \to f(r)\, e^{im\theta}$, where $x = r\cos\theta$, $y = r\sin\theta$

and call $s = \rho\cos\alpha$, $\sigma = \rho\sin\alpha$ (*)

$$\Rightarrow F(\rho,\alpha) = \frac{1}{2\pi}\int_0^\infty r\,dr\int_0^{2\pi} d\theta\; f(r)\exp[ir\rho(\underbrace{\cos\theta\cos\alpha+\sin\theta\sin\alpha}_{``\cos(\theta-\alpha)}) + im\theta]$$

and the inverse of this Fourier transform is

$$f(r)e^{im\theta} = \frac{1}{2\pi}\int_0^\infty \rho\,d\rho\int_0^{2\pi} d\alpha\; F(\rho,\alpha)\, e^{-ir\rho\cos(\theta-\alpha)} \qquad (**)$$

Refer now to Gradsteyn + Ryzhik 8.411.1,

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta + iz\sin\theta}\, d\theta$$

and set $\beta \equiv \theta - \frac{\pi}{2}$ here

$$\Rightarrow J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos\beta - in\beta - in\frac{\pi}{2}}\, d\beta$$

comparing this with our original transform (*) gives

$$F(\rho,\alpha) = e^{im(\alpha+\frac{\pi}{2})}\int_0^\infty r f(r)\, J_m(\rho r)\, dr$$

and similarly (**) gives

$$f(r)e^{im\theta} = \frac{1}{2\pi}\int_0^\infty \rho\,d\rho \int_0^{2\pi} d\alpha \left\{ \frac{1}{2\pi}\int_0^\infty r'dr' \int_0^{2\pi} d\theta' \right.$$

$$\left[f(r')\, e^{i(r'\rho\cos(\theta'-\alpha)+m\theta')}\, e^{-ir\rho\cos(\theta-\alpha)} \right]$$

$$= \frac{1}{2\pi}\int_0^\infty \rho\, d\rho \int_0^\infty r'\, dr' \int_0^{2\pi} d\alpha\; e^{-ir\rho\cos(\theta-\alpha)}$$

$$\times \left\{ \frac{1}{2\pi}\int_0^{2\pi} d\theta'\; e^{i(r'\rho\cos(\theta'-\alpha)+m\theta')} \right\}$$

and $\hookrightarrow \{\ \} = J_m(r'\rho)\, e^{im\alpha}$

and $\frac{1}{2\pi}\int_0^{2\pi} d\alpha\; e^{-ir\rho\cos(\theta-\alpha)+im\alpha}$

$= J_m(-r\rho)\, e^{im\theta}$ by this logic

So putting it all together we see that

$$f(r)e^{im\theta} = e^{im\theta}\int_0^\infty r'\,dr' \underbrace{\left(\int_0^\infty \rho\, d\rho\, J_m(\rho r')\, J_m(\rho r)\right)}_{``\frac{\delta(r-r')}{r'}}$$

So we're done! $\Longrightarrow$

3.16(b) The Green's function with no conducting surfaces obeys

$$\nabla_x^2 G(\vec{x},\vec{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

To satisfy the δ-functions in ρ, ϕ, take G in the form (an ansatz):

$$G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^\infty k\,dk\, J_m(k\rho)\, J_m(k\rho')\, Z_{km}(z,z')$$

Plugging this in gives an equation in z:

$$-k^2 Z(z,z') + Z''(z,z') = -4\pi\delta(z-z')$$

and the appropriate homogeneous solutions are

$u_1(z) = e^{kz}$ = regular @ $z \to -\infty$, if $k \geq 0$

and $u_2(z) = e^{-kz}$ = regular at $z \to +\infty$ if $k \geq 0$

so $$Z(z,z') = -4\pi \frac{u_1(z_<)\,u_2(z_>)}{W(u_1,u_2)}$$

$$\Rightarrow Z(z,z') = \frac{4\pi}{2k} e^{k(z_< - z_>)} = \frac{2\pi}{k} e^{-|z-z'|}$$

and plugging this back in to our ansatz gives another identity:

$$G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k|z-z'|}$$

$$= \frac{1}{|\vec{x}-\vec{x}'|}$$

3.16(c) As a special case of this formula, set $\rho'=0 \Rightarrow$ all $J_m(k\rho') \longrightarrow 0$ unless $m=0$ and also set $z' \to 0$, which gives

$$\boxed{\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty e^{-k|z|} J_0(k\rho)\, dk} \qquad (1)$$

Next let $z' \to 0$ and $\phi' \to 0$ to see another special case:

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = (\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{-1/2}$$
$$= \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im\phi} J_m(k\rho) J_m(k\rho') e^{-k|z|}$$

and one more identity is found by replacing ρ in Eq.(1) above by $(\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{1/2}$, namely

$$\int_0^\infty e^{-k|z|} J_0[k(\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{1/2}]\,dk$$
$$= \int_0^\infty dk\, e^{-k|z|} \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho')$$

and since this holds for ALL ρ and z, the integrands must be equal

$$\Rightarrow J_0\left[k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}\right] = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho')$$

Next, prove that $e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)$

Here I will use the generating function for Bessel functions:

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{m=-\infty}^{\infty} t^m J_m(x)$$

In this identity let $t \rightarrow e^{i(\phi+\frac{\pi}{2})}$

$$\Rightarrow e^{\frac{x}{2}[2i\sin(\phi+\frac{\pi}{2})]} = e^{ix\cos\phi} = \sum_{m=-\infty}^{\infty} e^{im\phi} \overset{i^m}{e^{i\frac{m\pi}{2}}} J_m(x)$$

or setting $x = k\rho$ $\Rightarrow$ $\boxed{e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)}$

Alternative method to show orthonormality of continuous eigenfunctions more generally

[i.e. alternative solution of Jackson 3.16(a)]
consider p.99–100 as an "appendix", not covered in class

Start from the Sturm – Liouville operator

$$\boxed{\hat{\mathcal{L}} \equiv \frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}}$$

whose eigenfunctions are Bessel functions:

$$\hat{\mathcal{L}}\, J_\nu(k\rho) = -k^2 J_\nu(k\rho)$$

and $\hat{\mathcal{L}}\, J_\nu(k'\rho) = -k'^2 J_\nu(k'\rho)$

Then consider $(k^2-k'^2)\int_0^\infty \rho\, d\rho\, J_\nu(k\rho)\, J_\nu(k'\rho) \equiv I$

so

$$I = \int_0^\infty \rho\, d\rho \left\{ \left[\hat{\mathcal{L}}\, J_\nu(k'\rho)\right] J_\nu(k\rho) - J_\nu(k'\rho)\left[\hat{\mathcal{L}}\, J_\nu(k\rho)\right] \right\}$$

$$= \int_0^\infty d\rho \left\{ \left[\frac{d}{d\rho}\left(\rho\frac{d J_\nu(k'\rho)}{d\rho}\right)\right] J_\nu(k\rho) - J_\nu(k'\rho)\left[\frac{d}{d\rho}\left(\rho\frac{d J_\nu(k\rho)}{d\rho}\right)\right] \right.$$

$$\left. + \cancel{\rho\, J_\nu(k'\rho)\, J_\nu(k\rho)\left(\frac{\nu^2-\nu^2}{\rho^2}\right)} \rightarrow 0 \right\}$$

but observe that we can rewrite the nonvanishing part of the integrand as:

$$I = \int_0^\infty d\rho \frac{d}{d\rho}\left\{\rho\frac{d J_\nu(k'\rho)}{d\rho} J_\nu(k\rho) - \rho\, J_\nu(k'\rho)\frac{d J_\nu(k\rho)}{d\rho}\right\}$$

or $I = \lim_{R\to\infty} \left\{\rho\, W\left[J_\nu(k\rho), J_\nu(k'\rho)\right]\right\}\Big|_0^R$

Now, this vanishes at the lower limit, while the upper limit can be evaluated using the asymptotic expansion, namely

$$J_\nu(k\rho) \xrightarrow[|x|\to\infty]{} \left(\frac{2}{\pi k\rho}\right)^{1/2} \cos\Big(k\rho \underbrace{-\frac{\nu\pi}{2} - \frac{\pi}{4}}_{\equiv \beta}\Big)$$

So $$W[J_\nu(kR), J_\nu(k'R)] \xrightarrow[R\to\infty]{} \left(\frac{2}{\pi R}\right)(kk')^{-1/2}$$

$$\times \left\{ \cos(kR+\beta)(-k')\sin(k'R+\beta) + k\sin(kR+\beta)\cos(k'R+\beta)\right\}$$

now set $k'-k\equiv\Delta$, i.e. replace $k'\to k+\Delta$ everywhere expand in powers of small Δ:

$$\Rightarrow \lim_{R\to\infty} W = \frac{2}{\pi R}(-1)\underbrace{[k(k+\Delta)]^{-1/2}}_{\simeq \frac{1}{k}}\Big\{\underbrace{k\sin(\Delta R)} - \Delta\cos(kR+\beta)\sin((k+\Delta)R+\beta)\Big\}$$

only term relevant to the δ-function!

oscillates infinitely fast and vanishes at $R\to\infty$

$$\Rightarrow \int_0^\infty \rho J_\nu(k\rho)J_\nu(k'\rho)\,d\rho = \lim_{R\to\infty}\frac{2}{\pi}\frac{\sin R\Delta}{k^2-k'^2},$$ and $k+k'\simeq 2k$

$$= \frac{1}{2k}\lim_{R\to\infty}\frac{2}{\pi}\frac{\sin R\Delta}{\Delta} = \frac{\delta(k-k')}{k} \quad ✓$$

and we're done!

Chapter 4 – Multipole expansions + dielectrics

Next we consider:

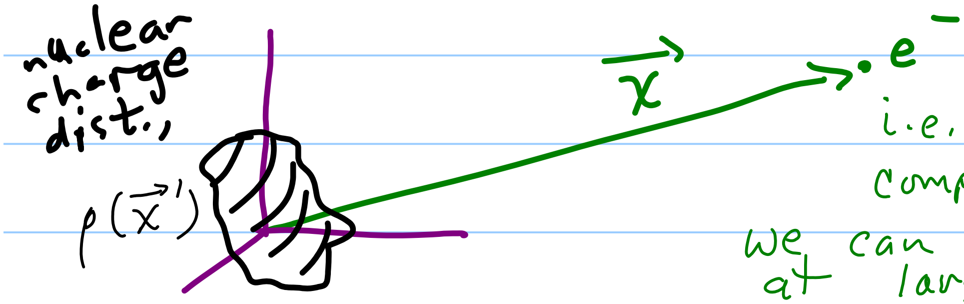
(i) Multipole expansion of $\Phi(\vec{x})$ produced by a localized $\rho(\vec{x})$

(ii) Electric fields and energy in macroscopic media, including charge rearrangement in response to an external field.

i.e. induced dipoles modify the electric field, as in $\vec{E} = \vec{E}_{ext} + \vec{E}_{induced}$

Multipole expansion First, our goal is to find properties like $\Phi(\vec{x})$, $\vec{E}(\vec{x})$, etc., produced by a localized $\rho(\vec{x}\,')$, e.g. when a nucleus produces a potential and then an e^- moves through it.

Concept Let's try to characterize effects of the charge distribution through a few parameters, = charge, electric dipole moment, electric quadrupole moment, etc.



i.e. the full $\rho(\vec{x}\,')$ could be complicated and detailed, but we can summarize its effects at large $|\vec{x}|$ more simply, For now, develop the theory for only localized charges in free space, i.e. with no boundaries:

$$\Rightarrow \quad \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

or if we plug in our spherical harmonic expansion, for $r > r'$, this gives

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{1}{4\pi\epsilon_0}\left[\int d^3x' \, r'^l \, Y^*_{lm}(\hat{x}')\rho(\vec{x}')\right] \times \frac{1}{r^{l+1}} Y_{lm}(\hat{x})$$

or $\Phi(\vec{x}) = \frac{1}{\epsilon_0}\sum_{lm} \frac{q_{lm}}{2l+1}\frac{Y_{lm}(\hat{x})}{r^{l+1}}$

where the $q_{lm} \equiv \int d^3x' \, r'^l \, Y^*_{lm}(\hat{x}')\rho(\vec{x}')$

are called the multipole moments of $\rho(\vec{x}')$

Some basic properties

(1) Note that $q_{l,-m} = (-1)^m q^*_{lm}$, simply from the Y_{lm} properties

(2) $q_{0,0} = \frac{1}{\sqrt{4\pi}}\int \rho(\vec{x}')\, d^3x' = \frac{q}{\sqrt{4\pi}}$ ← electric monopole moment

where q = total charge

(3) ELECTRIC DIPOLE, $l=1$

$$q_{11} = -\int d^3x' \, r' \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta' e^{-i\phi'} \rho(\vec{x}\,')$$

It is instructive to rewrite this in terms of the "SPHERICAL UNIT VECTORS", defined as $\hat{\epsilon}_{\pm 1} = \frac{\mp \hat{x} - i\hat{y}}{\sqrt{2}}$, $\hat{\epsilon}_0 = \hat{z}$

which shows that the $l=1$ spherical harmonics are in fact just:

$$Y_{1,\mu}(\theta',\phi') = \left(\frac{3}{4\pi}\right)^{1/2} \frac{\vec{x}\,'}{r} \cdot \hat{\epsilon}_\mu$$

Quick check

does $Y_{1,1}(\theta',\phi') \overset{?}{=} \left(\frac{3}{4\pi}\right)^{1/2} \left(\frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}\right) \cdot \left(\frac{-\hat{x} - i\hat{y}}{\sqrt{2}}\right)$

$$= -\left(\frac{3}{8\pi}\right)^{1/2} (\sin\theta\cos\phi + i\sin\theta\sin\phi)$$

$$= -\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\phi}$$ ✓ it checks!

Note also that the spherical unit vectors obey:

$$\hat{\epsilon}_\mu^* \cdot \hat{\epsilon}_\nu = \delta_{\mu\nu}$$

Addendum ANY vector in 3D can be written in terms of its spherical (tensor) components as follows:

$$\vec{P} = \sum_{\mu=-1}^{1} \hat{\epsilon}_\mu \left(\hat{\epsilon}_\mu^* \cdot \vec{P}\right)$$

which is formally analogous to the Hilbert Space expansion in quantum mechanics,

$$|\psi\rangle = \sum_\mu |\mu\rangle \left(\langle\mu|\psi\rangle\right)$$

Using this notation, we have

$$q_{11} = \left(\frac{3}{4\pi}\right)^{1/2} \hat{\epsilon}^*_{+1} \cdot \int d^3x' \vec{x}' \rho(\vec{x}')$$

$$= \sqrt{\frac{3}{4\pi}} \, \hat{\epsilon}^*_{+1} \cdot \vec{p}$$

where $\vec{p} \equiv \int d^3x' \, \vec{x}' \rho(\vec{x}')$ is the usual ELECTRIC DIPOLE MOMENT of ρ

e.g. $q_{11} = -\frac{1}{\sqrt{2}}(p_x - ip_y)\left(\frac{3}{4\pi}\right)^{1/2}$

$$q_{10} = \left(\frac{3}{4\pi}\right)^{1/2} \hat{\epsilon}^*_0 \cdot \vec{p} = \left(\frac{3}{4\pi}\right)^{1/2} p_z$$

$$q_{1,-1} = \left(\frac{3}{4\pi}\right)^{1/2} \hat{\epsilon}^*_{-1} \cdot \vec{p} = \left(\frac{3}{4\pi}\right)^{1/2} \frac{p_x + ip_y}{\sqrt{2}}$$

(4) Electric quadrupole moment ($l=2$)

$$q_{22} = \int d^3x' \, r'^2 \, Y^*_{22}(\hat{x}') \rho(\vec{x}')$$

and $Y^*_{22} = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \sin^2\theta' e^{-2i\phi'}$,

which we can also rewrite in terms of Cartesian components as

$$\frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2}\left(\frac{x'-iy'}{r}\right)^2$$

or $q_{22} = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \int (x'-iy')^2 \rho(\vec{x}') \, d^3x'$

It is conventional to write this in terms of a Cartesian quadrupole moment tensor, Q, i.e.

$$q_{22} = \frac{1}{12}\left(\frac{15}{2\pi}\right)^{1/2}\left(Q_{11} - 2iQ_{12} - Q_{22}\right)$$

and similarly $q_{21} = -\left(\frac{15}{8\pi}\right)^{1/2}\int z'(x-iy')\rho(\vec{x}')\,d^3x'$

$$\equiv -\frac{1}{3}\left(\frac{15}{8\pi}\right)^{1/2}\left(Q_{13} - iQ_{23}\right)$$

and

$$q_{20} = \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2}\int\left(3z'^2 - r'^2\right)\rho(\vec{x}')\,d^3x'$$

$$\equiv \frac{1}{2}\left(\frac{5}{4\pi}\right)^{1/2} Q_{33}$$

where

$$Q_{ij} \equiv \int\left(3x_ix_j - r'^2\overset{\text{makes } Q_{ij} \text{ traceless}}{\delta_{ij}}\right)\rho(\vec{x}')\,d^3x'$$

called the electric quadrupole moment tensor

Then the external electric potential beyond the range of ρ can be expanded as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{ij}\frac{\overbrace{x_iQ_{ij}x_j}^{\equiv\,\vec{x}\cdot\underline{Q}\,\vec{x}}}{r^5} + \dots\right.$$

Next, let's work out the electric field of a single l,m multipole, using $\vec{E} = -\nabla\Phi$:

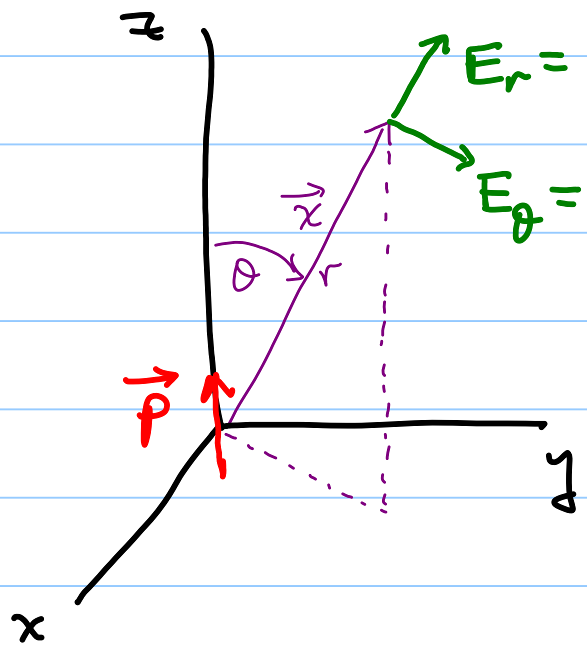
Here, use $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$

$$\Rightarrow \boxed{E_r = \frac{q_{lm}}{\epsilon_0}\,\frac{l+1}{2l+1}\,\frac{Y_{lm}(\hat{x})}{r^{l+2}}} \quad \boxed{E_\theta = \frac{-q_{lm}}{\epsilon_0(2l+1)}\,\frac{1}{r^{l+2}}\,\frac{\partial}{\partial\theta}Y_{lm}(\hat{x})}$$

$$\boxed{E_\phi = \frac{-q_{lm}}{\epsilon_0(2l+1)}\,\frac{im}{r^{l+2}\sin\theta}\,Y_{lm}(\hat{x})}$$

Specialize now to the case of an electric dipole and align it with the z-axis: $\vec{p} = p\,\hat{z}$

$\Rightarrow \ell = 1,\ m = 0$ only



$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \qquad E_\phi = 0$$

We can recast these expressions into a more general form, involving $\vec{x}$ and $\vec{p}$ only.

First observe that $\vec{p} = \hat{r}(\vec{p}\cdot\hat{r}) + \hat{\theta}(\vec{p}\cdot\hat{\theta})$

hence $\hat{\theta} = \dfrac{\vec{p} - \hat{r}(\vec{p}\cdot\hat{r})}{\vec{p}\cdot\hat{\theta}}$

whereby

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left\{2p\cos\theta\,\hat{r} + p\sin\theta\left(\frac{\vec{p} - \hat{r}(\vec{p}\cdot\hat{r})}{\vec{p}\cdot\hat{\theta}}\right)\right\}$$

and by inspection,

$$\vec{p}\cdot\hat{\theta} = p\cos\left(\tfrac{\pi}{2} + \theta\right) = -p\sin\theta\,, \quad \vec{p}\cdot\hat{r} = p\cos\theta$$

which simplifies $\vec{E}$ to the form:

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$

This form is no longer restricted to a dipole oriented along $\hat{z}$!

Or if $\vec{p}$ is located at $\vec{x}_0$, this reads,

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0 |\vec{x} - \vec{x}_0|^5}\left\{3\left[\vec{p}\cdot(\vec{x} - \vec{x}_0)\right](\vec{x} - \vec{x}_0) - \vec{p}\,|\vec{x} - \vec{x}_0|^2\right\}$$

It is worth noting that this expression gives 0 when $\vec{E}$ is integrated over a sphere centered on $\vec{p}$:

$$\text{i.e.} \quad \int_{r<R} \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}\, d^3r = 0$$

This is because the components of $\vec{E}$ are proportional to $Y_{2m}(\theta,\phi)$ and of course $\int Y_{lm}\, d\Omega = 0$ unless $l = m = 0$

$\Rightarrow$ To see this explicitly, we use a few identities to work out the Cartesian components of $\vec{E}$. First, observe that

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

and

$$\hat{\phi} = \frac{\partial\hat{r}}{\partial\phi} \Big/ \left|\frac{\partial\hat{r}}{\partial\phi}\right|$$

$$= -\hat{x}\sin\phi + \hat{y}\cos\phi$$

and $\hat{\theta} = \hat{\phi}\times\hat{r} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$

So for a dipole aligned with $\hat{z}$, i.e. $\vec{p} = p\hat{z}$

$$\Rightarrow 4\pi\epsilon_0 r^3 \vec{E} = 2p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$$

$$= 2p\cos\theta(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)$$

$$+ p\sin\theta(\hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta)$$

or comparing to the $l = 2$ spherical harmonics gives

$$\boxed{\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(\frac{8\pi}{15}\right)^{1/2}\left\{\hat{x}\,\frac{3}{2}\left(-Y_{2,1}+Y_{2,-1}\right)+\hat{y}\,\frac{3}{2i}\left(-Y_{2,1}-Y_{2,-1}\right) + \hat{z}\,\sqrt{2}\,Y_{2,0}\right\}}$$

Comments

(1) In general (see problem 4.4a), all multipoles of rank l HIGHER than the lowest nonzero one (l_{min}) depend on the coordinate origin.

illustration: a charge q at the origin has only one nonvanishing moment, namely

$$q_{0,0} = \frac{q}{\sqrt{4\pi}} = \text{the monopole moment}$$

However a charge q placed at $\vec{x}_0$ has the same monopole moment q_{00}, but also the other $q_{lm} = q\, r_0^{l}\, Y^*_{lm}(\theta_0, \phi_0) \neq 0$ for all l, m, in general.

(2) We have not yet investigated carefully the behavior near $r=0$, i.e. at the singularity. In fact a careful analysis shows that there is an additional δ-contribution, giving

an analogous term is important in atomic hyperfine structure, for magnetic moments

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n})-\vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{x}-\vec{x}_0)\right]$$

Here of course $\hat{n} = \dfrac{\vec{x}-\vec{x}_0}{|\vec{x}-\vec{x}_0|}$

Proof Consider the volume integral of $\vec{E}(\vec{x})$ over a tiny spherical volume, $r<R$

$$\Rightarrow \int_{r<R} \vec{E}(\vec{x})\, d^3r = -\int_{r<R} \nabla\Phi(\vec{x})\, d^3x = \underbrace{-\int R^2\, d\Omega\, \Phi(\vec{x})\,\hat{n}}_{\text{surface integral of radius } R}$$

this identity can be proved by applying the divergence theorem to $\vec{b}\,\Phi(\vec{x})$ where $\vec{b}$ = any constant vector

i.e. consider

$$\int_V \nabla\cdot(\vec{b}\,\Phi(\vec{x}))\,d^3x = \vec{b}\cdot\int\vec{\nabla}\Phi\, d^3x = \oint_S \vec{b}\cdot\hat{n}\,\Phi(\vec{x})\,d^3x$$

So since $\vec{b}\cdot\int\vec{\nabla}\Phi\, d^3x = \vec{b}\cdot\oint_S \hat{n}\,\Phi(\vec{x})\,da$ holds for arbitrary constant $\vec{b}$

$$\Rightarrow \boxed{\int\nabla\Phi(\vec{x})\,d^3x = \oint_S \Phi(\vec{x})\,\hat{n}\,da}$$

And returning to our above derivation, we have

$$\int_{r<R}\vec{E}(\vec{x})\,d^3x = \frac{-R^2}{4\pi\epsilon_0}\int d^3x'\,\rho(\vec{x}')\int_{r=R} d\Omega\,\frac{\hat{n}}{|\vec{x}-\vec{x}'|}$$

(i.e. $|\vec{x}|=R$ on surface)

and $\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

then recall $\hat{n} = \left(\frac{4\pi}{3}\right)^{1/2}\sum_{\mu} Y^*_{1\mu}(\hat{x})\,\hat{\epsilon}_{\mu}$

$$\Rightarrow \int \frac{\hat n \, d\Omega}{|\vec x - \vec x'|} = \int d\Omega \sum_{lm} \frac{4\pi}{2l+1} \left(\frac{4\pi}{3}\right)^{1/2} \sum_\mu \underbrace{Y^*_{1\mu}(\hat x) \hat\epsilon_\mu \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\hat x') Y_{lm}(\hat x)}_{\delta_{l,1}\,\delta_{m,\mu}}$$

$$= \underbrace{\frac{4\pi}{3}\left(\frac{4\pi}{3}\right)^{1/2} \sum_\mu Y^*_{1\mu}(\hat x') \hat\epsilon_\mu}_{= \hat n'} \frac{r_<}{r_>^2}$$

and so we see that

$$\int_{r<R} \vec E(\vec x) d^3r = \left(-\frac{4}{3}\pi R^2\right) \frac{1}{4\pi\epsilon_0} \int \frac{r_<}{r_>^2} \rho(\vec x') \hat n' d^3x'$$

Thus if <u>all</u> $\rho(\vec x')$ is INSIDE the sphere, then $r_< = r'$, $r_> = R$, whereby

$$\int \vec E(\vec x) d^3x = -\frac{\left(\frac{4}{3}\pi R^2\right)}{4\pi\epsilon_0 R^2} \overset{= \vec p}{\left(\int \rho(\vec x') \vec x' d^3x'\right)}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{4\pi}{3} \vec p,$$

this is <u>independent</u> of the sphere radius R provided R is outside of the charges ρ

In conclusion, for a dipole $\vec p$ at $\vec x' = 0$,

$$\boxed{\vec E_{dipole}(\vec x) = \frac{3(\vec p \cdot \hat r)\hat r - \vec p}{4\pi\epsilon_0 r^3} - \frac{\vec p}{3\epsilon_0}\delta(\vec x)}$$

Corollary to the preceding derivation

Consider a different limit now, where <u>all</u> charge $\rho(\vec{x}')$ lies OUTSIDE the sphere,

$\Rightarrow$ the same math applies <u>except</u> now we have $r_< = R$, $r_> = r'$

$$\Rightarrow \int_{r<R} \vec{E}(\vec{x})\,d^3x = -\frac{4\pi}{3}R^3 \boxed{\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')\,\hat{n}'\,d^3x'}{r'^2}}$$

but this is precisely $\vec{E}(0)$!

$$\Rightarrow \boxed{\int_{r<R} \vec{E}(\vec{x})\,d^3x = \frac{4}{3}\pi R^3\,\vec{E}(0)}$$

this holds provided NO CHARGE lies inside the sphere

Multipole expansion of the interaction energy of $\rho(\vec{x})$ with an external Field

The interaction energy with the external $\Phi(\vec{x})$ is:

$$W = \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x$$

Suppose now that the potential $\Phi(\vec{x})$ is slowly varying throughout the region where $\rho(\vec{x}) \neq 0$, we can Taylor expand $\Phi(\vec{x})$ about a suitable origin inside ρ:

$$\Phi(\vec{x}) = \Phi(0) + \vec{x}\cdot\vec{\nabla}\Phi(0) + \frac{1}{2}\sum_{ij} x_i \frac{\partial^2\Phi(0)}{\partial x_i \partial x_j} x_j + \ldots$$

$$= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2}\sum_{ij} x_i x_j \frac{\partial E_j(0)}{\partial x_i} + \ldots$$

$$= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j(0)}{\partial x_i} + \ldots$$

$\Rightarrow$ In that last term we have added a term

$$0 = \frac{r^2}{6}\nabla\cdot\vec{E},$$

in order to simplify the final expression.

Next, plugging this into the above energy expression gives the following:

$$W = q_{tot}\,\Phi(0) - \vec{E}(0)\cdot\int \vec{x}\,\rho(\vec{x})\,d^3x$$
$$- \frac{1}{6}\sum_{ij}\frac{\partial E_j(0)}{\partial x_i}\int\left(3x_i x_j - r^2\delta_{ij}\right)\rho(\vec{x})\,d^3x$$

or, more compactly,

$$W = q_{tot}\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\left[\nabla\cdot\left(\underline{Q}\,\vec{E}(\vec{x})\right)\right]_0 + \ldots$$

where $\underline{Q}$ = constant

= electric quadrupole Cartesian matrix.

e.g. 2 dipoles have:

$$W_{12} = \frac{\left[\vec{p}_1\cdot\vec{p}_2 - 3(\hat{n}\cdot\vec{p}_1)(\hat{n}\cdot\vec{p}_2)\right]}{4\pi\epsilon_0 r^3} + \frac{\vec{p}_1\cdot\vec{p}_2}{3\epsilon_0}\,\delta(\vec{x}_1-\vec{x}_2)$$

Electrostatics in macroscopic media:

- how matter affects or is affected by fields.

e.g - why does light pass through a window, but not through wood?

Some challenging aspects:

- Matter is mostly neutral, but it consists of charged particles

- Matter is rearranged, or polarized by external fields

- The polarization (usually expressed as induced dipole moments) produces "internal fields"

- The total field = sum of external + internal fields

- We must average over a "microsopically large but macroscopically small" volume, i.e. over a volume that might contain $\simeq 1000$ atoms

e.g., call $\vec{E}_{micro}(\vec{x})$ the primitive or microscopic E-field that obeys Maxwell's equations in free space and includes ALL charges in the universe

picture this as fluctuating wildly in space + time on the atomic scale, e.g. inside a solid.

$\Rightarrow$ We want to find equations obeyed by a coarse-grained or averaged E-field,

$$\Rightarrow \boxed{\vec{E}(\vec{x}) = \frac{\int_{|\vec{x}-\vec{x}'|<R} d^3x' \, \vec{E}_{micro}(\vec{x}')}{\frac{4}{3}\pi R^3}}$$

Point (1) Since $\nabla \times \vec{E}_{micro} = 0$ everywhere in electrostatics, clearly $\underline{\nabla \times \vec{E} = 0}$ still after averaging

Hence $\vec{E}$ is still derivable from a potential

i.e. $\boxed{\vec{E} = -\nabla \Phi}$

Point (2) Next, determine the internal fields produced by electric dipole moments $\vec{p}$ induced in the medium. (Assume that higher multipoles only produce small corrections.)

$\Rightarrow$ Start from $\Phi(\vec{x}) = \dfrac{\vec{p} \cdot (\vec{x} - \vec{x}')}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|^3}$

for a dipole at $\vec{x}'$. We can rewrite this in differential form, for the potential $d\Phi$ produced by a dipole $d\vec{p}$ at $\vec{x}'$:

$$d\Phi(\vec{x}) = \frac{d\vec{p}}{4\pi\epsilon_0} \cdot \nabla' \frac{1}{|\vec{x} - \vec{x}'|} = \frac{-d\vec{p}}{4\pi\epsilon_0} \cdot \nabla \frac{1}{|\vec{x} - \vec{x}'|}$$

So if we have a continuous average distribution of dipole moment, we write $d\vec{p}(\vec{x}') = \vec{P}(\vec{x}') \, d^3x'$ where we have defined

$$\vec{P}(\vec{x}') \equiv \text{polarization} = \frac{\text{average dipole moment}}{\text{unit volume}}$$

$$= \sum_i N_i \langle \vec{p}_i \rangle$$

where N_i = number density of the i^{th} type of molecule, having an average electric dipole moment, $\langle \vec{p}_i \rangle$

Of course, in addition to the dipoles that represent "bound charge", we could also have some "additional" or "free" charge present, ρ_{free}

But for now, consider only the dipoles.

$$\Rightarrow \quad \Phi(\vec{x}) = \int_V \frac{\vec{P}(\vec{x}')}{4\pi\epsilon_0} \cdot \nabla' \frac{1}{|\vec{x}-\vec{x}'|} d^3x'$$

or recalling the identity:

$$\nabla' \cdot \left(\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) = \frac{1}{|\vec{x}-\vec{x}'|} \nabla' \cdot \vec{P}(\vec{x}') + \vec{P}(\vec{x}') \cdot \nabla' \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \nabla' \cdot \left(\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) d^3x' - \int_V \frac{d^3x'}{4\pi\epsilon_0} \frac{\nabla' \cdot \vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

and from the divergence theorem, of course, this becomes

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\left(-\nabla' \cdot \vec{P}(\vec{x}')\right) d^3x'}{(\vec{x}-\vec{x}')} + \frac{1}{4\pi\epsilon_0} \oint_S \frac{\vec{P}(\vec{x}') \cdot \hat{n}' da'}{|\vec{x}-\vec{x}'|}}$$

Interpretation These 2 integrals now have the SAME FORM as for the potential produced by a volume "bound charge" $\rho_b \equiv -\nabla' \cdot \vec{P}(\vec{x}')$ and/or a surface bound charge $\sigma_b \equiv \vec{P}(\vec{x}') \cdot \hat{n}'$

where "b" stands for "bound charges"

Thus, interpreted this way, our old formulas for $\Phi(\vec{x})$ still work in the presence of media, namely

$$\Phi(\vec{x}) = \int_V \frac{\rho(\vec{x}')\, d^3x'}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|} + \int_S \frac{\sigma(\vec{x}')\, da'}{4\pi\epsilon_0 |\vec{x} - \vec{x}'|}$$

provided we use:

$$\rho(\vec{x}') = \rho_{free}(\vec{x}') + \rho_b(\vec{x}') \xleftarrow{\quad} -\nabla' \cdot \vec{P}(\vec{x}')$$

and

$$\sigma(\vec{x}') = \sigma_{free}(\vec{x}') + \sigma_b(\vec{x}') \xleftarrow{\quad} \vec{P}(\vec{x}') \cdot \hat{n}'$$

And since this equation has the same form for $\Phi(\vec{x})$ as before (without media), we know:

$$-\nabla^2 \Phi(\vec{x}) = \nabla \cdot \vec{E} = \frac{4\pi}{4\pi\epsilon_0}(\rho_{free} + \rho_b)$$

or $\nabla \cdot \vec{E} + \frac{1}{\epsilon_0} \nabla \cdot \vec{P} = \frac{\rho_{free}}{\epsilon_0}$

Or defining the ELECTRIC DISPLACEMENT as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P},$$

we obtain the appropriate Maxwell equation in the presence of macrosopic media,

$$\boxed{\nabla \cdot \vec{D} = \rho_{free}} \Rightarrow \boxed{\oint_S \vec{D} \cdot d\vec{a} = Q^{free}_{enclosed}}$$

This, in addition to $\boxed{\nabla\times\vec{E}=0,}$ defines the electrostatic field.

Generally, we cannot solve the Maxwell equations

$$\nabla\cdot\vec{D}=\rho$$

$$\nabla\times\vec{E}=0$$

unless we know how $\vec{D}$ and $\vec{E}$ are related.

Often, it is a good approximation to assume, especially for weak fields, that the induced polarization is proportional to $\vec{E}$, i.e. $\vec{P}=\chi_e\epsilon_0\vec{E}$

where χ_e is the electric susceptibility of the medium = CONSTANT for an isotropic medium.

$\Rightarrow$ $\vec{D}=\epsilon\vec{E}$ => $\epsilon=\epsilon_0(1+\chi_e)$ is called the DIELECTRIC CONSTANT

An Important Special Case

Often ϵ = constant within some region. When this is true, we simply have

$$\nabla\cdot\vec{E}=\rho/\epsilon$$ WITHIN THAT REGION,

while different regions are connected by BOUNDARY CONDITIONS, namely

$$\boxed{(\vec{D}_2-\vec{D}_1)\cdot\hat{n}_{2\leftarrow 1}=\sigma_F}\quad\boxed{(\vec{E}_2-\vec{E}_1)\times\hat{n}_{2\leftarrow 1}=0}$$

Note $\vec{P}$ is often NOT proportional to $\vec{E}$, notably:

(i) in ferroelectric materials, where $\vec{P}$ can be nonzero even when $\vec{E}=0$

or

(ii) for strong E-field where $\vec{P}$ can be a more complicated, nonlinear function of E, e.g. in nonlinear optics

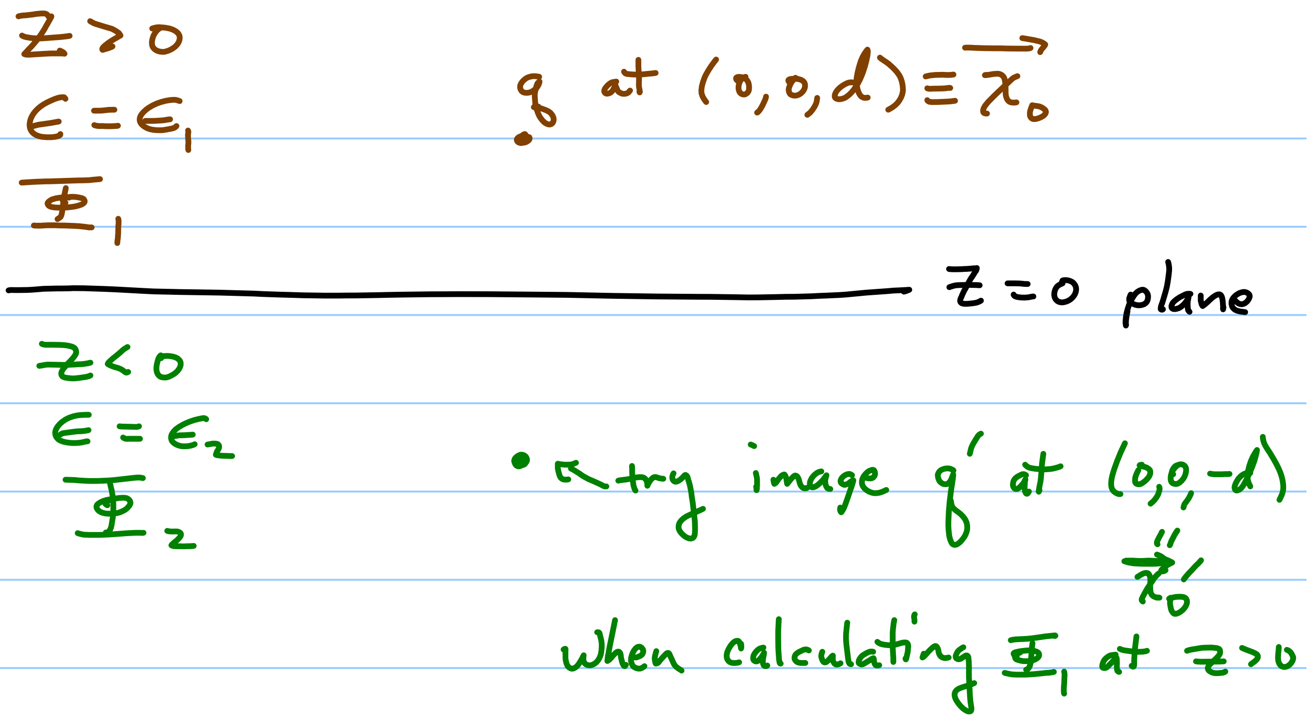
There is another important class of problems where the relationship between $\vec{P}$ and $\vec{E}$ is LINEAR but ANISOTROPIC, as in a birefringent crystal like calcite. For such a case, we can write:

$$\vec{P} = \epsilon_0 \underline{\chi_e} \vec{E}$$

where $\underline{\chi_e}$ is a 3x3 matrix rather than a scalar.

Boundary Value Problems with Dielectrics

Category I Problems solvable using image charge methods: e.g. a point charge q near a planar interface between 2 dielectrics.



Then a plausible hypothesis is that $\Phi(\vec{x})$ can be written as follows, for some q',

$$\Phi_1 = \frac{q}{4\pi\epsilon_1 |\vec{x} - \vec{x}_0|} + \frac{q'}{4\pi\epsilon_1 |\vec{x} - \vec{x}_0'|}$$

and another plausible guess/ansatz is that Φ_2 can be written in the form:

$$\Phi_2 = \frac{1}{4\pi\epsilon_2} \frac{q''}{|\vec{x} - \vec{x}_0|}$$

Recall that it is not allowed to put an image charge into the volume of interest!

Next, attempt to match the B.C.s for some choice of q', q'', using $\frac{1}{|\vec{x}-\vec{x}_0|} = [x^2+y^2+(z-d)^2\}^{-1/2}$

$$\frac{1}{|\vec{x}-\vec{x}_0'|} = [x^2+y^2+(z+d)^2]^{-1/2}$$

Since there are no free surface changes, the BCs are

$$\epsilon_1 E_z\Big|_{z=0^+} = \epsilon_2 E_z\Big|_{z=0^-}$$

and

$$E_x\Big|_{z=0^+} = E_x\Big|_{z=0^-}$$

i.e. $-\epsilon_1 \frac{\partial \Phi_1}{\partial z}(x,y,0) = -\epsilon_2 \frac{\partial \Phi_2}{\partial z}(x,y,0)$

$$\Rightarrow \frac{-\epsilon_1}{4\pi\epsilon_1}\left\{\frac{+qd}{(\rho^2+d^2)^{3/2}} - \frac{q'd}{(\rho^2+d^2)^{3/2}}\right\} = \frac{-\epsilon_2}{4\pi\epsilon_2}\frac{q''d}{(\rho^2+d^2)^{3/2}}$$

and $-\frac{\partial \Phi_1}{\partial x}(x,y,0) = -\frac{\partial \Phi_2}{\partial x}(x,y,0)$

or

$$-\frac{1}{4\pi\epsilon_1}\left[\frac{-qx}{(\rho^2+d^2)^{3/2}} - \frac{q'x}{(\rho^2+d^2)^{3/2}}\right] = -\frac{1}{4\pi\epsilon_2}\frac{(-q''x)}{(\rho^2+d^2)^{3/2}}$$

and simplifying gives:

$$\begin{cases} q - q' = q'' \\ \dfrac{q+q'}{\epsilon_1} = \dfrac{q''}{\epsilon_2} \end{cases}$$

2 equations & 2 unknowns to solve

and the solution is $\boxed{q' = -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right)q, \quad q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}q}$

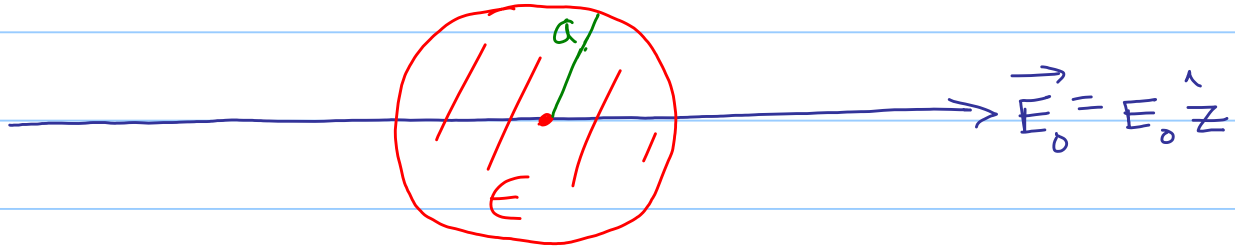
On your own, verify the following:

(1) The potential is <u>continuous</u> at the boundary ⇒ we could have imposed this as a requirement

(2) Compute the polarization surface charge density, $\sigma_b = \vec{P} \cdot \hat{n}$ ← Jackson calls this σ_{pol}

<u>Category II</u> – Boundary value problem solved by separation of variables

e.g. a dielectric sphere in a uniform E-field



⇒ Here we have AZIMUTHAL SYMMETRY and NO FREE CHARGES, $\rho_{free} = 0$

$$\Rightarrow \Phi_{in} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Phi_{out} = \sum_\ell \left(B_\ell r^\ell + \frac{C_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Boundary Conditions (1) $\lim_{r\to\infty} \Phi_{out} = -E_0 z = -E_0 r P_1(\cos\theta)$

i.e. $\boxed{B_1 = -E_0}$ and all other $B_\ell = 0$ for $\ell \neq 1$

(2) $E_{\parallel}$ is continuous $\Rightarrow -\frac{1}{r}\frac{\partial \Phi_{in}}{\partial\theta}\Big|_{r=a} = -\frac{1}{r}\frac{\partial \Phi_{out}}{\partial\theta}\Big|_{r=a}$

$$\Rightarrow A_\ell a^\ell = \frac{C_\ell}{a^{\ell+1}} \text{ if } \ell \neq 1, \text{ and } A_1 a = B_1 a + \frac{C_1}{a^2}$$

(3) $\epsilon E_\perp(in) = \epsilon_0 E_\perp(out)$

$$\Rightarrow -\epsilon \frac{\partial \Phi_{in}}{\partial r}\Big|_{r=a} = -\epsilon_0 \frac{\partial \Phi_{out}}{\partial r}\Big|_{r=a}$$

$$\Rightarrow -\epsilon \ell A_\ell a^{\ell-1} = -\epsilon_0(-\ell-1) C_\ell a^{-\ell-2}, \quad \ell \neq 1$$

and $-\epsilon A_1 = -\epsilon_0\left(-E_0\right) + \frac{2C_1}{a^2}, \quad \ell = 1$

These are 2 equations, 2 unknowns for each ℓ, whose solutions are:

$$A_\ell = C_\ell = 0, \quad \ell \neq 1$$

$$A_1 = \frac{-3E_0}{2 + \frac{\epsilon}{\epsilon_0}}, \quad C_1 = \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) a^3 E_0$$

$\Rightarrow$ the final solution is

$$\Phi_{in}(r,\theta) = -\frac{3}{2 + \frac{\epsilon}{\epsilon_0}} E_0 r \cos\theta, \quad r < a$$

$$\Phi_{out}(r,\theta) = -E_0 r\cos\theta + \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0 \frac{a^3}{r^2}\cos\theta, \quad r>a$$

Note that here again, Φ is continuous at a.

$$\Rightarrow \vec{E}_{inside} = \frac{3}{\frac{\epsilon}{\epsilon_0} + 2} E_0 \hat{z}$$

(observe that E_{inside} is smaller than E_0, if $\epsilon > \epsilon_0$)

and $\vec{E}_{outside} = E_0 \hat{z} + \vec{E}_{dipole}$

where

$\vec{E}_{dipole}$ is the field due to a dipole of magnitude $\vec{p} = \hat{z}\, 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{\frac{\epsilon}{\epsilon_0} + 2}\right) a^3 E_0$

i.e. $\vec{p} = \int d^3x\, \vec{P}(x)$ where
$$\vec{P}(\vec{x}) = (\epsilon - \epsilon_0)\vec{E} = 3\epsilon_0 \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2}\right)\vec{E}_0$$

or more directly using $\vec{E}_{out} = -\vec{\nabla}\Phi_{out}$,

$$\vec{E}_{out} = E_0\hat{z} - \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0\, a^3\, \nabla\left(\frac{\cos\theta}{r^2}\right)$$

$$= E_0\hat{z} - \left(\frac{\frac{\epsilon}{\epsilon_0} - 1}{\frac{\epsilon}{\epsilon_0} + 2}\right) E_0\, a^3 \left(-\frac{\cos\theta\, \hat{r}}{r^3}(2) - \frac{\sin\theta\ \hat{\theta}}{r^3}\right)$$

Compare with Eq. 4.12

Observe that the bound (polarization) charge density is 0 throughout the volume $r<a$, since there $\vec{P}(\vec{x})$ = constant,

$$\Rightarrow -\vec{\nabla}\cdot\vec{P}(\vec{x}) = 0$$

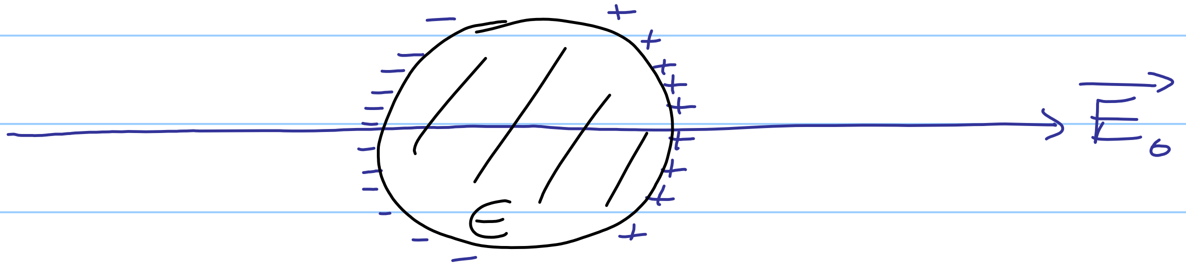
However, a surface bound charge remains,

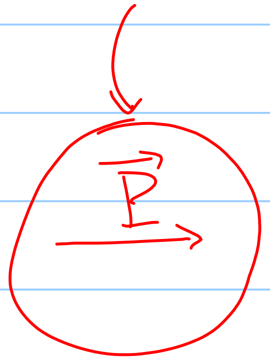
$$\sigma_b = \vec{P}\cdot\hat{r} = 3\epsilon_0\left(\frac{\frac{\epsilon}{\epsilon_0}-1}{\frac{\epsilon}{\epsilon_0}+2}\right)E_0\cos\theta$$

Comment The same math would apply to a spherical CAVITY in a dielectric medium, except that we must interchange $\epsilon \longleftrightarrow \epsilon_0$ everywhere!

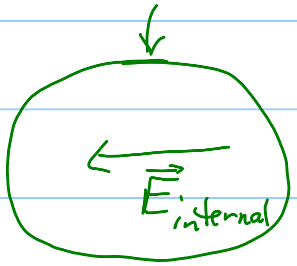
e.g. $\vec{E}_{in} = \frac{3\epsilon}{2\epsilon+\epsilon_0}\vec{E}_0$, which is $> E_0$ if $\epsilon > \epsilon_0$

Qualitative Picture for a dielectric sphere



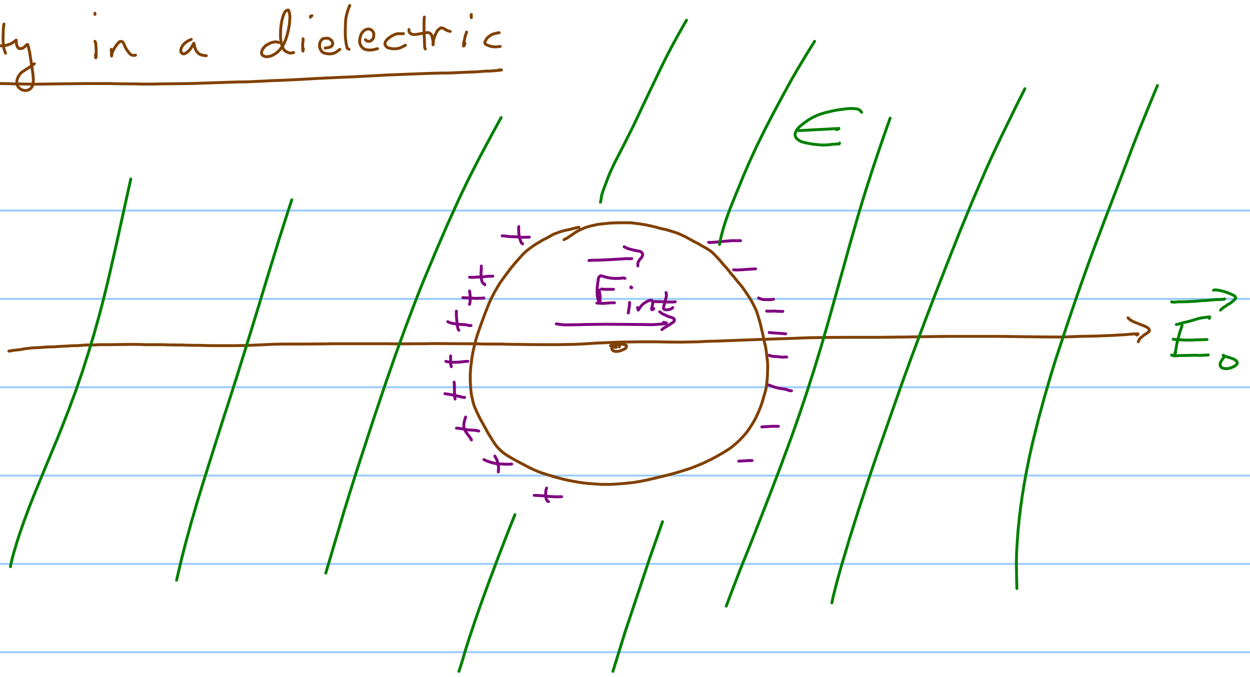


polarization induced by $\vec{E}_0$



$\vec{E}$-field induced by bound polarization charges σ_b REDUCES $\vec{E}$ when added to $\vec{E}_0$

Cavity in a dielectric



Now, for a cavity, the $\vec{E}$-field due to σ_b INCREASES the field higher than what it is in the dielectric far from the cavity.

IMPORTANT: When there are no free charges in a medium, we expect $\rho_b = -\nabla\cdot\vec{P} = 0$ since $\nabla\cdot\vec{E} = \frac{\rho_f}{\epsilon} \to 0$ and $\vec{P} = \epsilon_0\chi_e\vec{E}$

Microscopic Theory of Polarization

↪ The Clausius-Mossotti equation

1^{st}-order perturbation theory in quantum physics can be applied to an atom or molecule placed into an external electric field we will call $\vec{E}_{other}$, to calculate the induced electric dipole moment, written as $\vec{p} = \epsilon_0 \gamma_{mol} \vec{E}_{other}$

where γ_{mol} = molecular polarizability = dimensions = $(\text{length})^3$

IMPORTANT: $\vec{E}_{other}$ omits the $\vec{E}$-field due to the molecule itself

e.g. For a 1-electron atom or molecule, the perturbation term in the Hamiltonian is

$$H' = e E_{other} z \quad (\text{if } \vec{E}_{other} \text{ is along } \hat{z})$$

Then the 1^{st}-order perturbed wavefunction is

$$|\psi_{gnd}^{(1)}\rangle = |\psi_{gnd}^{(0)}\rangle + \sum_{k \neq gnd} |\psi_k^{(0)}\rangle \frac{\langle \psi_k^{(0)} | e E_{other} z | \psi_{gnd}^{(0)}\rangle}{E_{gnd}^{(0)} - E_k^{(0)}}$$

And the 1^{st}-order electric dipole moment is

$$p_z = -\langle \psi_{gnd}^{(1)} | e z | \psi_{gnd}^{(1)} \rangle$$

or

$$p_z = \underbrace{\left(\frac{2e^2}{\epsilon_0} \sum_{k \neq gnd} \frac{|\langle \psi_k^{(0)} | z | \psi_{gnd}^{(0)} \rangle|^2}{E_k^{(0)} - E_{gnd}^{(0)}} \right)}_{\equiv \gamma_{mol}} \epsilon_0 E_{other} ,$$

⇒ This is the microscopic induced dipole moment.

On the other hand, we have the macroscopic relation in the dielectric, $\vec{P} = \epsilon_0 \chi_e \vec{E}$

$$\boxed{\text{OUR MISSION: relate } \chi_e \text{ to } \gamma_{mol}}$$

where $\vec{E}$ = the TOTAL macroscopic field

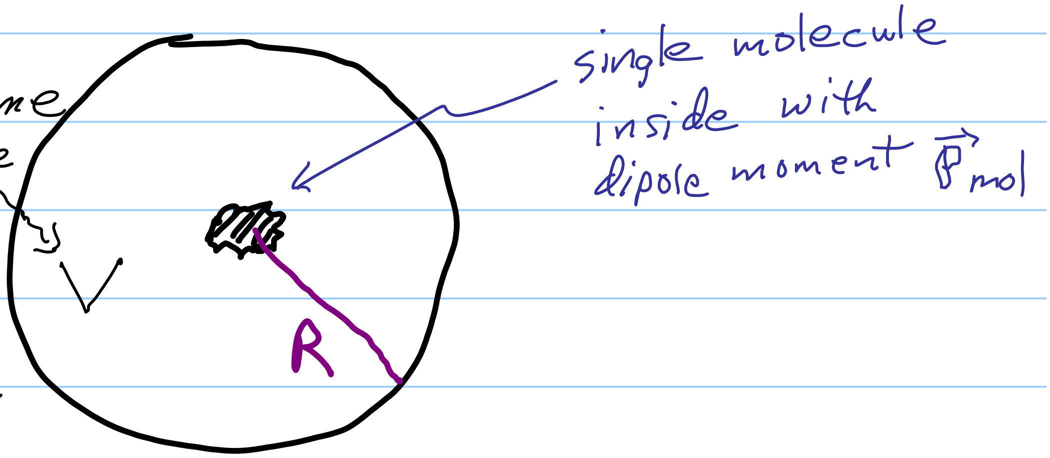
$\Rightarrow$ Let's try to keep separate the 2 contributions by writing $\vec{E} = \vec{E}_{other} + \vec{E}_{self}$

Consider a spherical volume V of radius R that is centered on, and includes, a single molecule

V = average volume per molecule

$\equiv \frac{1}{N}$, where

N = molecular number density



We need to find out the average field produced by the molecule itself, averaged over its own sphere.

$\Rightarrow$ Call this

$$\vec{E}_{self} = \left\langle \vec{E}^{microscopic}_{self} \right\rangle = \frac{1}{V} \int_V \vec{E}^{microscopic}_{self} \, d^3x$$

$$= \frac{1}{V} \int_V \left(-\frac{\vec{p}_{mol}}{3\epsilon_0} \delta(\vec{x}) \right) d^3x = -\frac{\vec{p}_{mol}}{V} \frac{1}{3\epsilon_0}$$

or $\vec{E}_{self} = -\frac{N}{3\epsilon_0}\vec{p}_{mol} = -\frac{\vec{P}}{3\epsilon_0}$, recalling that POLARIZATION is the average dipole moment per unit volume

Now, at the same time, a linear dielectric obeys

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \epsilon_0 \chi_e \left(\vec{E}_{other} + \vec{E}_{self}\right) \quad (*)$$

$$" \quad = -3\epsilon_0 \vec{E}_{self}$$

But we also know that $\vec{E}_{self} = -\frac{1}{V}\frac{1}{3\epsilon_0}\vec{p}_{mol}$

and $\vec{p}_{mol} = \epsilon_0 \gamma_{mol} \vec{E}_{other}$

$$\Rightarrow \boxed{\vec{E}_{self} = -\frac{1}{3}\gamma_{mol} N \vec{E}_{other}}$$

plug in here

Plugging this last into (*) gives

$$\cancel{\epsilon_0}\chi_e\left(\cancel{\vec{E}_{other}} - \frac{1}{3}\gamma_{mol} N \cancel{\vec{E}_{other}}\right) = \left(-3\cancel{\epsilon_0}\right)\left(-\frac{\gamma_{mol}}{3}N\right)\cancel{\vec{E}_{other}}$$

or

$$\chi_e\left(1 - \frac{1}{3}\gamma_{mol} N\right) = \gamma_{mol} N$$

giving finally

$$\boxed{\chi_e = \frac{\gamma_{mol} N}{1 - \frac{1}{3}\gamma_{mol} N}}$$

or the more common form, called the CLAUSIUS-MOSSOTTI equation, is

$$\boxed{\gamma_{mol} = \frac{3}{N}\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right)}$$

Although our model/"derivation" was admittedly rather crude, this is not a bad approximation for dilute gases. It is not quantitatively accurate for liquids or solids.

A crude microscopic model of γ_{mol}

A macroscopic polarization can occur via two separate mechanisms:

(I) $\vec{E}_{other}$ distorts the charge distribution in matter and induces dipoles $\vec{p}_{mol}$

or (II) For polar molecules with an intrinsic dipole moment $\vec{p}_0$, when $\vec{E}_{other}=0$, the dipole orientations are RANDOM, $\Rightarrow \langle\vec{p}\rangle = 0$.
But when $\vec{E}_{other} \neq 0$, dipoles tend to orient along $\vec{E}_{other} \Rightarrow \langle\vec{p}\rangle \neq 0$.

Case I Induced dipole moment - a model

Imagine crudely that an atomic or molecular electron is bound in a harmonic oscillator potential, with restoring force $\vec{F} = -m\omega_0^2\vec{x}$

$\Rightarrow$ In an external field $\vec{E}_0$, the oscillator stretches until the forces balance on $q=-e$.

$$\text{i.e. } q\vec{E}_0 - m\omega_0^2\vec{x} = 0$$

$$\Rightarrow \vec{p}_{mol} = q\vec{x} = \frac{q^2}{m\omega_0^2}\vec{E}_0$$

But $\vec{p}_{mol} = \epsilon_0 \gamma_{mol} \vec{E}_0 \Rightarrow \boxed{\gamma_{mol} = \frac{e^2}{\epsilon_0 m\omega^2}}$

We can test this against some numerical values: i.e. for a typical gas molecule an electronic excitation energy might be about

$$\hbar\omega_0 \simeq 4 eV \times 1.6\times10^{-19} \frac{J}{eV} = 6.4\times10^{-19} J$$

and $$\omega_0 = \frac{6.4\times10^{-19} J}{1.054\times10^{-34} J\text{-}s} \simeq 6\times10^{15} \text{ rad/sec}$$

and we would thus predict that in this model,

$$\gamma_{mol} \simeq \frac{(1.6\times10^{-19} C)^2}{\left(8.85\times10^{-12}\frac{C^2}{N\text{-}m^2}\right)\left(9.1\times10^{-31} kg\right)\left(6\times10^{15}\frac{rad}{s}\right)^2}$$

$$\simeq \underset{\sim\sim\sim\sim\sim\sim\sim}{8.6\times10^{-29} m^3}$$

← Jackson's numerics give one order of magnitude smaller, in an apparent numerical error — see p. 163.

Or setting $\left(\frac{\gamma_{mol}}{\frac{4}{3}\pi}\right)^{1/3} \simeq R \simeq 5 Å$, we see that γ_{mol} is the right order-of-magnitude to be interpreted as the "molecular volume"

Experiment for χ_e and ϵ/ϵ_0?

Let's use $N \simeq 2.5\times10^{25} \frac{\text{molecules}}{m^3}$ @ NTP, (20°C, 1 atm)

$\Rightarrow \chi_e \simeq N\gamma_{mol} \simeq 0.0022$ from the Clausius-Mossotti eqn

$\Rightarrow \frac{\epsilon}{\epsilon_0} \simeq 1.0022$, versus expt for air @ NTP, which is $\left.\frac{\epsilon}{\epsilon_0}\right|_{expt} = 1.00054$

On the other hand condensed liquids or solids have densities more like

$$N \simeq 10^{28} - 10^{29} \ m^{-3}$$

$\Rightarrow$ χ_e can be of order 1-10, which is the correct order of magnitude for condensed matter.

(II) Orientation of permanent moments:

In a field $\vec{E}_0$ the energy of a permanent dipole $\vec{p}_0$ is $\quad H = H_0 - \vec{p}_0 \cdot \vec{E}_0$

And according to stat. mech., different configurations occur with relative probabilities governed by the Boltzman factor,

$$f(H) = e^{-H/kT}$$

$$\Rightarrow \quad \langle p_z \rangle = \frac{\int_{-1}^{1} d(\cos\theta)\, p_0 \cos\theta \, e^{p_0 E_0 \cos\theta / kT}}{\int_{-1}^{1} d(\cos\theta)\, e^{p_0 E_0 \cos\theta / kT}}$$

These integrals are readily evaluated, giving

$$\langle p_z \rangle = -\frac{kT}{E_0} + p_0 \coth\left(\frac{p_0 E_0}{kT}\right)$$

$$\xrightarrow[p_0 E_0 << kT]{} \frac{1}{3} \frac{p_0^2 E_0}{kT}$$

(Look at Jackson Fig. 4.10)

Electrostatic Energy in Dielectrics

Recall from elementary physics:

A capacitor connected to a battery at voltage V can store MORE energy if you add a dielectric between its plates, since

$$C \rightarrow K C_0 \quad \text{with} \quad K = \frac{\epsilon}{\epsilon_0} = \underset{\text{constant}}{\text{dielectric}} > 1$$

and

$$W = \frac{1}{2} C V^2 \longrightarrow \frac{1}{2} K C_0 V^2$$

We would like to derive this result now, and generalize it, starting from the primitive definition for energy of a charge distribution ρ in free space:

$$W = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x$$

where $\rho(\vec{x})$ is the SAME distribution of charges that PRODUCES $\Phi(\vec{x})$

However, it is far from obvious at the outset whether ρ is the TOTAL charge, or FREE charge, or something else.

To analyze this more carefully, consider the change δW in the energy W caused by bringing up a tiny additional FREE charge density $\delta\rho_f$, i.e. determine $W \longrightarrow W+\delta W$ as $\rho_f \to \rho_f + \delta\rho_f$, and note that Jackson's notation for ρ_f is just ρ.

$$\Rightarrow \delta W = \int \delta\rho_f(\vec{x})\, \Phi(\vec{x})\, d^3x$$

But since $\nabla\cdot\vec{D} = \rho_f \Rightarrow \vec{\nabla}\cdot\delta\vec{D} = \delta\rho_f$

So we can write $\delta W = \int \Phi\, \nabla\cdot(\delta\vec{D})\, d^3x$

But using the identity $(\nabla\cdot\delta\vec{D})\,\Phi = \nabla\cdot(\Phi\delta\vec{D}) - \nabla\Phi\cdot\delta\vec{D}$

$$\Rightarrow \delta W = \underbrace{\oint_S \Phi\, \delta\vec{D}\cdot\hat{n}\, da}_{\to 0 \text{ if } S\to\infty} + \int \vec{E}\cdot\delta\vec{D}\, d^3x$$

So $\boxed{\delta W = \int_V \vec{E}\cdot\delta\vec{D}\, d^3x}$ is a generally valid result

or $W = \int d^3x \left(\int_0^{\vec{D}} \vec{E}\cdot\delta\vec{D} \right)$

where that last integral MAY depend on path, e.g. if there is hysteresis.

For a LINEAR DIELECTRIC MEDIUM, simpler expression obtain, since then $\delta\vec{D} = \epsilon\,\delta\vec{E}$ and $\vec{E}\cdot\delta\vec{D} = \frac{1}{2}\delta(\vec{E}\cdot\vec{D})$

giving

$$\delta W = \delta\left[\frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3x\right]$$

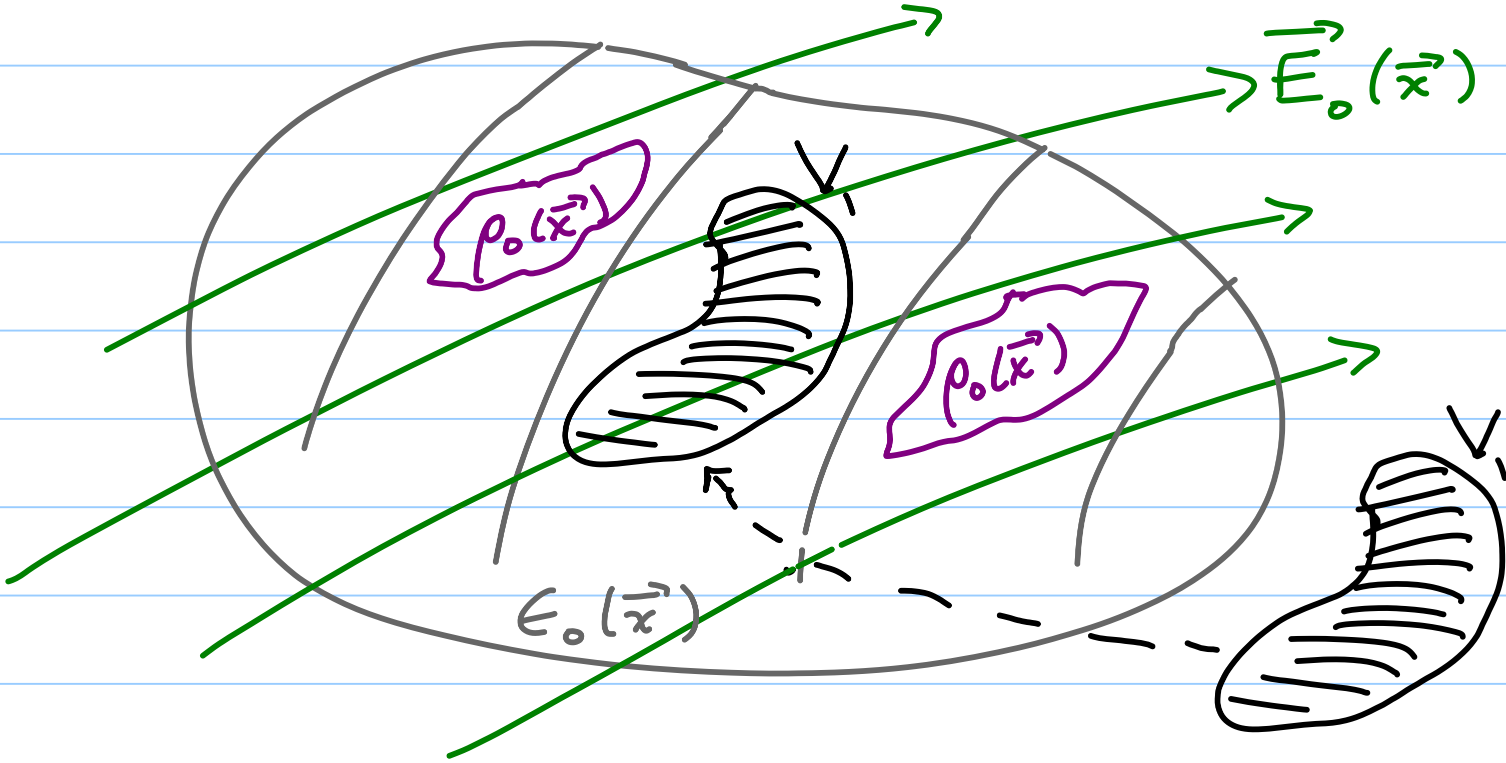
and $$\boxed{W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3x}$$ ← FOR LINEAR MEDIA ONLY!

Next, consider the change in energy when a dielectric is placed into an electric field created by fixed sources.

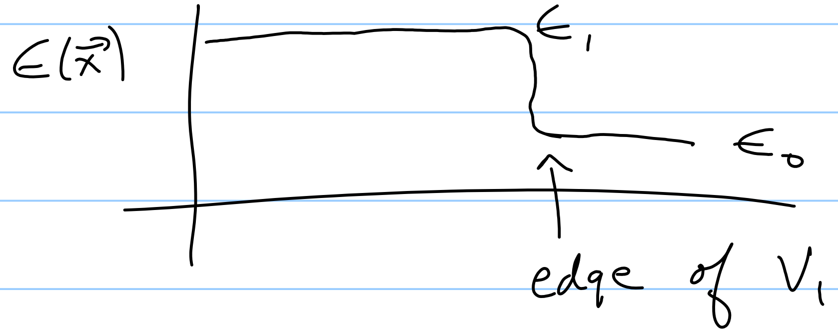
⟹ Suppose a field $\vec{E}_0(\vec{x})$ is produced by some charge distribution $\rho_0(\vec{x})$ in a medium of dielectric constant $\epsilon_0(\vec{x})$ not necessarily vacuum.

⟹ the initial energy is

$$W_0 = \frac{1}{2}\int \vec{E}_0\cdot\vec{D}_0\, d^3x$$



Now bring in an object of volume V_1 having dielectric constant $\epsilon_1(\vec{x}) = \begin{cases} \epsilon_1(\vec{x}) \text{ inside } V_1 \\ \epsilon_0(\vec{x}) \text{ outside } V_1 \end{cases}$



Then the new energy is $W_1 = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3x$ with $\vec{D} = \epsilon \vec{E}$. And the energy <u>change</u> is

$$\Delta W = W_1 - W_0 = \frac{1}{2}\int (\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0)\, d^3x$$

$$= \frac{1}{2}\int (\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0)\, d^3x + \underbrace{\frac{1}{2}\int (\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0)\, d^3x}$$

$$\| $$

$$-\frac{1}{2}\int [\nabla(\Phi+\Phi_0)\cdot(\vec{D}-\vec{D}_0)]\, d^3x$$

$\downarrow$ now integrate by parts, and assume surface terms $\rightarrow 0$

$$= \frac{1}{2}\int (\Phi+\Phi_0)\underbrace{\nabla\cdot(\vec{D}-\vec{D}_0)}_{=0}\, d^3x$$

$= 0$ since the sources of free change were assumed not to change.

$$\Rightarrow \Delta W = \frac{1}{2}\int_{\text{all space}} (\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0)\, d^3x$$

And since the integrand vanishes outside the dielectric (V_1), we can write this as

$$\Delta W = \frac{1}{2}\int_{V_1} (\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0)\, d^3x$$

$$= -\frac{1}{2}\int_{V_1} (\epsilon_1 - \epsilon_0)\vec{E}\cdot\vec{E}_0\, d^3x$$

and if the original dielectric was actually free space, then $(\epsilon_1 - \epsilon_0)\vec{E} = \epsilon_0 \chi_0 \vec{E} = \vec{P}$

whereby $\Delta W = -\frac{1}{2}\int_{V_1} \vec{P}\cdot\vec{E}_0\, d^3x$

This leads us to interpret the energy density of a dielectric placed into an $\vec{E}$-field with fixed sources is $w = -\frac{1}{2}\vec{P}\cdot\vec{E}_0$

Thus the energy is LOWERED when the dielectric is present

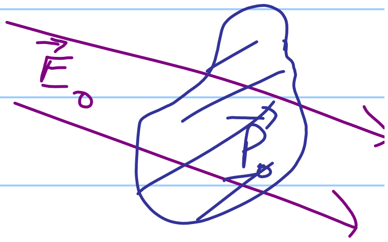
$\Rightarrow$ a dielectric with $\epsilon_1 > \epsilon_0$ is attracted inward

(analogously, $W = \frac{Q^2}{2C} < \frac{Q^2}{2C_0}$ since $C = K C_0 > C_0$)

Note: there is an important energy difference between induced versus permanent dipoles in a field:

Recall that a single permanent dipole in an electric field $\vec{E}_0$ has energy $W = -\vec{p}_0 \cdot \vec{E}_0$

$\Rightarrow$ The energy of a macroscopic body having PERMANENT polarization $\vec{P}_0$ in an $\vec{E}$-field $\vec{E}_0$ would be



$$W = -\int \vec{P}_0 \cdot \vec{E}_0 \, d^3x$$

or the corresponding energy density would be

$$\boxed{w = -\vec{P}_0 \cdot \vec{E}_0}$$

This is a universal factor of $\frac{1}{2}$ for the energy of an induced rather than permanent dipole.

This is also the case for a single atom or molecule, e.g.

$$\boxed{W = -\vec{p}_0 \cdot \vec{E}_0 \quad \text{for a } \underline{\text{permanent dipole}}}$$

whereas for an induced dipole, if $\vec{p} \to \vec{p} + \delta\vec{p}$

$\Rightarrow \delta W = -\delta\vec{p} \cdot \vec{E}$, where $\delta\vec{p} = \epsilon_0 \gamma_{mol} \, \delta\vec{E}$

$\Rightarrow \delta W = -\epsilon_0 \gamma_{mol} \vec{E} \cdot \delta\vec{E}$ and $\boxed{W = -\frac{1}{2}\epsilon_0 \gamma_{mol} E^2}$

i.e. $\boxed{W = -\frac{1}{2}\vec{P}(E) \cdot \vec{E}}$ for an induced dipole

Chapter 5 - Magnetostatics

Current = charges in motion,

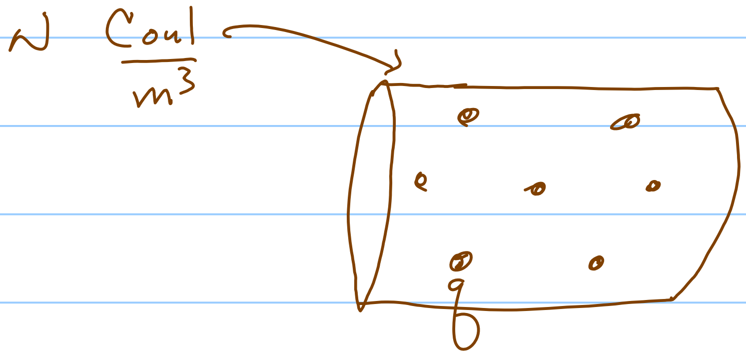
$$I = \frac{dQ}{dt} = \frac{\text{Coulombs}}{\text{second}} = \text{Amperes}$$

Current density If N charges q per unit volume move at velocity $\vec{v}$

$\Rightarrow$ $\vec{J} = Nq\vec{v}$ is the current density

$\Rightarrow$ The total amount of $\frac{\text{Coulombs}}{\text{sec}}$ crossing a differential area element $\vec{da}$ is: $dI = \vec{J}\cdot\vec{da}$

$$= \vec{J}\cdot\hat{n}\,da$$



Charge conservation

At every point in space, $\frac{\partial\rho(\vec{x},t)}{\partial t} + \nabla\cdot\vec{J}(\vec{x},t) = 0$

or in integral form,

$$\oint_S \vec{J}\cdot\vec{da} = -\frac{d}{dt}\int_V \rho(\vec{x},t)\,d^3x = -\frac{d}{dt}Q_{\text{enclosed}}$$

Here we focus on magnetostatics, where I and $\vec{J}$ are time-independent $\Rightarrow$ $\boxed{\frac{\partial\rho}{\partial t} = 0,\ \nabla\cdot\vec{J} = 0}$

Biot – Savart Law = magnetic "analogue" of Coulomb's Law

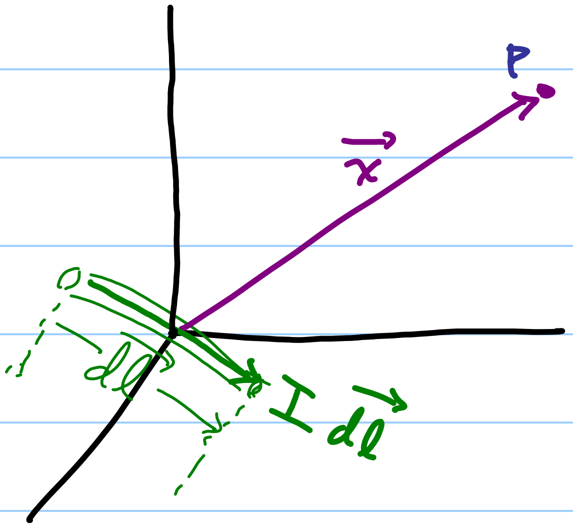
denote $\vec{B} = \begin{cases} \text{magnetic induction, or} \\ \text{magnetic flux density} \end{cases}$

Then consider an isolated, thin current element $I\vec{d\ell}$ of differential length $\vec{d\ell}$, located at the origin

Experiment: this $I\vec{d\ell}$ generates $\vec{dB}$ at $\vec{x}$, namely

$$\vec{dB} = kI\frac{\vec{d\ell} \times \vec{x}}{|\vec{x}|^3}$$

, in SI units, $k = \frac{\mu_0}{4\pi} \equiv 10^{-7}\frac{N}{A^2}$



$\vec{dB}$ at observation point P

and for a steady current that follows a complete closed circuit,

$$\boxed{\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi}\oint\frac{\vec{d\ell} \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}}$$

Units: $[B] = \frac{N}{A \cdot m}$ whereas $[E] = \frac{N}{C}$

$\Rightarrow [E] = [B] \cdot [\text{velocity}]$

You can also see this from the Lorentz force, $\vec{F} = q\vec{v}\times\vec{B} + q\vec{E}$

For some purposes it is desirable for $\vec{E}, \vec{B}$ to have the same dimensions, as in GAUSSIAN UNITS where

$$k = \frac{1}{c}, \quad c = \text{vacuum light speed}$$

e.g. the Lorentz force in Gaussian units is

$$\vec{F} = q\frac{\vec{v}}{c}\times\vec{B} + q\vec{E}$$

This is particularly helpful in relativity theory since the components of the field-strength tensor are made from $\vec{E}, \vec{B}$

Compare the SI expressions for $\vec{E}, \vec{B}$:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\,d^3x'$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}\,')\times(\vec{x}-\vec{x}\,')\,d^3x'}{|\vec{x}-\vec{x}\,'|^3}$$

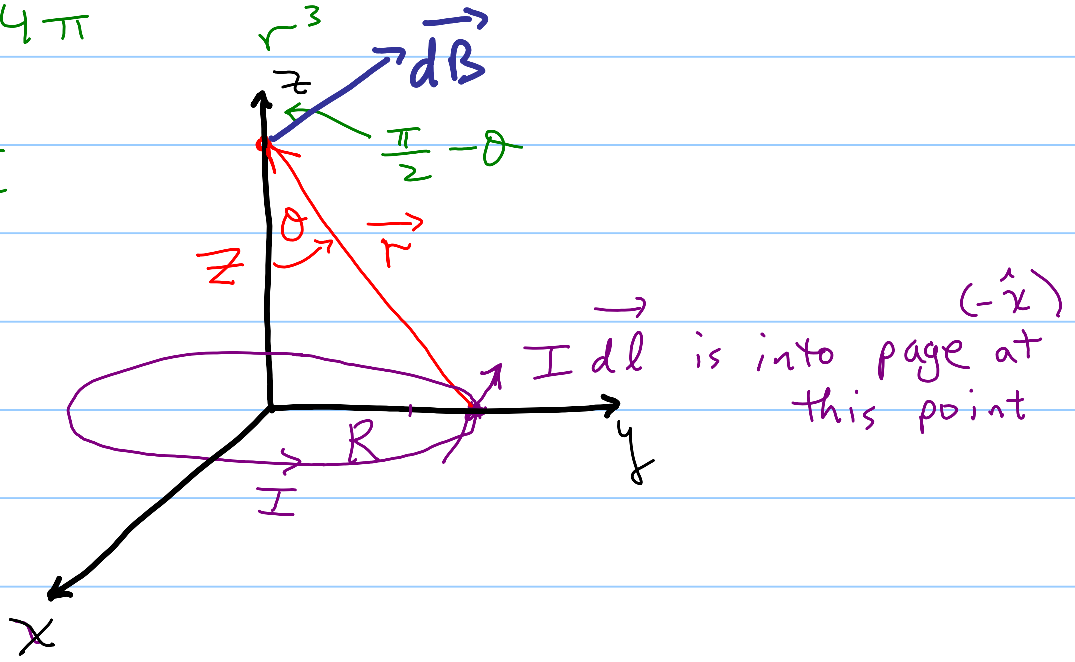
IMPORTANT: These two expressions have VERY different vectorial character!

Applications (a) $\vec{B}$ on axis of a circular current loop

We start from

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{\vec{dl}\times\vec{r}}{r^3} \qquad \vec{r} = \vec{x} - \vec{x}'$$

$$= \frac{\mu_0 I}{4\pi} \frac{\vec{dl}\times\hat{r}}{z^2+R^2}$$



$\Rightarrow$ By symmetry only B_z survives the integration

$$\Rightarrow \vec{B} = \hat{z}\,\frac{\mu_0 I}{4\pi}\,\frac{1}{z^2+R^2}\int dl\,\cos\left(\frac{\pi}{2}-\theta\right) \qquad \sin\theta = \frac{R}{(z^2+R^2)^{1/2}}$$

$$= \hat{z}\,\frac{\mu_0 I}{4\pi}\,\frac{R}{(z^2+R^2)^{3/2}}(2\pi R)$$

or finally

$$\boxed{\vec{B} = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}}\hat{z}}$$

Parametric integral for thin wire loops

Suppose we describe a 3D current loop parametrically, i.e. as $\vec{l} = \vec{l}(s)$

for $s = s_0$ to s_f

$\Rightarrow$ Then $\vec{dl} = \frac{\vec{dl}(s)}{ds}\, ds$

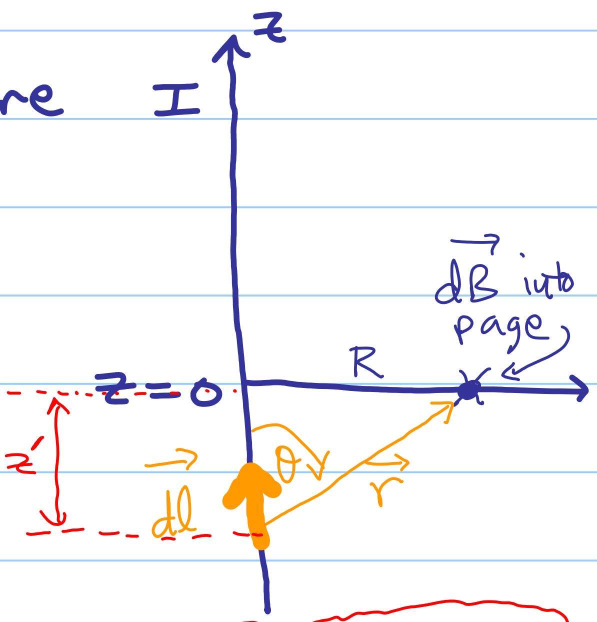
$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{4\pi} \int_{s_0}^{s_f} ds \; \frac{\vec{dl}(s)}{ds} \times \frac{[\vec{x} - \vec{l}(s)]}{|\vec{x} - \vec{l}(s)|^3}}$$

e.g. a helical current might be described by

helix $\quad \vec{l}(s) = R\cos(2\pi s)\hat{x} + R\sin(2\pi s)\hat{y} + sh\hat{z}$

where h = rise in height per helix turn

(b) A long straight wire



$$\Rightarrow \vec{dB} = \hat{\phi}\, \frac{\mu_0}{4\pi}\, \frac{dl}{z'^2 + R^2}\, \sin\theta$$

and $\sin\theta = \frac{R}{(z'^2+R^2)^{1/2}}$

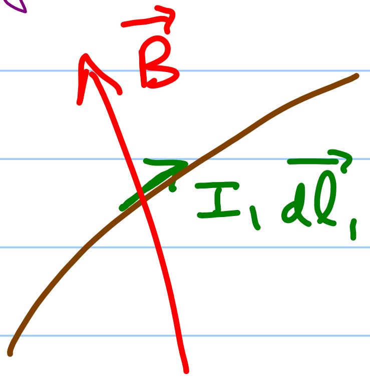
$$\Rightarrow B = \hat{\phi}\, \frac{\mu_0 I}{4\pi} \underbrace{\int_{-\infty}^{\infty} \frac{R\, dz'}{(z'^2+R^2)^{3/2}}}_{= \frac{2}{R}}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{2\pi R}\, \hat{\phi}}$$

Forces on currents

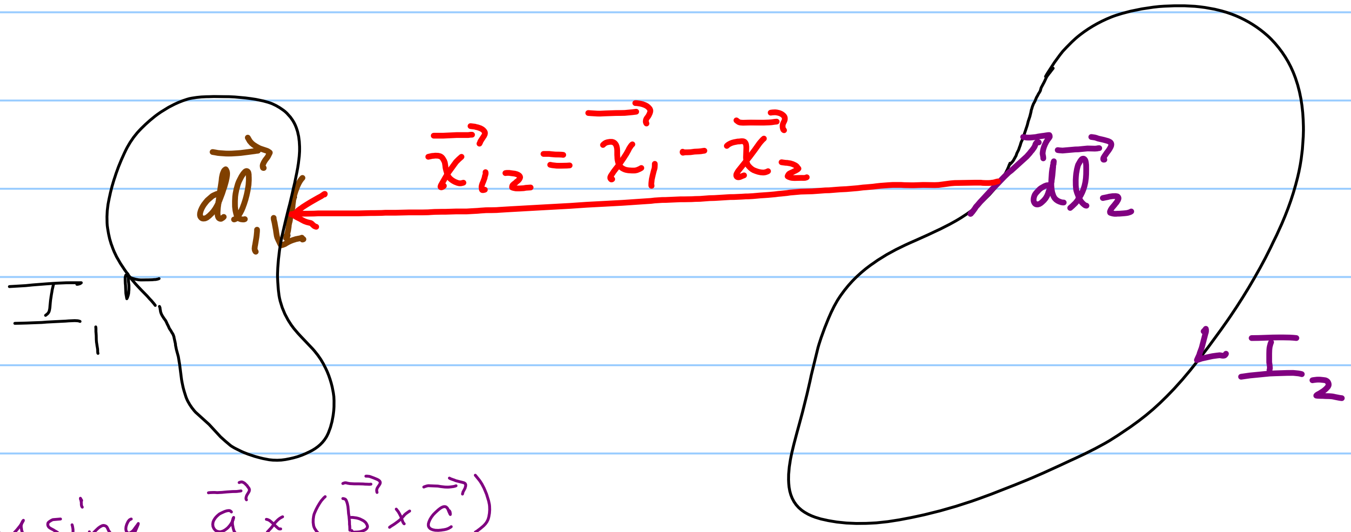
The force on a differential current element $I_1 \vec{dl}_1$ produced by another current's B-field is

$$\vec{dF}_1 = I_1 \vec{dl}_1 \times \vec{B}$$



If $\vec{B}$ is produced by a second current loop I_2, then the total force that I_2 causes on I_1 is:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\vec{dl}_1 \times (\vec{dl}_2 \times \vec{x}_{12})}{|\vec{x}_{12}|^3}$$



or using $\vec{a} \times (\vec{b} \times \vec{c})$
$= \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b})$

$$\Rightarrow d\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \left[\frac{(\vec{dl}_1 \cdot \vec{dl}_2)\,\vec{x}_{12}}{|\vec{x}_{12}|^3} + \underbrace{\frac{\vec{dl}_2\,(\vec{dl}_1 \cdot \vec{x}_{12})}{|\vec{x}_{12}|^3}}_{=} \right]$$

perfect differential gives 0 when integrated over closed loop! $\longrightarrow$ $\left(-\vec{dl}_1 \cdot \nabla_1 \frac{1}{|\vec{x}_{12}|}\right)$

Hence $$\boxed{\vec{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint\oint (d\vec{l}_1 \cdot d\vec{l}_2) \frac{\vec{x}_{12}}{|\vec{x}_{12}|^3}}$$

$$= -\vec{F}_{21} \text{ by inspection}$$

$\Rightarrow$ this obeys Newton's 3rd law

More generally the force on a current is

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$$

and the torque is

$$\vec{N} = \int \vec{x} \times (\vec{J} \times \vec{B})\, d^3x$$

Derivation the differential equations of magnetostatics

Begin from $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d^3x'$

or

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \nabla_x \frac{1}{|\vec{x} - \vec{x}'|} d^3x'$$

Aside: a vector identity is $\vec{\nabla}\psi \times \vec{a} = \nabla \times (\psi \vec{a}) - \psi \vec{\nabla} \times \vec{a}$

where $\vec{a} = \vec{J}(\vec{x}')$ $\quad \psi = 1/|\vec{x} - \vec{x}'|$

$$\Rightarrow \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \left[\int \vec{\nabla}_x \times \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' - \int \frac{\overset{0 \text{ since } \vec{J}(\vec{x}') \text{ is indep of } \vec{x}}{(\vec{\nabla}_x \times \vec{J}(\vec{x}'))}}{|\vec{x} - \vec{x}'|} d^3x' \right.$$

So $\boxed{\vec{B}(\vec{x}) = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,')\, d^3x'}{|\vec{x} - \vec{x}\,'|}}$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

Next, derive $\vec{\nabla} \times \vec{B}$:

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x' \right)$$

and use $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Hence

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla}_x \left[\int \vec{\nabla}_x \cdot \left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \right) d^3x' \right]$$

$$- \frac{\mu_0}{4\pi} \int \nabla_x^2 \left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \right) d^3x'$$

$$= \frac{\mu_0}{4\pi} \nabla_x \int \vec{J}(\vec{x}\,') \cdot \nabla_x \frac{1}{|\vec{x} - \vec{x}\,'|} d^3x' - \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \underbrace{\nabla_x^2 \frac{1}{|\vec{x} - \vec{x}\,'|}}_{-4\pi\delta(\vec{x} - \vec{x}\,')} d^3x'$$

$$= -\frac{\mu_0}{4\pi} \nabla_x \int \vec{J}(\vec{x}\,') \cdot \nabla' \frac{1}{|\vec{x} - \vec{x}\,'|} d^3x' \longleftarrow$$ we will now demonstrate that this integral Vanishes!

$$+ \mu_0 \int \vec{J}(\vec{x}\,') \, \delta(\vec{x} - \vec{x}\,') d^3x'$$

$$= \boxed{\mu_0 J(\vec{x}) = \nabla \times \vec{B}(\vec{x})}$$

To show that the aforementioned integral vanishes, use the identity $\vec{a}\cdot\nabla\psi = \nabla\cdot(\vec{a}\,\psi) - \psi\,\nabla\cdot\vec{a}$

$$\Rightarrow \int \vec{J}(\vec{x}\,')\cdot\nabla'\frac{1}{|\vec{x}-\vec{x}\,'|}\,d^3x'$$

$$= \int d^3x'\left\{\nabla'\cdot\left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\right) - \frac{1}{|\vec{x}-\vec{x}\,'|}\nabla'\cdot\vec{J}(\vec{x}\,')\right\}$$

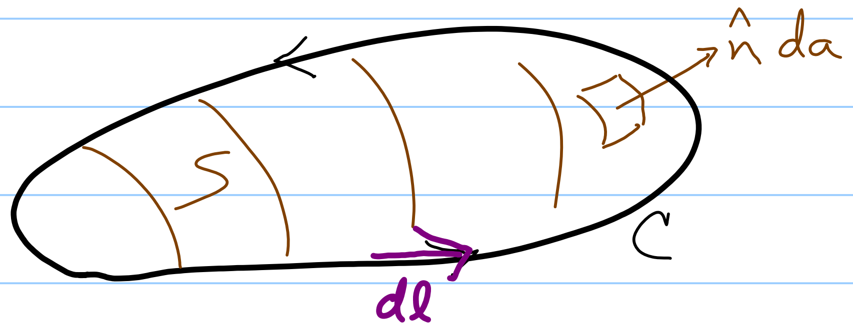
$$= \underbrace{\oint_S \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\cdot d\vec{a}\,'}_{\text{"0 if } S\to\infty \text{ for finite sources}} - \int_V \frac{\cancelto{0}{\nabla'\cdot\vec{J}(\vec{x}\,')}}{|\vec{x}-\vec{x}\,'|}\,d^3x' \quad \text{in magneto-statics!}$$

$= 0$, as we wished to prove.

$$\Rightarrow \boxed{\begin{array}{c}\nabla\times\vec{B} = \mu_0\vec{J}\\ \nabla\cdot\vec{B} = 0\end{array}}$$

Ampere's Law

Consider an arbitrary open surface S with outward normal $\hat{n}$ and a closed loop C bounding it, traversed in the right-hand rule direction:



Now integrate $\nabla \times \vec{B} = \mu_0 \vec{J}$ over such an open surface S,

$$\Rightarrow \int \nabla \times \vec{B} \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

or by Stokes Theorem,

$$\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encircled}}}$$

⇒ This is useful for problems of high symmetry, as with Gauss's Law in electrostatics

Solutions to differential laws of magnetostatics

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

Case #1 – Source-Free regions

If $\vec{J} = 0$ in some region of space, then $\nabla \times \vec{B} = 0$ ⇒ we can write

$$\vec{B} = -\nabla \Phi_M$$

where $\Phi_M \equiv$ magnetic scalar potential, which obeys

$$\nabla \cdot \vec{B} = 0 \Rightarrow \boxed{\nabla^2 \Phi_M = 0}$$

⇒ We can use all we know already about how to solve Laplace's equation!

However, the boundary conditions are different than electrostatics – discuss later.

Case #2 – General treatment based only on $\nabla \cdot \vec{B} = 0$

We can ALWAYS say that

$$\vec{B} = \nabla \times \vec{A}$$

for some vector field $\vec{A}(\vec{x}) \equiv$ vector potential

And, in fact we saw previously that

$$\vec{B} = \nabla \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|}$$

So we expect that the vector potential should have the form:

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'} + \cancel{\nabla \Psi(\vec{x})}$$

set to 0 for now, revisit in Chap. 6.

where $\Psi(\vec{x})$ = an arbitrary scalar function

Note that $\vec{A}$ is simpler than $\vec{B}$ usually, because lines of $\vec{A}$ are mostly "parallel to $\vec{J}$" while the twists of $\vec{B}$ about $\vec{J}$ come from taking $\nabla \times \vec{A} = \vec{B}$.

$\vec{A}$ and $\vec{B}$ for a circular current loop, radius a

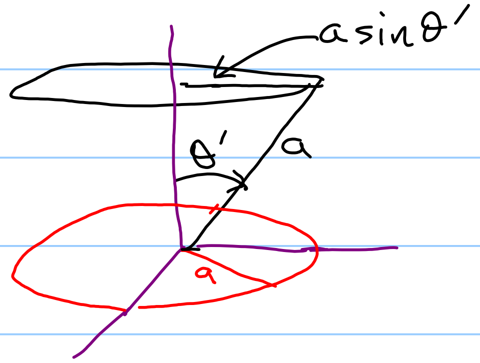
$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} \longrightarrow \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}'}{|\vec{x}-\vec{x}'|}$$

used $\vec{J}(\vec{x}') = \hat{\phi}' I \, \dfrac{\sin\theta' \,\delta(\cos\theta')\,\delta(r'-a)}{a}$

or in cylindrical coordinates $\vec{J}(\vec{x}') = \hat{\phi}' I \,\delta(z')\,\delta(\rho'-a)$

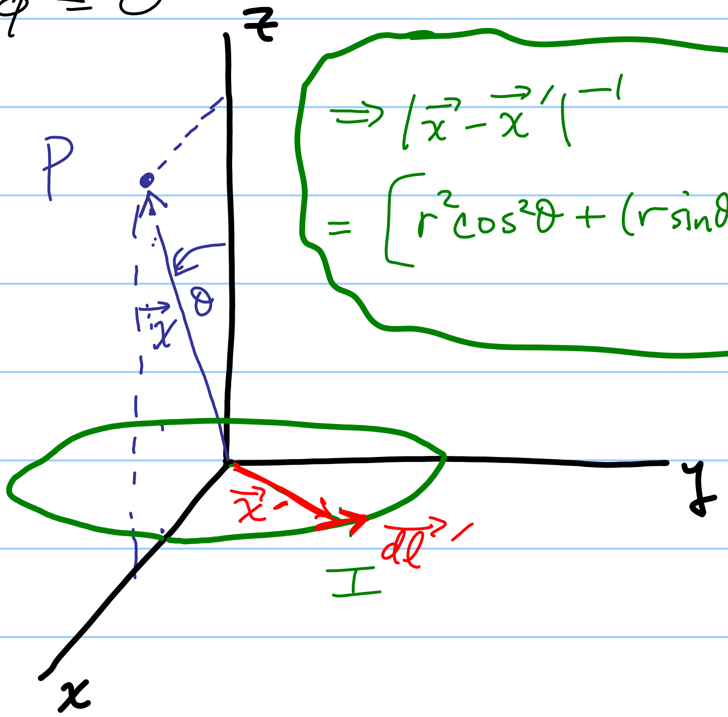
$\Rightarrow$ arranged such that

$$\underset{\substack{\text{area} \perp \\ \text{to } \vec{J}}}{\int} \vec{J}\, d^3x' = I\, d\vec{l}' = I a \sin\theta' \,d\phi' \,\hat{\phi}'$$



Because this problem has cylindrical symmetry, we may choose the field/observation point $\vec{x}$ to lie in the xz-plane, i.e. $\phi = 0$

$$\boxed{\begin{aligned} &\Rightarrow |\vec{x}-\vec{x}'|^{-1} \\ &= \left[r^2\cos^2\theta + (r\sin\theta - a\cos\phi')^2 + a^2\sin^2\phi'\right]^{-1/2} \end{aligned}}$$



One might treat this problem directly, as we will tackle it here, or by expanding $\frac{1}{|\vec{x}-\vec{x}'|}$ in spherical harmonics (see Jackson pp 183–184) or as a cylindrical expansion, etc.

Solution $\vec{l}\,'(\phi') = a\,\hat{x}\cos\phi' + a\,\hat{y}\sin\phi'$

$$\Rightarrow \vec{dl}\,' = -a\,\hat{x}\sin\phi'\,d\phi' + a\,\hat{y}\cos\phi'$$

$$= a\,\hat{\phi}'\,d\phi', \text{ since } \hat{\phi}' = -\hat{x}\sin\phi' + \hat{y}\cos\phi'$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{(-\hat{x}\sin\phi' + \hat{y}\cos\phi')\,d\phi'}{(r^2+a^2-2ar\sin\theta\cos\phi')^{1/2}}$$

$\Rightarrow$ the $\hat{y}$ integral survives but the $\hat{x}$ integrand is odd, so the integral vanishes

Thus, for $\vec{x} = (x, 0, z) = r(\sin\theta, 0, \cos\theta)$

$$\Rightarrow \vec{A}(\vec{x}) = \hat{y}\,\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{(r^2+a^2-2ar\sin\theta\cos\phi')^{1/2}}$$

Do the integral, get =

$$\vec{A}(\vec{x}) = \overset{\hat{\phi}}{\cancel{\hat{y}}}\;\mu_0\frac{I a^2\pi r\sin\theta}{4\pi(a^2+r^2)^{3/2}}\;{}_2F_1\left(\frac{3}{4},\frac{5}{4};2;\frac{4a^2r^2\sin^2\theta}{(a^2+r^2)^2}\right)$$

Or see Jackson Eq. 5.37

Hypergeometric2F1[a,b,c,z] (Mathematica)

Note that $\vec{A}$ is in the $\hat{\phi}$ direction, which makes sense since the current is in the $\hat{\phi}$ direction.

Next, evaluate $\vec{B} = \nabla \times \vec{A}$

$$\Rightarrow \quad B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi)$$

$$B_\phi = 0$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)$$

Note: ${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c\,1!}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \ldots$

So the large-r behavior is:

$$A_\phi(r,\theta) \xrightarrow[r\to\infty]{} \frac{\mu_0 I a^2 r\sin\theta}{4(a^2+r^2)^{3/2}} + O(r^{-5})$$

$$\simeq \frac{\mu_0 I a^2 \sin\theta}{4r^2} \quad \text{as } r\to\infty$$

and after evaluating the curl,

$$B_r \xrightarrow[r\to\infty]{} \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\cos\theta}{r^3}$$

$$B_\theta \xrightarrow[r\to\infty]{} \frac{\mu_0}{2\pi}(I\pi a^2)\frac{\sin\theta}{r^3}$$

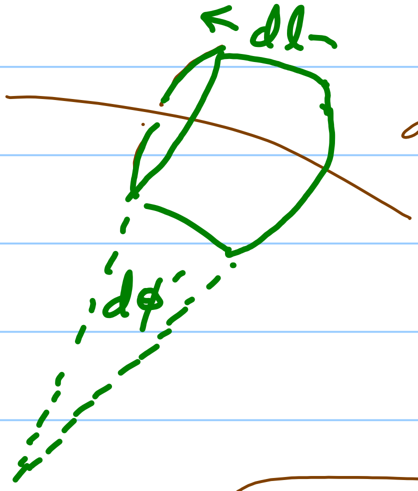
Now, these look very similar to the E-field produced by an electric dipole!

$\Rightarrow$ Call $\boxed{m = \pi a^2 I}$ the magnetic dipole moment

Aside on current δ-functions: previously we stated

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|} \longrightarrow \frac{\mu_0 I}{4\pi}\oint \frac{\vec{dl}'}{|\vec{x}-\vec{x}'|}$$

More careful discussion, integrate over a small volume δV oriented a differential length parallel to $\vec{J}$



$dl = a\,d\phi$

$$\Rightarrow \int_{\delta V} \vec{J}\, d^3x' =$$

$$\int \hat{\phi}' I \frac{\delta(r'-a)}{a}\frac{\delta(\theta'-\frac{\pi}{2})}{\sin\theta'}\, r'^2 \sin\theta'\, d\theta'\, d\phi'\, dr'$$

$$= I\, a\, d\phi'\, \hat{\phi}'$$

$$= I\,\vec{dl}' \text{ as desired if we set } \boxed{\vec{J}(\vec{x}') = I\hat{\phi}'\frac{\delta(r'-a)\delta(\theta'-\frac{\pi}{2})}{a}}$$

Back to magnetic dipoles:

Observe that $\hat{k} \times \frac{\vec{x}}{|\vec{x}|} = \hat{\phi}\sin\theta$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}}$$

where $\vec{m} = \pi a^2 I \hat{z}$ for a circular current loop in the xy-plane

and

$$\nabla\times\vec{A} = \frac{\mu_0}{4\pi}\left(\vec{m}\left(\vec{\nabla}\cdot\frac{\vec{x}}{|\vec{x}|^3}\right) - (\vec{m}\cdot\vec{\nabla})\frac{\vec{x}}{|\vec{x}|^3}\right)$$

$$\nabla\cdot\frac{\vec{x}}{|\vec{x}|^3} = \frac{1}{|\vec{x}|^3}(\nabla\cdot\vec{x}) + \vec{x}\cdot\nabla\frac{1}{|\vec{x}|^3} = \frac{3}{|\vec{x}|^3} - \frac{3}{2}\vec{x}\cdot\frac{(2x\hat{x}+2y\hat{y}+2z\hat{z})}{|\vec{x}|^5} = 0$$

and $(\vec{m}\cdot\nabla)\frac{\vec{x}}{|\vec{x}|^3} = m\frac{\partial}{\partial z}\frac{\vec{x}}{|\vec{x}|^3} = m\frac{\hat{z}}{|\vec{x}|^3} - m\frac{\vec{x}\,z}{|\vec{x}|^5}3$

$$= -\frac{3(\vec{m}\cdot\vec{x})\vec{x}}{|\vec{x}|^5} + \frac{\vec{m}}{|\vec{x}|^3}$$

define $\hat{n} \equiv \frac{\vec{x}}{|\vec{x}|}$

derive this term later!

So $$\boxed{\vec{B} = \nabla\times\vec{A} = \frac{\mu_0}{4\pi}\left(\frac{3\hat{n}(\vec{m}\cdot\hat{n}) - \vec{m}}{|\vec{x}|^3} + \frac{8\pi}{3}\vec{m}\,\delta(\vec{x})\right)}$$

Multipole expansion of the $\vec{B}$- field

As in Jackson, only expand up to the dipole term

Assume all currents $\vec{J}$ are localized, and use $$\frac{1}{|\vec{x}-\vec{x}'|} \xrightarrow[r>>r']{} \frac{1}{|\vec{x}|} + \frac{\vec{x}'\cdot\vec{x}}{|\vec{x}|^3} + \dots$$

(this is from the Taylor expansion, $\psi(\vec{x}+\vec{h}) \simeq \psi(\vec{x}) + \vec{h}\cdot\nabla\psi(\vec{x}) + \dots$

Consider the i^{th} Cartesian component of $\vec{A}$:

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left(\frac{1}{|\vec{x}|}\underbrace{\int_V J_i(\vec{x}')\,d^3x'} + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int_V J_i(\vec{x}')\,\vec{x}'\,d^3x' + \dots\right)$$

use the identity: $\int_V J_i(\vec{x}')d^3x' = \int_V (\nabla' x_i')\cdot\vec{J}(\vec{x}')d^3x'$

$$= \int_V \left[\nabla'\cdot(x_i'\vec{J}(\vec{x}')) - x_i'\underbrace{\nabla'\cdot\vec{J}(\vec{x}')}_{\to 0 \text{ in magnetostatics}}\right]d^3x'$$

$$= \oint_S x_i'\,\vec{J}(\vec{x}')\cdot d\vec{a}' = 0$$

since the surface S is beyond the range of $\vec{J}$

Conclusion: There is no MONOPOLE contribution to $\vec{A}$ $\Longrightarrow$ no magnetic charge!

Now consider the 2^{nd} term: $\int \vec{x}\cdot\vec{x}'\,\vec{J}(\vec{x}')\,d^3x'$

First, we must derive a useful identity:

$$\int_V \nabla\cdot(f\vec{J}g)\,d^3x = \oint_S fg\,\vec{J}\cdot d\vec{a} = 0$$

since $\vec{J}$ = localized,

assuming f, g are arbitrary scalar functions

$$\Rightarrow 0 = \int [\nabla(fg)\cdot\vec{J} + fg\,\underbrace{\nabla\cdot\vec{J}}_{\to 0 \text{ in magnetostatics}}]\,d^3x$$

$$\Rightarrow \boxed{\int_V (\nabla f\cdot\vec{J}g + f\vec{J}\cdot\nabla g)\,d^3x = 0}$$

Now apply this identity, set $f = x_i'$, $g = x_j'$

$$\Longrightarrow \int (x_j' J_i + x_i' J_j)\,d^3x' = 0$$

$$\Rightarrow \vec{x}\cdot\int \vec{x}'J_i\,d^3x' = \sum_j x_j \int x_j' J_i\,d^3x'$$

which we can rewrite as

$$= -\frac{1}{2}\sum_j x_j \int (x_i' J_j - x_j' J_i)\,d^3x'$$

which now looks like a cross product, $(\vec{x}'\times\vec{J})_k\,\epsilon_{ijk}$

$$= -\frac{1}{2}\sum_{j,k}\epsilon_{ijk}x_j\int(\vec{x}'\times\vec{J})_k\,d^3x' \quad (*)$$

$$= -\frac{1}{2}\left[\vec{x}\times\int(\vec{x}'\times\vec{J}(\vec{x}'))\,d^3x'\right]_i$$

e.g. suppose 'i' = 'x-component'

$$\Rightarrow (\text{above expression})_i = \cancelto{0}{-\tfrac{1}{2}\, x \int (x' J_x - x' J_x)\, d^3x'}$$

$$-\tfrac{1}{2}\, y \int (x' J_y - y' J_x)\, d^3x'$$

$$-\tfrac{1}{2}\, z \int (x' J_z - z' J_x)\, d^3x'$$

$$= -\tfrac{1}{2}\, y \int (\vec{x}\,' \times \vec{J}\,)_z\, d^3x'$$

$$+\tfrac{1}{2}\, z \int (\vec{x}\,' \times \vec{J}\,)_y\, d^3x'$$

$$= -\tfrac{1}{2}\left(\vec{x} \times \int (\vec{x}\,' \times \vec{J}\,)\, d^3x'\right)_x$$

Note that Eq. (*) above has used the Levi-Civita antisymmetric symbol ϵ_{ijk}, also called the "completely antisymmetric unit tensor"

<u>Definition</u>

$$\epsilon_{ijk} \equiv \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3) \text{ or any cyclic permutation} \\ -1, & \text{if an anticyclic permutation of } (1,2,3) \\ 0, & \text{if 2 or more indices are equal.} \end{cases}$$

The bottom line is that we have proved above that

$$\int (\vec{x} \cdot \vec{x}\,')\, \vec{J}(\vec{x}\,')\, d^3x' = -\tfrac{1}{2}\, \vec{x} \times \int (\vec{x}\,' \times \vec{J}(\vec{x}\,'))\, d^3x'$$

whereby we now have for an ARBITRARY current loop,

$$\boxed{\vec{A}_{\text{dipole}}(\vec{x}) = \frac{\mu_0}{4\pi}\, \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}}$$

, where in general the magnetic dipole moment of the current distribution is

$$\boxed{\vec{m} = \tfrac{1}{2} \int \vec{x}\,' \times \vec{J}(\vec{x}\,')\, d^3x'}$$

This generalizes our earlier result for a circular current loop, and we see that, as before, we have

$$\vec B_{dipole} = \nabla\times\vec A_{dipole} = \frac{3(\vec m\cdot\hat n)\hat n - \vec m}{r^3}, \quad \text{at } r\neq 0$$

Next, we consider small $r = |\vec x|$ <u>carefully</u>

Consider $\displaystyle\int_{r<R} \vec B(\vec x)\, d^3x = \int_{r<R} \nabla\times\vec A\, d^3x = R^2 \oint d\Omega\, \hat n\times\vec A$

$$= -\frac{R^2\mu_0}{4\pi}\int d^3x'\, \vec J(\vec x')\times \underbrace{\int \frac{\hat n\, d\Omega}{|\vec x-\vec x'|}}_{=\frac{4\pi}{3}\hat n'\frac{r_<}{R^2}}$$

r' here

$$= -\frac{R^2\mu_0}{4\pi}\int d^3x'\, \vec J(\vec x')\times\hat n'\left(\frac{4\pi r'}{3R^2}\right)$$

(refer to pp. 109–110)

but of course $\hat n' r' = \vec x'$, hence:

$$\int_{r<R} \vec B(\vec x)\, d^3x = -\frac{\mu_0}{3}\underbrace{\int d^3x'\, \vec J(\vec x')\times\vec x'}_{=-2\vec m}$$

$$= \frac{2\mu_0}{3}\vec m$$

and so finally we have for a magnetic dipole,

$$\boxed{\vec B(\vec x) = \frac{\mu_0}{4\pi}\left(\frac{3\hat n(\hat n\cdot\vec m) - \vec m}{|\vec x|^3} + \underline{\frac{8\pi}{3}\vec m\,\delta(\vec x)}\right)}$$

this term is important in atomic hyperfine interaction!

Other implications

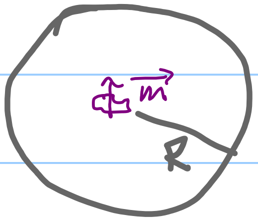
That last derivation implies also that

$$\int_{r<R} \vec{B}(\vec{x})\, d^3x = \frac{4}{3}\pi R^3 \vec{B}(0)$$ if all current $\vec{J}(\vec{x})$ lies OUTSIDE the sphere, just like we found for $\vec{E}$

i.e. there are two separate cases of interest:

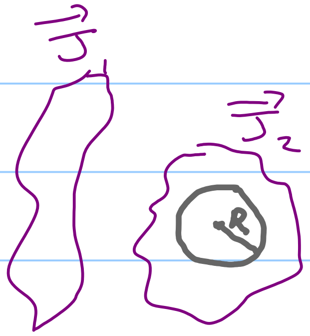
Case (1) — sphere outside a tiny dipole:

$$\int_{r<R} \vec{B}(\vec{x})\, d^3x = \frac{2\mu_0}{3}\vec{m}$$



Case (2) Sphere inside of ALL currents:

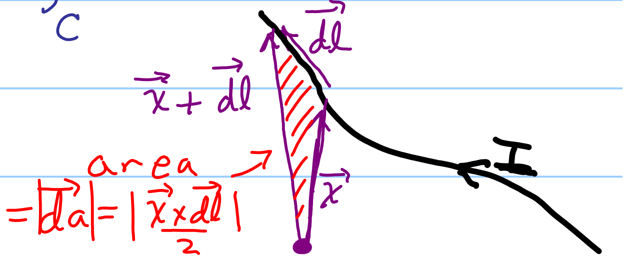
$$\int_{r<R} \vec{B}(\vec{x})\, d^3x = \frac{4}{3}\pi R^3 \vec{B}(0)$$



Quick check of our formula for $\vec{m}$, for a planar loop:

$$\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J}(\vec{x})\, d^3x = \frac{1}{2} I \oint_C \vec{x} \times \vec{dl}$$

So from the construction →
we find for a planar loop,
$\oint \frac{\vec{x} \times \vec{dl}}{2} = a\hat{n}$, where a = loop area
and $\vec{m} = I\vec{a}$



Next, Jackson Secs 5.8, 5.9

(temporarily skip Sec 5.7 concerning force, torque, energy)

Magnetostatics in Macroscopic Matter

We can visualize $\vec{B}$ as inducing magnetic dipoles $\vec{m}$ in matter, much as $\vec{E}$ induces electric dipoles $\vec{p}$. $\Rightarrow$ Thus, introduce the MAGNETIZATION,

$$\vec{M} \equiv \frac{\text{magnetic dipole moment}}{\text{unit volume}}$$

i.e. $\vec{M} = \sum_i N_i \langle \vec{m}_i \rangle$ where N_i = number density of dipoles of type i.

$\Rightarrow$ We can calculate the magnetic induction produced by a volume distribution of dipoles, starting with the vector potential,

$$d\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \vec{M}(\vec{x}\,') \times \frac{(\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} d^3x'$$

(this is merely an application of $\vec{A}_{dipole} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$)

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int_V \vec{M}(\vec{x}\,') \times \nabla' \frac{1}{|\vec{x} - \vec{x}\,'|} d^3x'$$

$$= -\frac{\mu_0}{4\pi} \int_V \nabla' \times \left(\frac{\vec{M}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \right) d^3x' + \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times \vec{M}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\underbrace{\cancel{\oint_S \frac{\vec{M}(\vec{x}')\times\hat{n}'\,da'}{|\vec{x}-\vec{x}'|}}}_{0} + \frac{\mu_0}{4\pi}\int_V \frac{\nabla'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

surface integral vanishes if surface S is outside the medium.

So comparing this last term with $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$ it suggests that we might define

$$\boxed{\nabla\times\vec{M} \equiv \vec{J}_{bnd}} \equiv \text{"volume bound current density"}$$

And for some problems there could be a bound surface current density,

$$\boxed{\vec{M}\times\hat{n} \equiv \vec{K}_{bnd}}$$

Next, calculate $\vec{B} = \nabla\times\vec{A}$, returning to the differential equations, $\Rightarrow \nabla\times\vec{B} = \mu_0\vec{J}_{total} = \mu_0\vec{J}_{free} + \mu_0\nabla\times\vec{M}$

or if we define

$$\boxed{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}} \equiv \text{MAGNETIC FIELD}$$

Then the macroscopic equations of magnetostatics are

$$\underset{\text{magnetostatics}}{\boxed{\begin{aligned}\nabla\times\vec{H} &= \vec{J}_{free}\\ \nabla\cdot\vec{B} &= 0\end{aligned}}} \quad \xrightarrow{\text{analogous to}} \quad \underset{\text{electrostatics with } \vec{D} = \epsilon_0\vec{E} + \vec{P}}{\boxed{\begin{aligned}\nabla\cdot\vec{D} &= \rho_{free}\\ \nabla\times\vec{E} &= 0\end{aligned}}}$$

and as with $\vec{D}, \vec{E}$, we require a constitutive relation between $\vec{B}, \vec{H}$ in order to solve these equations.

For many substances, like isotropic paramagnetic or diamagnetic materials, the relationship is linear,

$$\Rightarrow \quad \vec{B} = \mu \vec{H}$$

where μ = MAGNETIC PERMEABILITY

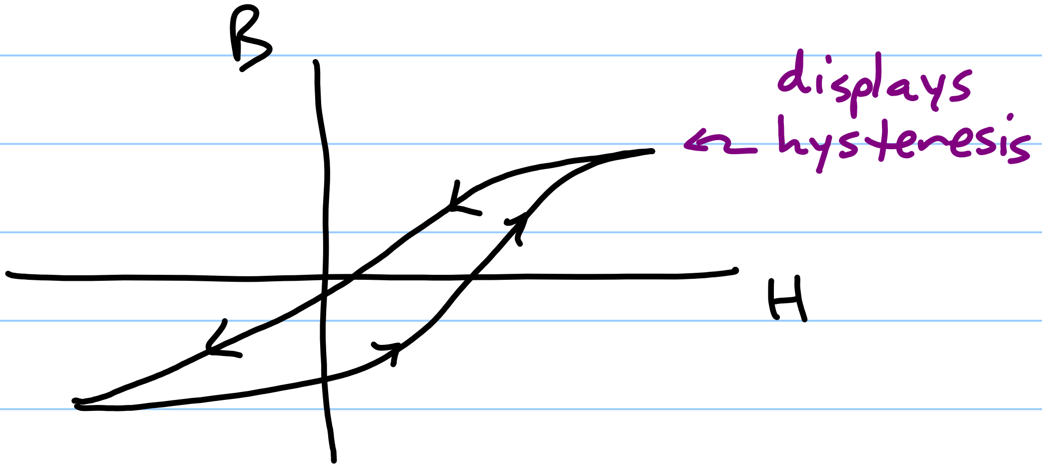
Typical values, $\left|\frac{\mu}{\mu_0} - 1\right| \approx 10^{-5}$

and $\frac{\mu}{\mu_0} - 1 < 0$, diamagnetic substances $(\mu < \mu_0)$

$\frac{\mu}{\mu_0} - 1 > 0$, paramagnetic " $(\mu > \mu_0)$

In a ferromagnet the relationship is not linear, i.e. $\mu = \mu(H) \neq$ constant

AND μ can depend on the history of preparation



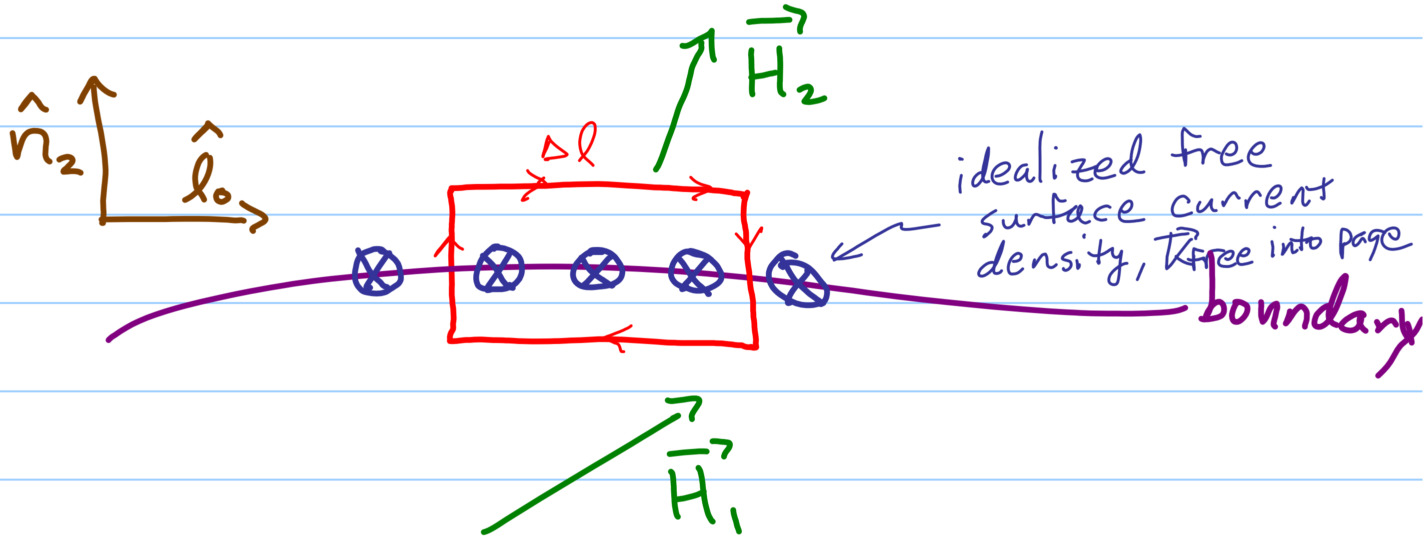
order of magnitude,

$$\frac{\mu}{\mu_0} \simeq 10 - 10^6 \text{ for ferromagnets}$$

Behavior of $\vec{B}$, $\vec{H}$ at boundaries

(1) $\nabla \cdot \vec{B} = 0 \implies B_\perp$ is continuous at any boundary, <u>always</u>

(2) $\nabla \times \vec{H} = \vec{J}_{\text{free}} \implies \int \nabla \times \vec{H} \cdot \vec{da} = \oint \vec{H} \cdot \vec{dl} = I_{\text{free}}^{\text{enclosed}}$



$$\implies (\vec{H}_2 - \vec{H}_1) \cdot \hat{l}_0 \cancel{\Delta l} = (\vec{K}_f \cancel{\Delta l}) \cdot (\hat{n}_2 \times \hat{l}_0)$$

$$\implies (\vec{H}_2 - \vec{H}_1) \cdot \hat{l}_0 = \vec{K}_f \times \hat{n}_2 \cdot \vec{l}_0$$

or $(\vec{H}_2 - \vec{H}_1) = \vec{K}_f \times \hat{n}_2$

or $\boxed{\hat{n}_2 \times (\vec{H}_2 - \vec{H}_1) = \vec{K}_f}$

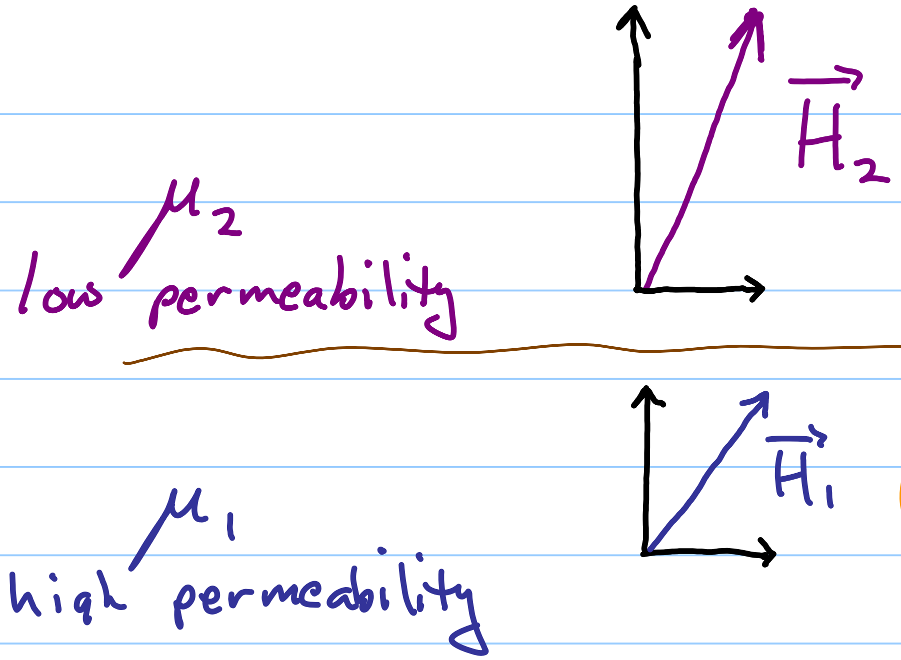
and in particular, if $\vec{K}_f = 0 \implies H_\parallel =$ continuous

For linear media, these BCs simplify to:

$$\vec{B}_2 \cdot \hat{n}_2 = \vec{B}_1 \cdot \hat{n}_2 \quad , \quad \vec{B}_2 \times \hat{n}_2 = \frac{\mu_2}{\mu_1} \vec{B}_1 \times \hat{n}_2$$

OR $\vec{H}_2 \cdot \hat{n}_2 = \dfrac{\mu_1}{\mu_2} \vec{H}_1 \cdot \hat{n}_2 \quad , \quad \vec{H}_2 \times \hat{n}_2 = \vec{H}_1 \times \hat{n}_2$

e.g.



Observe that if $\mu_1 >> \mu_2$, then

$$\boxed{|H_{2,\perp}| >> |H_{1,\perp}|}$$

$\Rightarrow \vec{H}_2$ is oriented NORMAL to the surface in the limit $\frac{\mu_1}{\mu_2} \to \infty$

Exception: UNLESS $\vec{H}_1$ is accurately parallel to the surface

ALSO, since $B_{2,\parallel} = \frac{\mu_2}{\mu_1} B_{1,\parallel} \Rightarrow B_{2,\parallel} \simeq 0$

$\Rightarrow H_\parallel$ and $B_\parallel$ vanish just outside an infinitely permeable material, just as $E_\parallel, D_\parallel$ vanish just outside a conductor surface.

Boundary Value Problems in Magnetostatics

We can ALWAYS write $\vec{B} = \nabla \times \vec{A}$

Ⓐ Most general case has $\vec{H} = \vec{H}(\vec{B}) \Rightarrow \nabla \times \vec{H}(\nabla \times \vec{A}) = \vec{J}_f$

That last equation = hard to solve!

But for a linear medium, $\vec{B} = \mu \vec{H}$

$$\Rightarrow \quad \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = \vec{J}_f$$

or throughout any region where $\mu = \text{constant}$,

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}_f$$

$$\text{or} \quad \boxed{\nabla^2 \vec{A} = -\mu \vec{J}} \longleftarrow \text{in the Coulomb gauge where } \nabla \cdot \vec{A} = 0$$

Next solve the PDE, match solutions at boundaries using the BCs derived above.

Ⓑ Suppose $\vec{J}_f = 0$ in some region of space

$$\Rightarrow \nabla \times \vec{H} = 0$$

$\Rightarrow$ In any such region, we can introduce a MAGNETIC SCALAR POTENTIAL Φ_M

i.e. $\vec{H} = -\nabla \Phi_M \quad \Rightarrow \nabla \cdot \vec{B} = 0$ or $\nabla \cdot (\mu \vec{H}) = 0$

$$\Rightarrow \boxed{\nabla^2 \Phi_M = 0}$$

in regions where $\mu = \text{constant}$, so we can use all the standard methods, and the BCs above.

ⓒ Hard Ferromagnets

– Treatment when $\vec{J}_f = 0$, $\vec{M}(\vec{x}) =$ given

$\Rightarrow$ Here of course, $\vec{B} \neq \mu \vec{H}$

$$\Rightarrow \nabla \cdot \vec{B} = 0 = \mu_0 \nabla \cdot (\vec{H} + \vec{M})$$

$$\Rightarrow \nabla \cdot (-\nabla \Phi_M) = -\nabla \cdot \vec{M}$$

or introducing $\rho_M = -\nabla \cdot \vec{M} =$ "effective magnetic charge density

$\Rightarrow$ we have $\boxed{\nabla^2 \Phi_M = -\rho_M}$

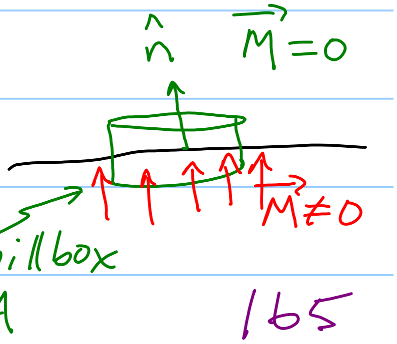
and, by analogy with electrostatics the solution is

$$\Phi_M = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

Next, observe that we can sometimes usefully treat a discontinuity of $\vec{M}$ at a surface like a "surface magnetization,

i.e. consider "Gauss's Law for $\vec{M}$" in a problem with no free charge or current

$\Rightarrow$ Start from $\nabla \cdot \vec{M} = -\rho_M$



Then $\oint \vec{M} \cdot d\vec{a} = -\int d^3x \, \rho_M(\vec{x})$

or $-(\vec{M} \cdot \hat{n}) A = -A \, \sigma_M(\vec{x})$

i.e. $\boxed{\sigma_M = \vec{M} \cdot \hat{n}}$ acts like an EFFECTIVE magnetic surface-charge density

Hence we can write separate volume and surface terms:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi} \int_V \frac{\nabla' \cdot \vec{M}(\vec{x}') \, d^3x'}{|\vec{x} - \vec{x}'|} + \frac{1}{4\pi} \int_S \frac{\hat{n}' \cdot \vec{M}(\vec{x}') \, da'}{|\vec{x} - \vec{x}'|}$$

And of course in many problems, $\vec{M}$ = constant inside, so Φ_M will then be determined ONLY by the <u>surface</u> term.

<u>Example</u> - the uniformly magnetized sphere

$$\vec{M} = M_0 \, \hat{z}$$

(i) Scalar potential method:

$$\nabla \times \vec{H} = 0 \implies \vec{H} = -\nabla \Phi_M$$

volume term vanishes since $\nabla \cdot \vec{M} = 0 \implies \Phi_M(\vec{x}) = \oint_S \frac{\hat{n}' \cdot \vec{M}(\vec{x}') \, d^3x'}{4\pi |\vec{x} - \vec{x}'|}$

So from our previous work, we can write

$$\Phi_M(\vec{x}) = \frac{M_0 a^2}{4\pi}\int d\Omega' \frac{\cos\theta'}{|\vec{x}-\vec{x}'|},$$

integrated over all $|\vec{x}'|=a$ on S

$\Rightarrow$ plug in our expansion into Y_{lm}'s:

$$\Phi_M = \frac{M_0 a^2}{4\pi}\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}\left(\frac{4\pi}{3}\right)^{1/2}\overbrace{\int d\Omega'\, Y_{10}(\hat{x}')\,Y^*_{lm}(\hat{x}')}^{''\ \delta_{l,1}\,\delta_{m,0}}\,Y_{lm}(\hat{x})$$

giving finally $\Phi_M = \dfrac{M_0 a^2}{3}\dfrac{r_<}{r_>^2}\left(\dfrac{4\pi}{3}\right)^{1/2} Y_{10}(\hat{x})$

$$\Phi_M = \frac{M_0 a^2}{3}\frac{r_<}{r_>^2}\cos\theta$$

with $r_< = \min(r,a)$, $r_> = \max(r,a)$

$$\Rightarrow \Phi_M(r<a) = \frac{M_0 a^2}{3a^2}z = \frac{M_0 z}{3}$$

whereby and

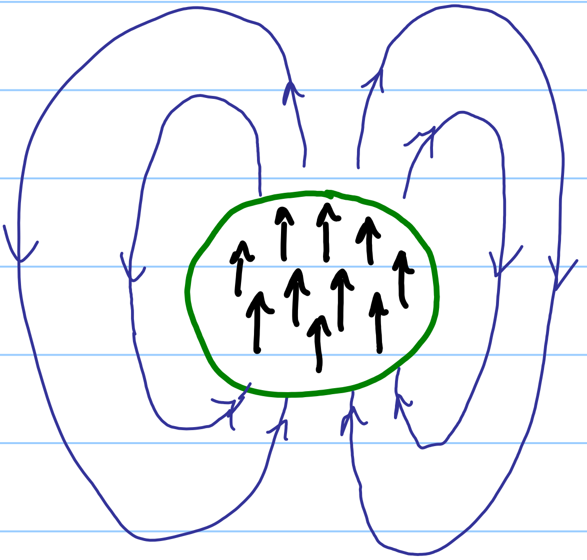
$$\boxed{\begin{aligned}\vec{H}_{in} &= -\frac{M_0\hat{z}}{3} = -\frac{\vec{M}}{3} \quad \text{at } r<a.\\ \vec{B}_{in} &= \mu_0(\vec{H}+\vec{M}) = \frac{2\mu_0}{3}\vec{M}\end{aligned}}$$

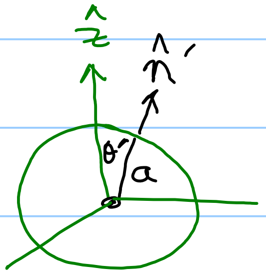
Whereas $\boxed{\Phi_M^{\text{outside}} = \dfrac{M_0 a^3}{3r^2}\cos\theta = \dfrac{\vec{m}\cdot\vec{x}}{4\pi r^3}}$,

after setting $\vec{m} = \frac{4}{3}\pi a^3 \vec{M}$

= magnetic dipole moment of entire sphere.

Conclusion: there is a constant $\vec{B}, \vec{H}$ inside a uniformly magnetized sphere, while outside the sphere you get simply a pure dipole field.





(ii) Alternative solution using vector potential

Start from

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\cancelto{0}{\int_V \frac{\nabla' \times \vec{M}(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|}} + \frac{\mu_0}{4\pi}\oint_S \frac{\vec{M}(\vec{x}') \times \hat{n}'\, da'}{|\vec{x}-\vec{x}'|}$$

and use $\vec{M}(\vec{x}') \times \hat{n}' = M_0 \hat{z} \times \hat{n}' = M_0 \sin\theta' \hat{\phi}'$

$$= M_0 \sin\theta' (-\sin\phi' \hat{x} + \cos\phi' \hat{y})$$

Recalling that $\sin\theta' \sin\phi' = -\left(\frac{8\pi}{3}\right)^{1/2}\left(\frac{Y_{11}+Y_{1,-1}}{2i}\right)$

and

$$\sin\theta' \cos\phi' = -\left(\frac{8\pi}{3}\right)^{1/2}\left(\frac{Y_{11}-Y_{1,-1}}{2}\right)$$

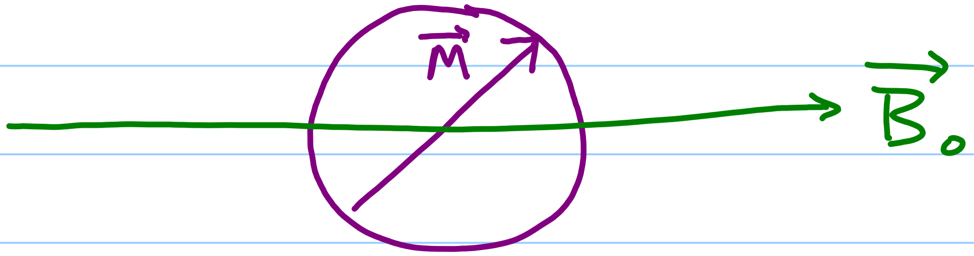
$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0 M_0 a^2}{4\pi}\int d\Omega' \sum_{l,m} \frac{4\pi}{2l+1} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x}) \frac{r_<^l}{r_>^{l+1}}$$

$$\times \left(\frac{8\pi}{3}\right)^{1/2}\left\{-\hat{x}(-1)\frac{Y_{11}(\hat{x}')+Y_{1-1}(\hat{x}')}{2i} + \hat{y}(-1)\frac{Y_{11}(\hat{x}')-Y_{1-1}(\hat{x}')}{2}\right\}$$

⇒ In the end, we see simply that

$$\boxed{\vec{A} = \frac{\mu_0 M_0 a^2}{3} \frac{r_< \sin\theta}{r_>^2} \hat{\phi} \text{ , etc.}}$$

Uniformly magnetized sphere in an external $\vec{B}$-field



Because we have linear equations, we can simply add a constant $\vec{B}_0$ everywhere, i.e.

$$\vec{B}_{in} = \vec{B}_0 + \frac{2\mu_0}{3}\vec{M} \qquad (1)$$

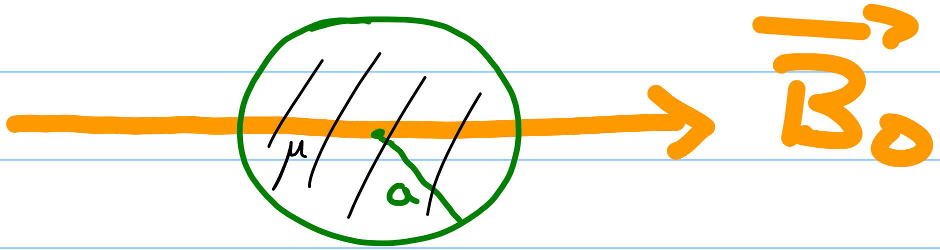
$$\vec{B}_{outside} = \vec{B}_0 + \frac{\mu_0}{4\pi}\left(\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{r^3}\right) \qquad (2)$$

where $\vec{m} = \frac{4}{3}\pi a^3 \vec{M}$ (3)

and $\vec{H}_{in} = \frac{\vec{B}_{in}}{\mu_0} - \vec{M} = \frac{\vec{B}_0}{\mu_0} - \frac{1}{3}\vec{M}$

and $\vec{H}_{out} = \frac{\vec{B}_{out}}{\mu_0}$ (since $\vec{M} = 0$ outside the sphere)

Sphere of linear magnetic permeability μ in an external $\vec{B}$-Field, $\vec{B}_0 = B_0 \hat{z}$



There are no free currents, so

$$\nabla \times \vec{H} = 0 \qquad \Rightarrow \vec{H} = -\nabla \Phi_M$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{and } \nabla^2 \Phi_M = 0, \text{ inside \& out}$$

$$\Rightarrow \Phi_M^{\text{inside}} = \sum_{l=0}^{\infty} b_l r^l P_l(\cos\theta)$$

$$\Phi_M^{\text{outside}} = c_1 r P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{d_l}{r^{l+1}} P_l(\cos\theta)$$

at $r \to \infty$, we must have $-\nabla \Phi_M \to H_0 \hat{z} = \frac{B_0}{\mu_0} \hat{z}$

$$\Rightarrow \boxed{c_1 = -\frac{B_0}{\mu_0}}$$

Next demand continuity of Φ_M at $r=a$

$$\Rightarrow \quad b_l a^l = c_1 a \delta_{l,1} + \frac{d_l}{a^{l+1}}$$

Next demand continuity of B_r,

$$\Rightarrow \mu H_r^{in}\Big|_{r=a} = \mu_0 H_r^{out}\Big|_{r=a} \Rightarrow -\frac{\partial \Phi_M^{in}}{\partial r}\Big|_{r=a} = -\frac{\mu_0}{\mu} \frac{\partial \Phi_M^{out}}{\partial r}\Big|_{r=a}$$

$$\Rightarrow \ell\, b_\ell\, a^{\ell-1} = \frac{\mu_0}{\mu}\left[c_1 \delta_{\ell,1} - (\ell+1)\, d_\ell\, a^{-\ell-2}\right]$$

Note that this system of equations

is linear, homogeneous for $\ell \neq 1 \Rightarrow b_\ell = 0 = d_\ell$ for $\ell \neq 1$

" inhomogeneous for $\ell = 1$

$\Rightarrow$ solution works out to be

$$b_1 = \frac{3\mu_0}{\mu+2\mu_0} c_1 \,, \quad d_1 = -a^3 \frac{\mu-\mu_0}{\mu+2\mu_0} c_1$$

So putting this together, we have

$$\Phi_M^{\text{inside}} = \frac{-3B_0}{\mu+2\mu_0} r\cos\theta$$

$$\Phi_M^{\text{outside}} = -\frac{B_0}{\mu_0} r\cos\theta + \frac{B_0}{\mu_0} \frac{a^3\cos\theta}{r^2} \frac{\mu-\mu_0}{\mu+2\mu_0}$$

and so $\boxed{\vec{H}^{\text{inside}} = -\nabla \Phi_M^{\text{in}} = \frac{3B_0}{\mu+2\mu_0}\hat{z}}$

and $\boxed{\vec{B}^{\text{inside}} = \mu \vec{H}^{\text{inside}} = \frac{3\mu}{\mu+2\mu_0} B_0 \hat{z}}$ = constant inside!

while $\vec{H}^{\text{outside}} = -\nabla \Phi_M^{\text{outside}} = \frac{B_0 \hat{z}}{\mu_0} + \frac{\mu_0[3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}]}{\mu_0 4\pi r^3}$

where $\boxed{\vec{m} = 4\pi a^3 \vec{B_0} \frac{\mu-\mu_0}{\mu+2\mu_0}}$ = the induced magnetic dipole moment of the sphere

On your own, read Sec. 5.11, 5.12 and pp. 187 – 191

Relation between $\vec{m}$ and the orbital angular momentum $\vec{L}$

For a discrete set of moving particles,

$$\vec{J} = \sum_i q_i \vec{v}_i \delta(\vec{x} - \vec{x}_i)$$

If constant velocity $\vec{x}_i(t) = \vec{x}_i(0) + \vec{v}_i t$

since for 1-particle, $\vec{J} = \rho \vec{v}$ and $\rho = q\,\delta(\vec{x} - \vec{x}_i)$,

Now recall that $\vec{L} = M_i \vec{x}_i \times \vec{v}_i$

So $\vec{m} = \frac{1}{2} \int \vec{x} \times \vec{J}\, d^3x = \frac{1}{2} \sum_i q_i \vec{x}_i \times \vec{v}_i = \sum_i \frac{q_i \vec{L}_i}{2M_i}$

and if all particles are identical, e.g. $\frac{q_i}{M_i} = \frac{-e}{M}$

$\Rightarrow \boxed{\vec{m} = \frac{-e}{2M} \vec{L}}$ where $\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots + \vec{L}_N$

= total orbital angular momentum

orbital magnetic moment

whereas in quantum physics one sees that the <u>spin</u> angular momentum of an e^- has a magnetic moment too:

$$\vec{m}_s = 2(1.00116\ldots)\left(\frac{-e}{2M_e}\right)\vec{S}$$

(this is derived from the Dirac equation + quantum electrodynamics)

Force $\vec{F}$ and torque $\vec{N}$ on a localized current distribution:

Start from our earlier force expression,

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$$

and insert a Taylor series for $\vec{B}$, produced by $\underline{\underline{\text{other}}}$ currents outside of $\vec{J}$

$$\Longrightarrow \quad \vec{B}(\vec{x}) = \vec{B}(0) + (\vec{x}\cdot\nabla)\,\vec{B}\Big|_{\vec{x}=0} + \ldots$$

$$\text{i.e.} \quad B_i(\vec{x}) = B_i(0) + \underbrace{(\vec{x}\cdot\nabla)B_i}$$

derivatives of B_i are evaluated at $\vec{x} \longrightarrow 0$, which we write as a shorthand as $(\vec{x}\cdot\nabla)\vec{B}(0)$

$$\text{Then } \vec{F} = -\vec{B}(0)\times\int \vec{J}(\vec{x})\, d^3x + \int \vec{J}(\vec{x})\times\left[(\vec{x}\cdot\nabla)\vec{B}(0)\right] d^3x$$

Moreover, $\int \vec{J}(\vec{x})\, d^3x = 0$ in magnetostatics (proved on p. 154) of these notes

Or using the Levi-Civita symbol, we have

Cartesian unit vectors, $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$, $\hat{e}_3 = \hat{z}$

$$\vec{F} = \sum_{ijk} \epsilon_{ijk}\, \hat{e}_k \int J_i\, (\vec{x}\cdot\nabla) B_j(0)\, d^3x$$

but notice that

$$\int J_i\,(\vec{x}\cdot\nabla)B_j(0)\, d^3x = \sum_\ell (\nabla B_j)^{(0)}_\ell \int x_\ell J_i\, d^3x$$

$$= \frac{1}{2}\sum_\ell (\nabla B_j)^{(0)}_\ell \int \underbrace{(x_\ell J_i - x_i J_\ell)}_{= \sum_{k'} (\vec{x}\times\vec{J})_{k'}\, \epsilon_{\ell i k'}} d^3x,$$

(using Eq. 5.52 and below)

Hence $\vec{F} = \frac{1}{2} \sum_{\substack{ijk \\ \ell k'}} \hat{e}_k \epsilon_{ijk} (\nabla B_j^{(0)})_\ell \, \epsilon_{\ell i k'} \left[\int \vec{x} \times \vec{J} \, d^3x \right]_{k'}$

Now we have a key identity we can use:

$$\sum_i \epsilon_{ijk} \epsilon_{\ell i k'} = \sum_i \epsilon_{ijk} \epsilon_{ik'\ell} = \delta_{jk'} \delta_{k\ell} - \delta_{j\ell} \delta_{kk'}$$

also note: $\sum_{ij} \epsilon_{ijk} \epsilon_{ijn} = 2\delta_{kn}$ and $\sum_{ijk} \epsilon_{ijk} \epsilon_{ijk} = 6$

So $\vec{F} = \frac{1}{2} \sum_{jk\ell k'} \hat{e}_k (\nabla B_j^{(0)})_\ell \left(\int \vec{x} \times \vec{J} \, d^3x \right)_{k'} (\delta_{jk'} \delta_{k\ell} - \delta_{j\ell} \delta_{kk'})$

and so, after evaluating these sums over ℓ, k', we get

$$\vec{F} = \frac{1}{2} \sum_{jk} \hat{e}_k \left[(\nabla B_j^{(0)})_k \left(\int \vec{x} \times \vec{J} \, d^3x \right)_j - \cancel{(\nabla B_j^{(0)})_j} \left(\int \vec{x} \times \vec{J} \, d^3x \right)_k \right]$$

$\to 0$, since $\nabla \cdot \vec{B} = 0$

Thus $\vec{F} = \sum_{jk} \hat{e}_k \frac{\partial}{\partial x_k} B_j^{(0)} m_j = \boxed{\nabla (\vec{m} \cdot \vec{B}) = \vec{F}}$

and this result is analogous to $\vec{F} = \nabla(\vec{p} \cdot \vec{E})$ for an electric dipole

Also, since we expect $\vec{F} = -\nabla W_{dipole}$, we can interpret $\boxed{W_{dipole} = -\vec{m} \cdot \vec{B}}$ as the energy of a magnetic dipole $\vec{m}$ in an external $\vec{B}$-field

NOTE: This is not actually the TOTAL ENERGY because it neglects the energy needed to maintain steady currents as the dipole is brought in from ∞.

Similarly, relevant equations for torque on a localized current:

$\vec{N} = \int \vec{x}' \times (\vec{J} \times \vec{B}(0)) \, d^3x'$ + higher order terms

or $\vec{N} = \int \left[(\vec{x}' \cdot \vec{B}(0)) \vec{J} - (\vec{x}' \cdot \vec{J}) \vec{B}(0) \right] d^3x'$

Faraday's Law of Induction

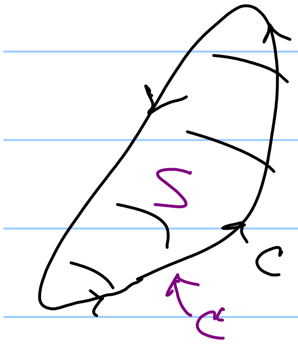
Now allow time-dependence, i.e. $\frac{\partial \rho}{\partial t} \neq 0, \frac{\partial \vec{J}}{\partial t} \neq 0$

Experiment: Start by defining

(1) $\mathcal{E} = \oint_C \vec{E} \cdot d\vec{l}$ = "electromotive force" or emf

(2) $\Phi_B = \int_S \vec{B} \cdot d\vec{a}$ = magnetic flux, Jackson's F

recall — The sense of direction of $d\vec{a}$ is in accordance with the right-hand rule



Now, Faraday's experiments showed that

$$\mathcal{E} = -k \frac{d\Phi}{dt}, \begin{cases} k = 1, & \text{SI units} \\ k = \frac{1}{c}, & \text{Gaussian units} \end{cases}$$

Let's work in SI units,

$$\Rightarrow \boxed{\underset{\text{volts}}{\mathcal{E}} = \underset{\frac{\text{Tesla-m}^2}{\text{sec}}}{\frac{-d\Phi_B}{dt}}}$$

On your own, read Jackson's Sec 5.15 on Galilean invariance concept.

Observations (a) In $\oint_C \vec{E} \cdot d\vec{l}$, C can be any closed path in space, i.e it need not be a wire, though it could be one.

(b) The magnetic flux can change by changing $\vec{B}$ or the shape, size, or orientation of C

$\Rightarrow$ of course physics does not depend on the reference frame.

(c) It is understood that $\vec{E}$ is the electric field at $\vec{dl}$ in the coordinate system in which $\vec{dl}$ is at rest

Galilean Transformation

Let $\vec{E}, \vec{B}$ be the fields in the laboratory frame and let $\vec{E}'$ be the E-field in a frame moving at velocity $\vec{v}$ relative to the laboratory, i.e. along with the charge q whose velocity is $\vec{v}$ in the laboratory frame.

$\Rightarrow$ Then we must have $\boxed{\vec{E}' = \vec{E} + \vec{v} \times \vec{B}}$,

in order to be consistent with the Lorentz force,

$$\vec{F}' = q\vec{E}' \Longleftrightarrow \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Differential Form of Faraday's Law

Consider an arbitrary time-independent loop C

$$\Rightarrow -\frac{d}{dt}\Phi_B = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{da} = \oint_C \vec{E} \cdot \vec{dl} = \int_S \nabla \times \vec{E} \cdot \vec{dl}$$

and since the loop C is arbitrary, the integrands are equal

$$\Rightarrow \boxed{\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0}$$

(which generalizes our result $\nabla \times \vec{E} = 0$ from before in electrostatics)

Energy in the magnetic field

Let's now discuss what happens to the energy as we increase $\vec{J}$ up from 0, and use Faraday's Law to account for the fact that we perform work against the induced emf as the current is increased.

$\Rightarrow$ We interpret this work as the "ENERGY STORED IN THE MAGNETIC FIELD"

Then the rate at which work is done by the battery or power supply, in the course of changing the flux and inducing currents & emf, is

$$P = \frac{dW}{dt} = -I\mathcal{E}, \quad \text{but } \mathcal{E} = -\frac{d\Phi_B}{\partial t}$$

Hence $\frac{dW}{dt} = I \frac{d\Phi_B}{dt}$

As the flux through a circuit carrying current I changes from Φ_B to $\Phi_B + \delta\Phi_B$, the work done by external sources is equal to

$$\delta W = I\,\delta\Phi_B$$ ← no ohmic losses have been included here

But $\Phi_B = \int \vec{B}\cdot\hat{n}\,da = \int \nabla\times\vec{A}\cdot\vec{da} = \oint_C \vec{A}\cdot\vec{dl}$

whereby

$$\delta W = I\oint_C \delta\vec{A}\cdot\vec{dl} = \int \delta\vec{A}\cdot\vec{J}\,d^3x$$

and using $\nabla\times\vec{H} = \vec{J}$,

$$\Rightarrow \delta W = \int \delta\vec{A}\cdot(\nabla\times\vec{H})\,d^3x$$

Recall the following identity, here with $\vec{P}\to\vec{H}$, $\vec{Q}\to\delta\vec{A}$,

$$\nabla\cdot(\vec{P}\times\vec{Q}) = \vec{Q}\cdot(\nabla\times\vec{P}) - \vec{P}\cdot(\nabla\times\vec{Q})$$

$$\Rightarrow \delta W = \int\left[\vec{H}\cdot(\nabla\times\delta\vec{A}) + \underbrace{\nabla\cdot(\vec{H}\times\delta\vec{A})}\right]d^3x$$

vanishes for localized fields by the divergence theorem

or finally $\boxed{\delta W = \int \vec{H}\cdot\delta\vec{B}\,d^3x}$

which is valid for any kind of magnetic system, the MOST GENERAL formula

But if the material is linear, paramagnetic or diamagnetic, $\Rightarrow \vec{H}\cdot\delta\vec{B} = \frac{1}{2}\delta(\vec{H}\cdot\vec{B})$

and in that case we have

$$\boxed{W = \frac{1}{2}\int \vec{H}\cdot\vec{B}\, d^3x}$$

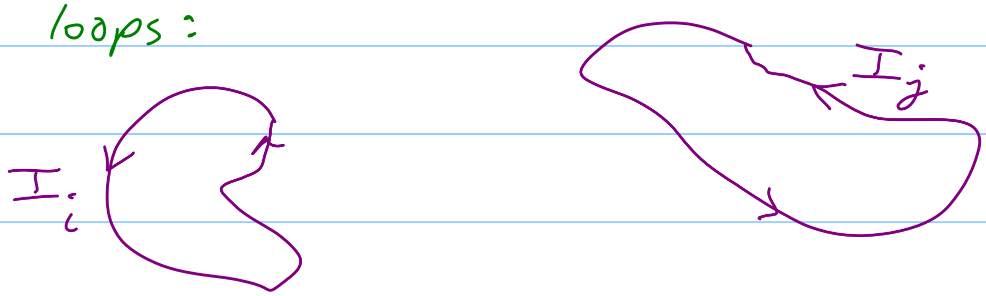
Alternatively, when $\vec{J}$ and $\vec{A}$ are linearly related, (e.g. when no magnetic media are present and $\nabla\times(\nabla\times\vec{A}) = \mu_0\vec{J}$)

$$\Rightarrow \quad \delta\vec{A}\cdot\vec{J} = \frac{1}{2}\delta(\vec{A}\cdot\vec{J}) \Rightarrow \underline{W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\, d^3x}$$

So for a fixed current loop, $W = \frac{I}{2}\oint_C \vec{A}\cdot d\vec{l}$

or $$W = \frac{I}{2}\int_S \nabla\times\vec{A}\cdot d\vec{a} = \boxed{\frac{I\,\Phi_B}{2} = W}$$

This leads us to the idea of INDUCTANCE, i.e. for a situation where we have some current loops:



Then the flux in that last equation includes BOTH the flux through DIFFERENT current loops $i \neq j$, AND through the SAME loop itself. Hence we can keep track of these separate contributions by writing, for N loops:

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i} M_{ij} I_i I_j$$

Justification:

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\,d^3x = \frac{1}{2}\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x})\times\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x\,d^3x'$$

$$= \frac{\mu_0}{8\pi}\sum_{i=1}^{N}\sum_{j=1}^{N} I_i I_j \underbrace{\oint_{C_i}\oint_{C_j}\frac{\vec{dl}_i\cdot\vec{dl}_j\,'}{|\vec{x}-\vec{x}\,'|}}$$

Observe that this integral is a purely GEOMETRICAL quantity

$\Rightarrow$ Define $$L_i \equiv \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_i}\frac{\vec{dl}_i\cdot\vec{dl}_i\,'}{|\vec{x}-\vec{x}\,'|} = \text{SELF-INDUCTANCE}$$

and

$$M_{ij} \equiv \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\frac{\vec{dl}_i\cdot\vec{dl}_j\,'}{|\vec{x}-\vec{x}\,'|} = \text{MUTUAL-INDUCTANCE}$$

Quasi-static B-fields in conductors

Observation: A changing B-field induces $\vec{E}$ according to Faraday's Law, but if

$$\left|\frac{\partial \vec{B}}{\partial t}\right| = \text{"small"}$$

then B-field physics still dominates

$\Rightarrow$ This is called the QUASI-STATIC regime

Under these conditions we can:

(i) ignore the finite value of light speed

(ii) neglect Maxwell's displacement current (i.e. neglect the effect on $\vec{B}$ of a changing $\vec{E}$-field)

Conducting Media

– To treat a conductor in this regime, begin by assuming that Ohm's Law is valid,

$$\vec{J} = \sigma \vec{E}, \quad \sigma = \text{conductivity}$$

$$\Rightarrow \nabla \times \vec{H} = \vec{J} \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

$$\Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

(Note: (a) if there is no free charge, $\vec{E}$ is produced only by $\partial\vec{B}/\partial t$.)

Hence we can introduce a potential Φ such that

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi$$

And for a linear, homogeneous medium,

$$\vec{H} = \frac{\vec{B}}{\mu} \Rightarrow \nabla\times\vec{B} = \mu\vec{J} = \mu\sigma\vec{E}$$

hence we can take $\Phi = 0 \Rightarrow \vec{E} = \frac{-\partial\vec{A}}{\partial t}$,
and in the gauge where $\nabla\cdot\vec{A} = 0$ (Coulomb gauge)
and recalling that $\nabla\cdot\vec{E} = 0$ here too,
then (b):

$$\nabla\times(\nabla\times\vec{A}) = \mu\sigma\left(-\frac{\partial\vec{A}}{\partial t}\right) = \nabla(\cancelto{0}{\nabla\cdot\vec{A}}) - \nabla^2\vec{A}$$

or

$$\boxed{\nabla^2\vec{A} = \mu\sigma\frac{\partial\vec{A}}{\partial t}}$$

This is a diffusion equation obeyed by each Cartesian component of the vector potential.

Diffusion equation solution by separation of variables

For just one component of $\vec{A}$, this PDE has the structure

$$\nabla^2 u(\vec{x},t) = \gamma\frac{\partial u(\vec{x},t)}{\partial t}$$

Look for separable space/time solutions like:

$$u(\vec{x},t) = F(\vec{x})\,T(t)$$

Plug this in and divide by FT

$$\Rightarrow \frac{\nabla^2 F(\vec{x})}{F(\vec{x})} = \frac{\gamma}{T(t)} \frac{dT(t)}{dt} = -k^2$$

$$\Rightarrow \frac{dT(t)}{dt} = -\frac{k^2}{\gamma} T(t) \text{, solution is } T(t) = e^{-\frac{k^2 t}{\gamma}}$$

and

$F(\vec{x})$ obeys the <u>Helmholtz</u> PDE

$$\boxed{(\nabla^2 + k^2) F(\vec{x}) = 0}$$

This equation has solutions in spherical coordinates of the form

$$F_{klm}(\vec{x}) = Y_{lm}(\hat{x}) \begin{cases} \text{either} \\ j_l(kr) \\ \text{or} \\ n_l(kr) \end{cases}$$

spherical Bessel functions

$\Rightarrow$ See Jackson pp. 425 – 427, which has the same spatial equation, but a different time-dependent equation there.

e.g. $j_l(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{l+\frac{1}{2}}(kr)$, which is regular at $r \to 0$, oscillatory at $r \to \infty$.

Hence, in the context of $\vec{A}$ diffusing in conducting media, each component can be expanded as

$$A_i(\vec{x}, t) = \sum_{\ell m} \int_0^\infty dk \; C^{(i)}_{\ell m}(k) \, Y_{\ell m}(\hat{x}) \, j_\ell(kr) \, e^{-k^2 \frac{t}{\gamma}}$$

the solution <u>must</u> have this form if the volume includes $r = 0$ and the full angular range

<u>TERMINOLOGY</u>: Jackson calls this "A Laplace transform representation of the solution"

How can we use this? One might be given $A_i(\vec{x}, t=0)$. Then one uses "Fourier's trick" to determine the time-independent coefficients $C^{(i)}_{\ell m}(k)$, which then determine the solution at all later times.

Hint: See the orthogonality relations in Jackson's problem #3.16(a).

Addendum about inductance and energy sign conventions

Recall from basic electrical circuit theory that the energy stored in an inductor is

$W = \frac{1}{2} L I^2$, whereby the rate at which power is delivered to it is

$$\frac{dW}{dt} = I\left(L\frac{dI}{dt}\right)$$

But the instantaneous flux through the inductor is $\Phi_B = LI$, so the induced emf according to Faraday's Law is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

So $\boxed{\frac{dW}{dt} = -I\mathcal{E}}$ where $\mathcal{E} = \oint \vec{E} \cdot d\vec{l}$

$= \text{power}, P$

How can relate this to the voltage change as one goes around the loop?

Recall $V = -\int_{\text{loop}} \vec{E} \cdot d\vec{l} = -\mathcal{E}$

whereby $\underline{P = IV} = -I\mathcal{E}$

the usual expression from circuit theory

Chapter 6 Maxwell's equations

To this point we have 4 incomplete maxwell equations,

$$\nabla\cdot\vec{D}=\rho \qquad\qquad \boxed{\nabla\times\vec{H}=\vec{J}}$$

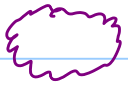
$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \qquad\qquad \nabla\cdot\vec{B}=0$$

apply $\nabla\cdot(\nabla\times\vec{H})=\underline{0=\nabla\cdot\vec{J}}$

But we know that the general form of the continuity equation for charge and current reads

$$\boxed{\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0}$$

not consistent! in general!

To see how to make them consistent, rewrite  as:

$$\nabla\cdot\vec{J}+\nabla\cdot\frac{\partial\vec{D}}{\partial t}=0=\nabla\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right),$$

which suggests that we should replace $\vec{J}$ (in the Maxwell equation $\nabla\times\vec{H}=\vec{J}$) by $\vec{J}+\frac{\partial\vec{D}}{\partial t}$.

$$\Longrightarrow \boxed{\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}}$$

$$\boxed{\nabla\cdot\vec{D}=\rho} \qquad \textbf{MAXWELL'S EQUATIONS} \qquad \boxed{\nabla\cdot\vec{B}=0}$$

$$\boxed{\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0}$$

Electromagnetic Potentials

Start by solving the 2 homogeneous Maxwell equations "automatically"

Since $\nabla \cdot \vec{B} = 0 \implies \boxed{\vec{B} = \nabla \times \vec{A}}$

and Faraday's Law $\implies \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$

which implies that $\rightarrow \boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \Phi}$

or $\boxed{\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}}$

Now, specialize to a vacuum, no media

$$\implies \vec{D} = \epsilon_0 \vec{E}$$
$$\vec{H} = \vec{B}/\mu_0$$

Next, look at the 2 inhomogeneous equations:

$\implies \nabla \cdot \left(-\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}$ (Gauss' Law)

$$\implies \boxed{\nabla^2 \Phi + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}}$$

and

$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ (Ampere's Law + Maxwell's term)

$$\implies \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} - \frac{1}{c^2} \left(\nabla \frac{\partial \Phi}{\partial t} + \frac{\partial^2 \vec{A}}{\partial t^2} \right)$$

which has used $\epsilon_0 \mu_0 = \frac{1}{c^2}$

$$\implies \boxed{\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla \left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \vec{J}}$$

Gauge Choices

Observe that $\vec{B}$ will not be changed if we modify $\vec{A} \longrightarrow \vec{A} + \nabla\Lambda \equiv \vec{A}'$, where Λ = any scalar function

However, under this change $\vec{A} \longrightarrow \vec{A}'$, $\vec{E}$ will change UNLESS we simultaneously change $\Phi \longrightarrow \Phi - \frac{\partial \Lambda}{\partial t} \equiv \Phi'$

Throughout most of this course we will choose our gauge such that $(\vec{A}, \Phi)$ obey the LORENTZ CONDITION,

$$\boxed{\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0}$$

which is advantageous because the wave equations for $(\vec{A}, \Phi)$ then look symmetrical AND uncoupled:

$$\boxed{\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\rho/\epsilon_0 \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \end{aligned}}$$

These are the inhomogeneous wave equations for $(\vec{A}, \Phi)$ in the LORENZ GAUGE

A different gauge sometimes adopted is characterized by $\boxed{\nabla \cdot \vec{A} = 0}$,

which is called by different names: the Coulomb gauge
" radiation "
" transverse "

$\Rightarrow$ $\nabla^2 \Phi = -\rho/\epsilon_0$

whose solution we already know to be:

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}\,', t)\, d^3x'}{|\vec{x} - \vec{x}\,'|}$$

whereas the vector potential equation is more complicated in the Coulomb gauge, namely:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \frac{1}{c^2}\nabla \frac{\partial \Phi}{\partial t}$$

__Aside__: It is well-known that ANY vector field can be written as the sum of a __longitudinal__ term $\vec{J}_l$ obeying $\nabla \times \vec{J}_l = 0$ PLUS a __transverse__ term $\vec{J}_t$ obeying $\nabla \cdot \vec{J}_t = 0$ (see Jackson, top of p. 242)

$$\Rightarrow \quad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}_t$$

Green Function for the inhomogeneous wave equation

We see that we must solve wave equations having the form:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\vec{x},t) = -4\pi f(\vec{x},t)$$

where

$f(\vec{x},t) =$ a known, specified source

This is a linear, homogeneous equation, so it can be solved using Green's function methods!

W00T!

Specifically, start by solving for G, obeying

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{x},t;\vec{x}\,',t') = -4\pi\,\delta(\vec{x}-\vec{x}\,')\,\delta(t-t')$$

and once we know G, we can solve the original PDE in the form

$$\Psi(\vec{x},t) = \iint G(\vec{x},t;\vec{x}\,',t')\,f(\vec{x}\,',t')\,d^3x'\,dt'$$

(assumes that G obeys the same BCs as Ψ)

Considerations relevant to finding G:

(1) Only RELATIVE times (not absolute) should matter $\Rightarrow$ $G(\vec{x},t;\vec{x}\,',t') \rightarrow G(\vec{x},\vec{x}\,';t-t')$

(2) And in free space with no boundaries, only RELATIVE positions should matter,

$$\Rightarrow G(\vec{x},\vec{x}\,';t-t') \longrightarrow G(\vec{x}-\vec{x}\,',t-t') \equiv G(\vec{R},\tau)$$

where
$$\vec{R} \equiv \vec{x} - \vec{x}'$$
$$\tau \equiv t - t'$$

We can solve this using Fourier analysis in time, τ by writing

$$G(\vec{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\, \mathcal{G}(\vec{R},\omega)$$

After plugging this into the PDE, we get the following equation for $\mathcal{G}(\vec{R},\omega)$:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\mathcal{G}(\vec{R},\omega)$$
$$= -4\pi\delta(\vec{R})\,\delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\left(-4\pi\delta(\vec{R})\right)$$

and this is obeyed provided $\mathcal{G}$ obeys:

$$\boxed{\left(\nabla^2 + k^2\right)\mathcal{G}(\vec{R},\omega) = -4\pi\delta(\vec{R})}\,, \quad \left(\text{where } k = \frac{\omega}{c}\right)$$

This equation can be solved by various methods, but let's use the method of eigenfunction expansion in ALL coordinates. Recall that, in free space, an orthonormal set of eigenfunctions of $-\nabla^2$ are:

$$u_{\vec{k}'}(\vec{R}) = (2\pi)^{-3/2} e^{i\vec{k}'\cdot\vec{R}}$$

which obey $$-\nabla^2 u_{\vec{k}'}(\vec{R}) = k'^2\, u_{\vec{k}'}(\vec{R})$$

and $$\int u^*_{\vec{k}'}(\vec{R})\, u_{\vec{k}''}(\vec{R})\, d^3R = \delta(\vec{k}'-\vec{k}'')$$

Since these states are orthonormal & complete, we can expand as

$$G(\vec{R},\omega) = \int d^3k' \, \frac{e^{i\vec{k}'\cdot\vec{R}}}{(2\pi)^{3/2}} \, a_\omega(\vec{k}')$$

Plugging this into the PDE for G gives:

$$(\nabla^2+k^2)\int d^3k' \, \frac{e^{i\vec{k}'\cdot\vec{R}}}{(2\pi)^{3/2}} \, a_\omega(\vec{k}')$$

$$= \int d^3k' \, (-k'^2+k^2) \frac{e^{i\vec{k}'\cdot\vec{R}}}{(2\pi)^{3/2}} \, a_\omega(\vec{k}') = -4\pi\,\delta(\vec{R})$$

and since we know that

$$(2\pi)^{-3}\int d^3k' \, e^{i\vec{k}'\cdot\vec{R}} = \delta(\vec{R})$$

we deduce that $a_\omega(\vec{k}') = \dfrac{4\pi}{k'^2-k^2} \dfrac{1}{(2\pi)^{3/2}}$

$$\Rightarrow G(\vec{R},\omega) = \frac{4\pi}{(2\pi)^3}\int d^3k' \, \frac{e^{i\vec{k}'\cdot\vec{R}}}{k'^2-k^2}$$

This integral can be evaluated by choosing spherical coordinates in k'-space, with the k'_z-axis chosen along $\hat{R}$, whereby we have axial symmetry, an integrand independent of $\varphi_{k'}$,

so $$\int d^3k' \longrightarrow 2\pi\int k'^2dk'\int dx' \qquad \text{with } x' \equiv \cos\theta_{k'} = \hat{k}'\cdot\hat{R}$$

and $$\vec{k}'\cdot\vec{R} = k'Rx'$$

Thus $G(\vec{R},\omega) = \frac{4\pi}{(2\pi)^2}\int_0^\infty k'^2 dk' \frac{k'^2}{k'^2-k^2}\int_{-1}^{1} dx' e^{ik'Rx'}$

$$= \frac{4\pi}{(2\pi)^2}\int_0^\infty dk' \frac{k'^2}{k'^2-k^2}\left(\frac{e^{ik'R}-e^{-ik'R}}{ik'R}\right)$$

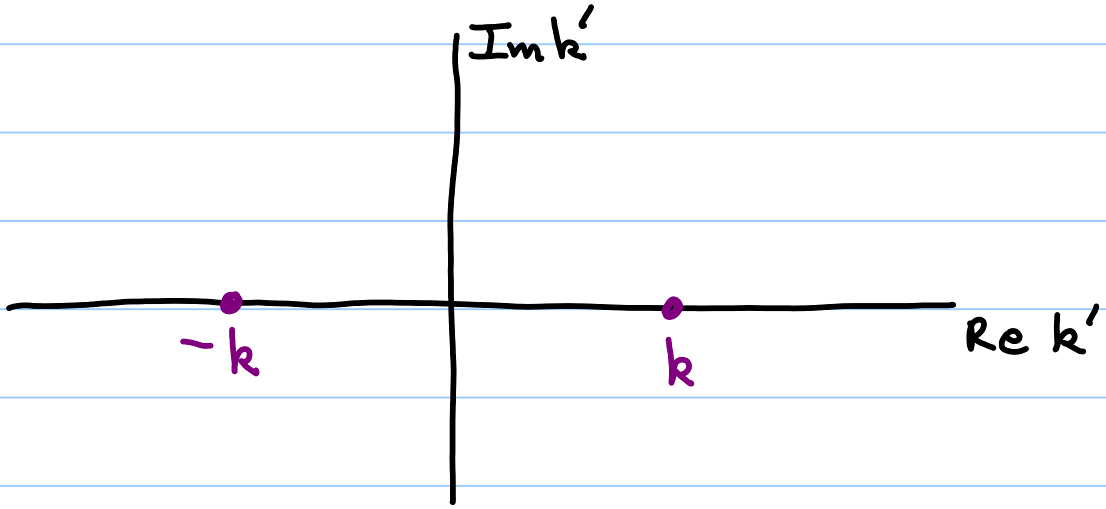
$$= \frac{1}{\pi R}\int_0^\infty dk' \frac{k'}{k'^2-k^2} \sin k'R$$

and since this integrand is an even function of k', we can rewrite it as:

$$\boxed{G(\vec{R},\omega) = \frac{1}{i\pi R}\int_{-\infty}^{\infty} dk' \frac{k' e^{ik'R}}{k'^2-k^2}}$$

A problem ..... (is it a "bug" or a "feature"?):

This integrand has two poles on the real axis, i.e. on the integration contour!



Hence, this integral is not uniquely defined as written above.

This type of ambiguity in the meaning of an integral occurs whenever a pole occurs exactly ON the integration path/contour.

Reference: Arfken + Weber, 5th Edition
Sec 7.2, esp. pp 456 - 459

Note that there are different ways to interpret such a nonunique integral, and we need to invoke some physics to choose the right interpretation.

i.e. 3 standard interpretations to invoke:

- Principal value
- Move the poles infinitesimally into either the upper or lower half-plane
- Deform the integration contour to go above or below each pole

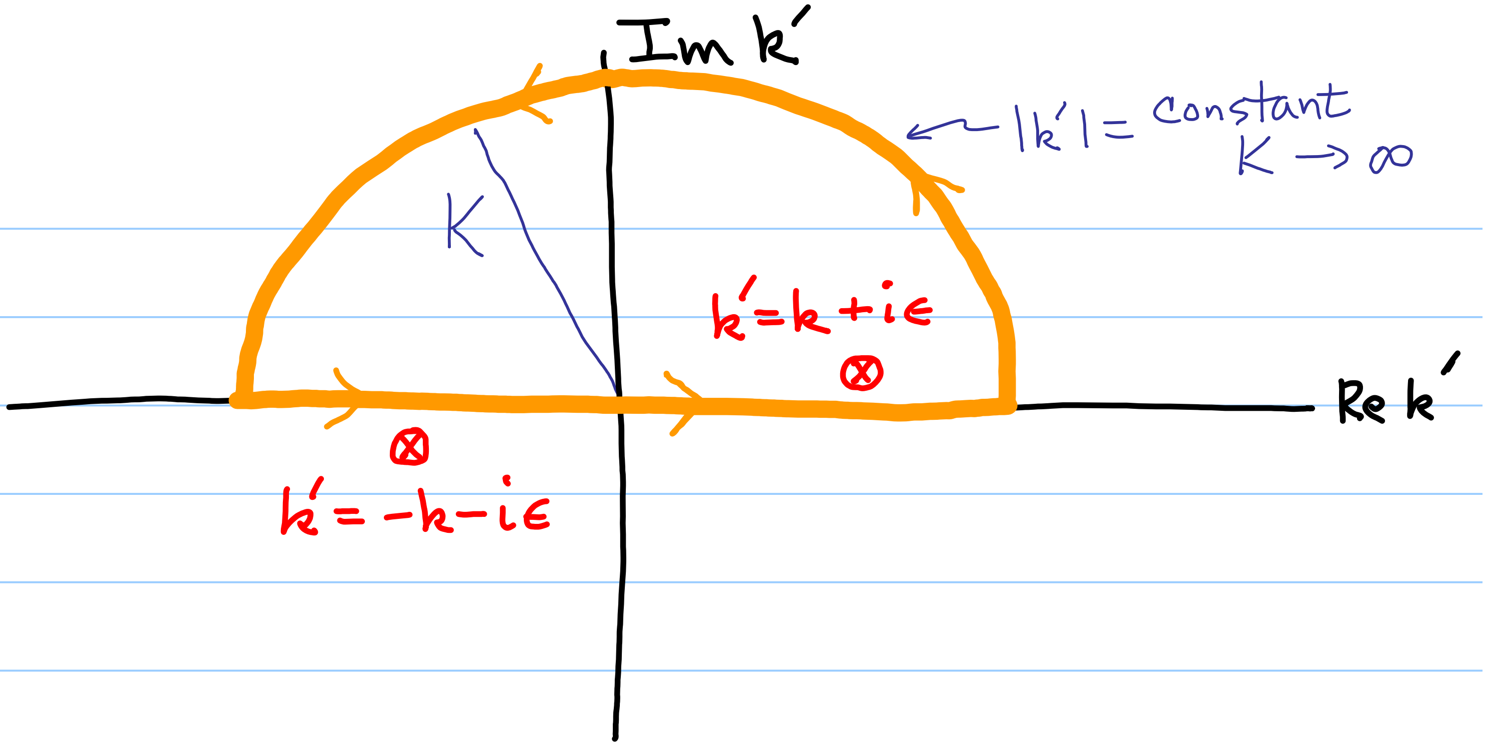
we will use THIS method

e.g. let $k \to k + i\epsilon$ and consider the above integral in the limit $\epsilon \to 0^+$

⇒ Thus our two poles at $k' = \pm k$ now move to

$$k' = k + i\epsilon$$

and $$k' = -k - i\epsilon$$



Thus the above integral is transformed into:

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} dk' \frac{k' e^{ik'R}}{(k'-k-i\epsilon)(k'+k+i\epsilon)} = \lim_{\epsilon\to 0^+}\lim_{K\to\infty}\oint_C \frac{dk'\, k' e^{ik'R}}{(k'-k-i\epsilon)(k'+k+i\epsilon)}$$

this equality holds because the semicircular contribution vanishes as K → ∞

Next evaluate the $\oint_C$ using Cauchy's Integral Formula,

$$\oint_C \frac{f(z)\,dz}{z-z_0} = 2\pi i \,\text{Res}\left(\frac{f(z)}{z-z_0}\right)_{z_0} = 2\pi i f(z_0)$$

$$= \lim_{\epsilon\to 0^+}\lim_{k'\to k+i\epsilon}\left(\frac{k' e^{ik'R}}{k'+k+i\epsilon}\right)(2i\pi) = i\pi e^{ikR}$$

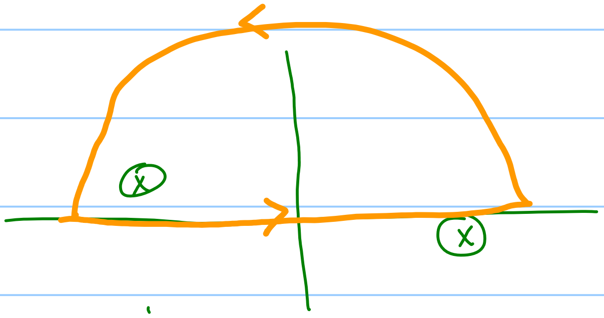
Putting this into our formula,

$$\mathcal{G}(\vec{R},\omega) = \frac{1}{i\pi R}\int_{-\infty}^{\infty} dk' \, \frac{k' e^{ik'R}}{k'^2 - k^2}$$

and our choice of moving the poles to $k \to k + i\epsilon$ gives finally a remarkably simple result:

$$\boxed{\mathcal{G}(\vec{R},\omega) = \frac{e^{ikR}}{R}}$$

If instead we had chosen $k \to k - i\epsilon$ with $\epsilon \to 0^+$



we would obtain

$$\oint_C dk' \, \frac{k' e^{ik'R}}{(k'-k+i\epsilon)(k'+k-i\epsilon)} = \frac{2\pi i(-k)}{(-2k)} e^{-ikR}$$

$$= \pi i e^{-ikR}$$

Thus we have obtained two different possible solutions,

$$\boxed{\mathcal{G}^{(\pm)}(\vec{R},\omega) = \frac{e^{\pm ikR}}{R}} \quad \text{with } k = \frac{\omega}{c}$$

Recalling that the time dependence for each ω is $e^{-i\omega\tau}$, we can see that the $G^{(+)}$ solution corresponds to an OUTGOING spherical wave propagating outward from a source disturbance.

And the $G^{(-)}$ solution represents an INCOMING spherical wave.

To see this in more detail, evaluate the final ω-integral,

$$G^{(\pm)}(\vec{R},\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\tau \pm i\omega\frac{R}{c}}}{R}$$

which is trivially evaluated, giving

$$G^{(\pm)}(\vec{R},\tau) = \frac{1}{R}\,\delta\left(\tau \mp \frac{R}{c}\right)$$

or recalling the earlier variable change, we finally have the infinite space Green function for the wave equation in 3D:

$$G^{(\pm)}(\vec{x}-\vec{x}\,', t-t') = \frac{\delta\left(t - t' \mp \frac{1}{c}|\vec{x}-\vec{x}\,'|\right)}{|\vec{x}-\vec{x}\,'|}$$

sometimes convenient to rewrite as

$$\boxed{G^{(\pm)}(\vec{x}-\vec{x}\,',t-t') = \frac{\delta\left[t'-\left(t\mp\frac{|\vec{x}-\vec{x}\,'|}{c}\right)\right]}{|\vec{x}-\vec{x}\,'|}}$$

Using this, the resulting solutions of the original inhomogeneous wave equation are:

$$\Psi(\vec{x},t) = \iint d^3x'\,dt'\,f(\vec{x}\,',t')\,\frac{\delta\left[t'-\left(t\mp\frac{|\vec{x}-\vec{x}\,'|}{c}\right)\right]}{|\vec{x}-\vec{x}\,'|}$$

or

$$\Psi(\vec{x},t) = \int d^3x'\,\frac{f\left(\vec{x}\,',t\mp\frac{|\vec{x}-\vec{x}\,'|}{c}\right)}{|\vec{x}-\vec{x}\,'|}$$

Interpretation

The $G^{(+)}$ solution at $t'=t-\frac{|\vec{x}-\vec{x}\,'|}{c}$ implies that a wave disturbance that is FELT at time t must have originated from a SOURCE disturbance at an EARLIER time, $t'=t-\frac{|\vec{x}-\vec{x}\,'|}{c}$

It is earlier by exactly the light transit time from $\vec{x}\,'$ to $\vec{x}$.

Retarded solutions for the potentials (Lorentz gauge)

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{[\rho(\vec{x}',t')]_{ret}}{R}$$

and

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{[\vec{J}(\vec{x}',t')]_{ret}}{R}$$

- where $[\ \ ]_{ret}$ means to evaluate at the retarded time $t' = t - \frac{R}{c}$
- and where $\vec{R} \equiv \vec{x} - \vec{x}'$,

$$R = |\vec{x} - \vec{x}'| , \qquad \hat{R} = \frac{\vec{R}}{R}$$

$\vec{E}$- and $\vec{B}$-fields if no media are present

Start from Maxwell's equations $\nabla\cdot\vec{E} = \rho/\epsilon_0$, $\nabla\cdot\vec{B} = 0$

and

$$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \nabla\times(\nabla\times\vec{E}) + \frac{\partial}{\partial t}\nabla\times B = 0$$

$$\Rightarrow \nabla(\underbrace{\cancel{\nabla\cdot}\vec{E}}_{\rho/\epsilon_0}) - \nabla^2\vec{E} + \frac{\partial}{\partial t}(\underbrace{\cancel{\nabla\times}\vec{B}}_{\mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}) = 0$$

$$\Rightarrow \nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial\vec{J}}{\partial t} + \frac{1}{\epsilon_0}\nabla\rho$$

or

$$\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{\epsilon_0}\left(-\nabla\rho - \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t}\right)}$$

and similarly for $\vec{B}$:

$$\nabla\times(\nabla\times\vec{B}) = \mu_0\nabla\times\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\underbrace{(\nabla\times\vec{E})}_{-\frac{\partial\vec{B}}{\partial t}}$$

$$\Rightarrow \nabla(\underbrace{\nabla\cdot\vec{B}}_{0}) - \nabla^2\vec{B} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = \mu_0\nabla\times\vec{J}$$

$$\Rightarrow \boxed{\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = -\mu_0\nabla\times\vec{J}}$$

Since "the same equations have the same solutions" = Feynman's dictum, we can immediately solve these inhomogeneous wave equations as:

$$\boxed{\begin{aligned}\vec{E}(\vec{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{1}{R}\left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t'}\right]_{ret}\\ \vec{B}(\vec{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{R}\left[\nabla'\times\vec{J}\right]_{ret}\end{aligned}}$$

An important aside: working with retarded sources requires <u>care</u>, e.g. observe that

$$\left[\nabla' f(\vec{x}',t')\right]_{ret} \neq \nabla'\left[f(\vec{x}',t')\right]_{ret}$$

Why? Because $[\nabla' f(\vec{x}',t')]_{ret}$ means that the gradient is taken with t' held fixed

whereas $[f(\vec{x}',t')]_{ret} = f\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$ implies a gradient <u>with t fixed</u>.

Hence $$[\nabla'\rho]_{ret} = \nabla'[\rho]_{ret} - \left[\frac{\partial\rho}{\partial t'}\right]_{ret}\nabla'\left(t-\frac{R}{c}\right)$$

since,

e.g. $$\nabla'\rho\left(\vec{x}', t-\frac{R}{c}\right) = \left[\nabla'\rho(\vec{x}',t')\right]_{ret} + \left[\frac{\partial\rho(\vec{x}',t')}{\partial t'}\right]_{ret}\nabla'\left(t-\frac{R}{c}\right)$$

and $\boxed{\nabla' R = -\hat{R}}$

or $$\boxed{[\nabla'\rho]_{ret} = \nabla'[\rho]_{ret} - \frac{\hat{R}}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{ret}}$$

Next we need $[\nabla'\times\vec{J}]_{ret}$:

consider, with implied summation over repeated indices:

$$\epsilon_{ijk}[\partial_i' J_j]_{ret} = \epsilon_{ijk}\partial_i'[J_j]_{ret} - \epsilon_{ijk}\left[\frac{\partial J_j}{\partial t'}\right]_{ret}\partial_i'\left(t-\frac{R}{c}\right)$$

which is rewritten in vector notation as

$$[\nabla'\times\vec{J}]_{ret} = \nabla'\times[\vec{J}]_{ret} + \frac{1}{c}(\vec{\nabla}'R)\times\left[\frac{\partial\vec{J}}{\partial t'}\right]_{ret}$$

or Finally

$$\boxed{[\nabla'\times\vec{J}]_{ret} = \nabla'\times[\vec{J}]_{ret} + \frac{1}{c}\left[\frac{\partial\vec{J}}{\partial t'}\right]\times\hat{R}}$$

Next let's use these expressions in our solution for $\vec{E}$:

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\frac{-\nabla'[\rho]_{ret}}{R} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\vec{J}}{\partial t'}\right]_{ret}\right\}$$

Integration by parts can simplify this a bit,

e.g. $$\int d^3x'\,\frac{1}{R}\,\partial_i'[\rho]_{ret} = -\int d^3x'\,[\rho]_{ret}\overbrace{\left(\partial_i'\frac{1}{R}\right)}^{=\frac{\hat{R}}{R^2}} + \underbrace{(\text{surface terms})}_{\to 0 \text{ if a localized source}}$$

Giving

$$\boxed{\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}[\rho(\vec{x}',t')]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\vec{x}',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\right\}}$$

= generalized Coulomb Law

And tackling $\vec{B}$ similarly, i.e.

$$\int \left(\nabla' \times [\vec{J}]_{ret}\right) \frac{1}{R}\, d^3x' = -\int \left(\nabla' \frac{1}{R}\right) \times [\vec{J}]_{ret}\, d^3x' + (\text{surface terms}) \rightarrow 0$$

$$= -\int \frac{\hat{R}}{R^2} \times [\vec{J}]_{ret}$$

giving

$$\boxed{\vec{B}(\vec{x},t) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ [\vec{J}(\vec{x}',t')]_{ret} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret} \times \frac{\hat{R}}{cR} \right\}}$$

= generalized Biot-Savart Law

Special Case: A point charge q in motion, $\vec{r}_0(t)$:

$$\Longrightarrow \quad \rho(\vec{x}',t') = q\,\delta[\vec{x}' - \vec{r}_0(t')]$$

$$\vec{J}(\vec{x}',t') = \rho(\vec{x}',t') \frac{d\vec{r}_0(t')}{dt'} = \rho \vec{v}(t')$$

gives (show this on your own):

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left\{ \left[\frac{\hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{\kappa R}\right]_{ret} - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\vec{v}}{\kappa R}\right]_{ret} \right\}$$

$$\vec{B} = \frac{\mu_0 q}{4\pi} \left\{ \left[\frac{\vec{v}\times\hat{R}}{\kappa R^2}\right]_{ret} + \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\vec{v}\times\hat{R}}{\kappa R}\right]_{ret} \right\}$$

where $\kappa = 1 - \frac{\vec{v}\cdot\hat{R}}{c}$, and "Feynman's form" is

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \left\{ \left[\frac{\hat{R}}{R^2}\right]_{ret} + \frac{[R]_{ret}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{R^2}\right]_{ret} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[\hat{R}]_{ret} \right\}$$

$$\vec{B} = \frac{\mu_0 q}{4\pi} \left\{ \left[\frac{\vec{v}\times\hat{R}}{\kappa^2 R^2}\right]_{ret} + \frac{1}{c[R]_{ret}}\frac{\partial}{\partial t}\left[\frac{\vec{v}\times\hat{R}}{\kappa}\right]_{ret} \right\}$$

Poynting's Theorem - Conservation of energy & momentum

For a single charge q, the rate at which work is done by external $\vec{E}, \vec{B}$-fields is

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \vec{v} = q\,\vec{v} \cdot \vec{E}$$

and this is often stated as:

"B- fields do no work on q"

or if charge and current are distributed in space, then the rate of change of energy within any finite volume V is

$$\frac{dW}{dt} = \int_V \vec{J} \cdot \vec{E}\, d^3x$$

Next, explore explicitly how the EM-field energy within V changes, let's eliminate $\vec{J}$ using

$$\vec{J} = \nabla \times \vec{H} - \frac{\overrightarrow{\partial D}}{\partial t}$$

$$\Rightarrow \int_V \vec{J} \cdot \vec{E}\, d^3x = \int_V \left\{ \vec{E} \cdot \left(\nabla \times \vec{H}\right) - \vec{E} \cdot \frac{\overrightarrow{\partial D}}{\partial t} \right\} d^3x$$

$$\text{But} \quad \nabla \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \underset{\overset{''}{-\frac{\overrightarrow{\partial B}}{\partial t}}}{\left(\nabla \times \vec{E}\right)} - \vec{E} \cdot \left(\nabla \times \vec{H}\right)$$

whereby

$$\int_V \vec{J} \cdot \vec{E}\, d^3x = -\int_V \left[\nabla \cdot \left(\vec{E} \times \vec{H}\right) + \vec{E} \cdot \frac{\overrightarrow{\partial D}}{\partial t} + \vec{H} \cdot \frac{\overrightarrow{\partial B}}{\partial t} \right] d^3x$$

Suppose now that the medium is linear, so the total energy density becomes

$$u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$$

$$\Rightarrow \int_V \left[\frac{\partial u}{\partial t} + \nabla\cdot(\vec{E}\times\vec{H}) \right] d^3x = - \int_V \vec{J}\cdot\vec{E}\, d^3x$$

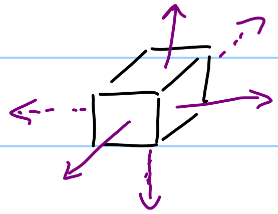
But since this holds for ANY volume V, the integrands must be equal, i.e.

$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = -\vec{J}\cdot\vec{E}}$$

where $\vec{S} \equiv \vec{E}\times\vec{H}$ is called the Poynting vector

Interpretation

$\boxed{\nabla\cdot\vec{S}}$ represents the flow of energy OUT FROM a unit volume d^3x



$\boxed{\vec{J}\cdot\vec{E}}$ is the WORK DONE (per unit volume, per unit time) by the EM FIELDS, which is converted into mechanical or heat energy of the charged particles i.e., $$\frac{dW^{mech}}{dt} = \int_V \vec{J}\cdot\vec{E}\, d^3x$$

$\Rightarrow W^{mech}$ = total energy of particles within V, assuming that no particles leave or enter V

And from previous analysis we know that

$$W^{fields} = \int_V u\, d^3x = \frac{1}{2}\int\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)d^3x$$

(for linear media)

no media → $\frac{\epsilon_0}{2}\int_V\left(\vec{E}^2 + c^2\vec{B}^2\right)d^3x$

Thus we see that Poynting's theorem expresses energy conservation for the combined system:

$$\boxed{\frac{dW^{total}}{dt} = \frac{d}{dt}\left(W^{fields} + W^{mech}\right) = -\boxed{\oint_S \hat{n}\cdot\vec{S}\, da}}$$

total field energy in volume V, $W^{fields} = \int_V u\, d^3x$

total mechanical energy of particles in V

rate of EM field energy flow out of V through the surface S

Conservation of Linear Momentum

Start from the total electromagnetic force on a charge q:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \longrightarrow \int_V \left(\rho\vec{E} + \rho\vec{v}\times\vec{B}\right) d^3x$$

Then if we call $\vec{P}_{mech}$ = sum of the momenta of all particles in a volume V, we have:

$$\frac{d}{dt}\vec{P}_{mech} = \int_V \left(\rho\vec{E} + \vec{J}\times\vec{B}\right) d^3x$$

Next, use Maxwell's equations to eliminate the sources $\rho, \vec{J}$, again assuming no media.

$$\Rightarrow \rho = \epsilon_0 \nabla\cdot\vec{E}, \quad \vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}(\nabla\cdot\vec{E}) + \frac{1}{\mu_0\epsilon_0}(\nabla\times\vec{B})\times\vec{B} - \frac{\partial\vec{E}}{\partial t}\times\vec{B}\right],$$

and observe that

$$\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

So we can write also:

we've added 0 here

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\Big[\vec{E}(\nabla\cdot\vec{E}) + c^2\vec{B}(\nabla\cdot\vec{B}) - \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times(\nabla\times\vec{E}) - c^2\vec{B}\times(\nabla\times\vec{B})\Big]$$

Thus we get:

$$\frac{d\vec{P}_{mech}}{dt} + \frac{d}{dt}\boxed{\int_V \epsilon_0 \vec{E}\times\vec{B}\, d^3x}$$

tentatively identify this as the total EM momentum, i.e. $\vec{P}_{field} = \int d^3x\, \vec{g}$, where $\vec{g} = \epsilon_0 \vec{E}\times\vec{B} = c^{-2}\vec{E}\times\vec{H}$

$$= \epsilon_0 \int_V \left\{ \vec{E}(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E}) + c^2\vec{B}(\nabla\cdot\vec{B}) - c^2\vec{B}\times(\nabla\times\vec{B}) \right\} d^3x$$

Logic: To establish this as a momentum conservation law, we want to convert the RHS of this preceding equation into a surface integral. We would then interpret the surface integral as a flow of field momentum through the closed surface S into the volume V.

$\Rightarrow$ Consider the Cartesian components, use the notations $\{1,2,3\} \longleftrightarrow \{x,y,z\}$, i.e. $\alpha,\beta,\gamma = 1,2,3$, etc.

$$\Rightarrow \left[\vec{E}(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E})\right]_\gamma = E_\gamma \sum_\beta \partial_\beta E_\beta - \sum_{\alpha\beta} \epsilon_{\alpha\beta\gamma} E_\alpha (\nabla\times\vec{E})_\beta$$

$$= E_\gamma \sum_\beta \partial_\beta E_\beta - \sum_{\alpha\beta}\sum_{ij} \epsilon_{\alpha\beta\gamma}\,\epsilon_{ij\beta}\, E_\alpha\, \partial_i E_j$$

(Now use $\sum_\beta \epsilon_{\beta\gamma\alpha}\epsilon_{\beta ij} = \delta_{\gamma i}\delta_{\alpha j} - \delta_{\gamma j}\delta_{\alpha i}$)

$$= E_\gamma \sum_\beta \partial_\beta E_\beta - \sum_{\alpha ij} \delta_{\gamma i}\delta_{\alpha j} E_\alpha \partial_i E_j + \sum_{\alpha ij} \delta_{\gamma j}\delta_{\alpha i} E_\alpha \partial_i E_j$$

$$= E_\gamma \sum_\alpha \partial_\alpha E_\alpha - \sum_\alpha E_\alpha \partial_\gamma E_\alpha + \sum_\alpha E_\alpha \partial_\alpha E_\gamma$$

or $$[\vec{E}(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E})]_\gamma$$

$$= \sum_\alpha \frac{\partial}{\partial x_\alpha}(E_\alpha E_\gamma) - \frac{1}{2}\sum_\alpha \frac{\partial}{\partial x_\gamma}(E_\alpha^2)$$

$$= \sum_\alpha \frac{\partial}{\partial x_\alpha}\Big[E_\alpha E_\gamma - \frac{1}{2}\delta_{\alpha\gamma}\vec{E}\cdot\vec{E}\Big]$$

Observe that for <u>each</u> component γ, this $[\;\;]_\alpha$ is a 3-component quantity i.e. a <u>vector</u> Q_α

In other words the RHS, for each γ, is the DIVERGENCE of the vector $\vec{Q}$, (i.e. $\nabla\cdot\vec{Q}$)

We can put this into the standard form by changing the index $\gamma \to \beta$, and now define the symmetric <u>MAXWELL STRESS TENSOR</u> as

$$\boxed{T_{\alpha\beta} \equiv \epsilon_0\Big[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\Big]}$$

We have thus proven that

$$\frac{d}{dt}(\vec{P}^{mech} + \vec{P}^{field}) = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\beta\alpha}\, d^3x$$

$$= \int_V (\nabla\cdot\overleftrightarrow{T})_\alpha d^3x = \oint_S n_\beta T_{\beta\alpha}\, da$$

components of outward normal $\hat{n}$

Or finally we have simply:

$$\boxed{\frac{d}{dt}\left(\vec{P}^{\,mech} + \vec{P}^{\,field}\right) = \oint_S da\,\hat{n}\cdot\overleftrightarrow{T}}$$

W00T!

(This looks slightly different than Jackson's 6.122, but owing to the symmetry of $T_{\alpha\beta}$, they are equivalent)

Notational Comments

$\vec{a}\cdot\overleftrightarrow{T}$ is a vector: $(\vec{a}\cdot\overleftrightarrow{T})_\alpha = \sum_\beta a_\beta T_{\beta\alpha}$

whereas

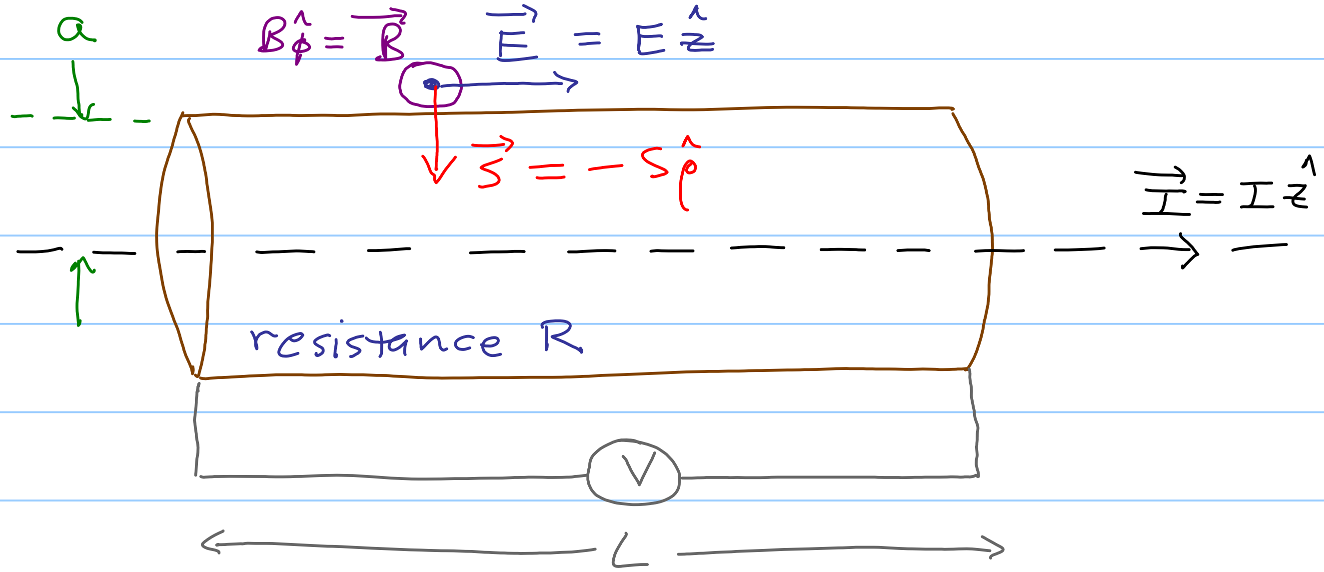
$\vec{a}\cdot\overleftrightarrow{T}\cdot\vec{b}$ is a scalar, $\sum_{\alpha\beta} a_\alpha T_{\alpha\beta} b_\beta$

Physical Interpretation

$(\hat{n}\cdot\overleftrightarrow{T})_\alpha = \sum_\beta n_\beta T_{\beta\alpha}$ is the α-th component of the EM momentum flow per unit area across the surface S and into volume V.

Moreover, since $\vec{F} = \frac{d\vec{p}}{dt}$, we can also view $(\hat{n}\cdot\overleftrightarrow{T})_\alpha$ as the α^{th} component of the FORCE PER UNIT AREA that is transmitted onto the particles and fields within V.

Example: Poynting's Theorem applied to a wire carrying steady current



i.e. $\vec{E} = \frac{V}{L}\hat{z}$, $\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}$ at the $\rho = a$ surface

$$\Rightarrow \vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left(\frac{V}{L}\right)\left(\frac{\mu_0 I}{2\pi a}\right)(-\hat{\rho})$$

$$\Rightarrow \frac{dW^{tot}}{dt} = \underbrace{\frac{dW^{mech}}{dt}}_{} + \underbrace{\frac{dW^{Fields}}{dt}}_{} = \underbrace{-\oint_S \hat{n}\cdot\vec{S}\,da = \left(\frac{VI}{2\pi aL}\right)2\pi aL = VI}_{}$$

I^2R = rate of appearance of mechanical (heat) energy in the wire

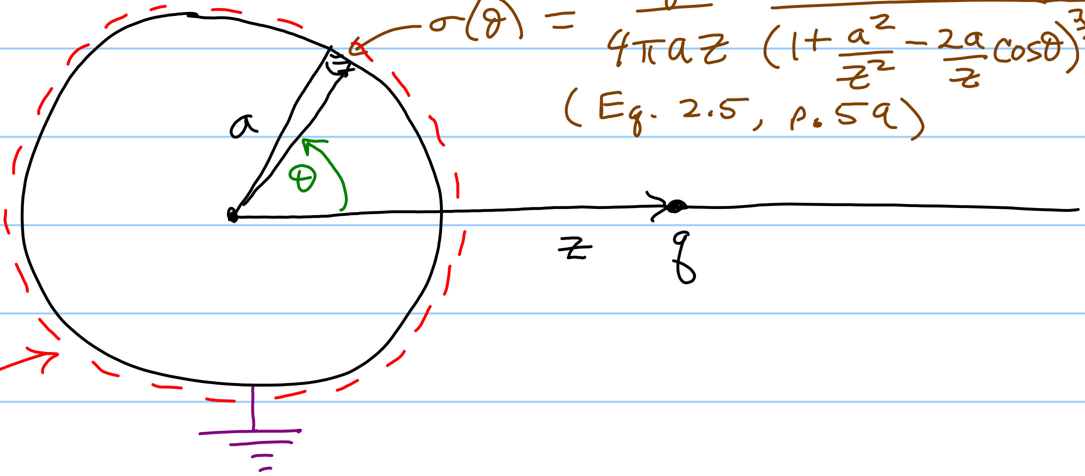
0 for a constant voltage source that produces a steady current

rate at which EM energy flows INTO the wire through the surface

$\Rightarrow VI = I^2R$ or $V = IR$, as expected from circuit theory

Check of the stress tensor calculation of force

e.g. – Find the force on a grounded, conducting sphere of radius a due to a nearby charge q at distance z from the center.



$$\sigma(\theta) = \frac{-q}{4\pi a z}\frac{1 - a^2/z^2}{\left(1 + \frac{a^2}{z^2} - \frac{2a}{z}\cos\theta\right)^{3/2}}$$

(Eq. 2.5, p. 59)

consider an imaginary sphere infinitesimally larger than the physical sphere

$$\Rightarrow \vec{E} = \frac{\sigma(\theta)}{\epsilon_0}\hat{r} = \frac{\sigma(\theta)}{\epsilon_0}\left[\hat{x}\, r\sin\theta\cos\phi + \hat{y}\, r\sin\theta\sin\phi + \hat{z}\, r\cos\theta\right]$$

and $E^2 = \dfrac{\sigma^2(\theta)}{\epsilon_0^{\,2}}$

So the stress tensor is $T_{\alpha\beta} = \epsilon_0\left(E_\alpha E_\beta - \frac{1}{2}E^2\delta_{\alpha\beta}\right)$

i.e.

$$\frac{\overleftrightarrow{T}}{\epsilon_0\left(\frac{\sigma(\theta)}{\epsilon_0}\right)^2} = \begin{array}{c|ccc} & x & y & z \\ \hline x & \sin^2\theta\cos^2\phi-\frac{1}{2}, & \sin^2\theta\cos\phi\sin\phi, & \sin\theta\cos\theta\cos\phi \\ y & \sin^2\theta\cos\phi\sin\phi, & \sin^2\theta\sin^2\phi-\frac{1}{2}, & \sin\theta\cos\theta\sin\phi \\ z & \sin\theta\cos\theta\cos\phi, & \sin\theta\cos\theta\sin\phi, & \cos^2\theta-\frac{1}{2} \end{array}$$

We expect $\vec{F} = F\hat{z}$ by symmetry, of course

$$\vec{F} = \oint_s \hat{r}\cdot\overleftrightarrow{T}\,da = a^2\int d\Omega\Big[\underset{\sin\theta\cos\phi}{\underbrace{(\hat{r}\cdot\hat{x})}}T_{xz}\hat{z} + \underset{\sin\theta\sin\phi}{\underbrace{(\hat{r}\cdot\hat{y})}}T_{yz}\hat{z} + \underset{\cos\theta}{\underbrace{(\hat{r}\cdot\hat{z})}}T_{zz}\hat{z}\Big]$$

$$= \epsilon_0 a^2\int_0^{2\pi} d\phi\int_{-1}^{1} d(\cos\theta)\left[\frac{q^2\left(1-\frac{a^2}{z^2}\right)^2}{(4\pi a)^2\epsilon_0^2 z^2}\;\frac{1}{\left(1+\frac{a^2}{z^2}-\frac{2a\cos\theta}{z}\right)^3}\right]$$

$$\cdot\underbrace{\left\{\sin^2\theta(\cos^2\phi+\sin^2\phi)+(\cos^2\theta-\tfrac{1}{2})\right\}}_{\{\}=\frac{1}{2}}\cos\theta$$

and using $\displaystyle\int_{-1}^{1}\frac{u\,du}{\left(1+\frac{a^2}{z^2}-\frac{2au}{z}\right)^3} = \frac{4az^7}{(z^2-a^2)^4}$

gives finally

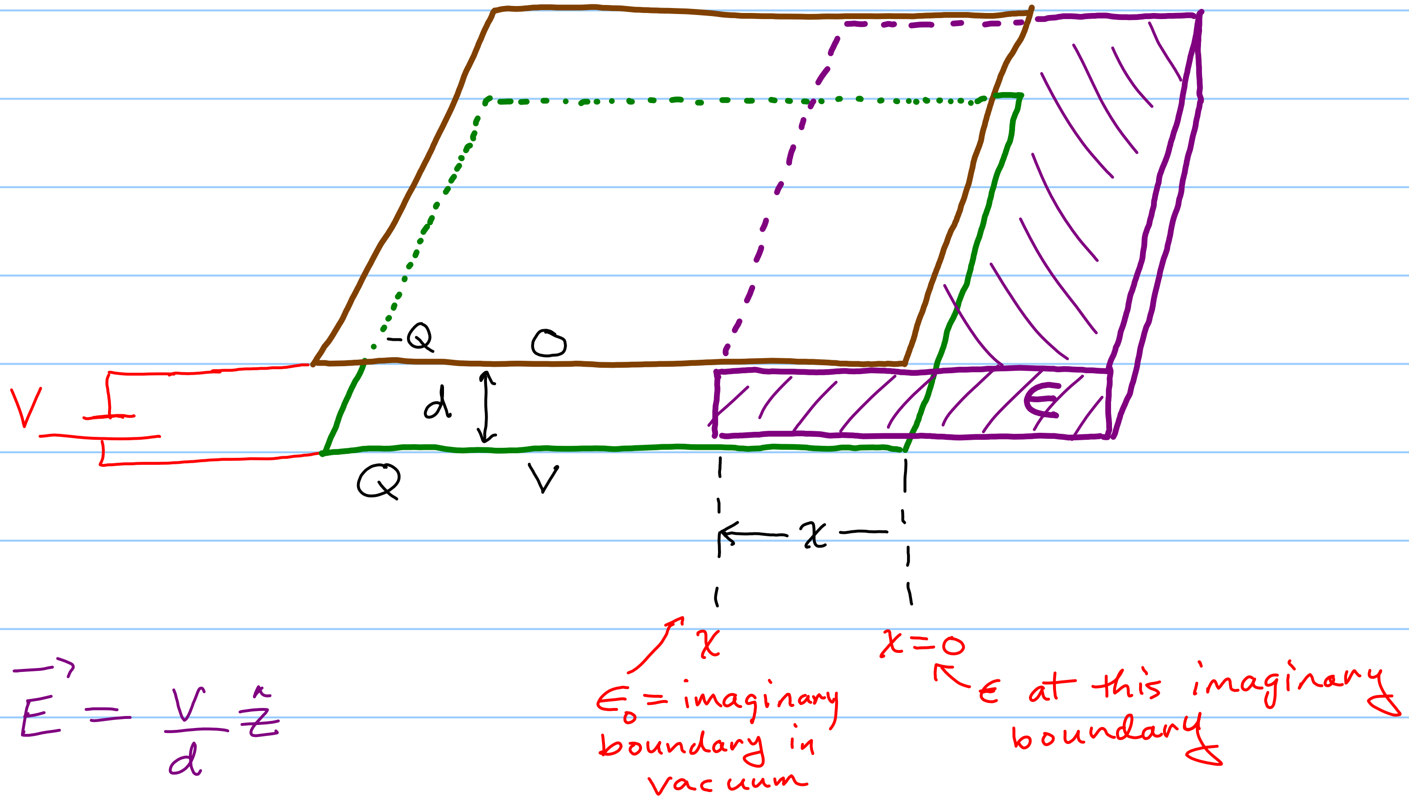
$$\vec{F} = \hat{z}\;\frac{q^2a^2}{4\pi\epsilon_0}(z^2-a^2)^2\,\frac{az^3}{z^2a^2}\;\frac{1}{(z^2-a^2)^4}$$

or $$\boxed{\vec{F} = \hat{z}\;\frac{q^2}{4\pi\epsilon_0}\;\frac{za}{(z^2-a^2)^2}}$$

which agrees with the elementary calculation in Eqs. 2.6, 2.7.

Note: We used the TOTAL FIELD to obtain $\vec{E}$ in $\overleftrightarrow{T}$, NOT just the field from the external charge q!

Example: Force on a dielectric slab for an "ideal" parallel plate capacitor



$$\vec{E} = \frac{V}{d}\hat{z}$$

We will need the generalization of the stress tensor to linear media; see, e.g., Problem 6.9

$$T_{\alpha\beta} = (E_\alpha D_\beta + H_\alpha B_\beta) - \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\delta_{\alpha\beta}$$

Then $T_{\alpha\beta} = \begin{cases} \epsilon_0 (E_\alpha E_\beta - \frac{1}{2}E^2\delta_{\alpha\beta}), & \text{at } x\text{-boundary} \\ \epsilon\,(E_\alpha E_\beta - \frac{1}{2}E^2\delta_{\alpha\beta}), & \text{at } x=0 \text{ boundary} \end{cases}$

$$\Rightarrow \overleftrightarrow{T} = \begin{pmatrix} \epsilon_0 \\ \text{or} \\ \epsilon \end{pmatrix} \frac{V^2}{d^2} \begin{array}{c|ccc|} & x & y & z \\ \hline x & -\frac{1}{2} & 0 & 0 \\ y & 0 & -\frac{1}{2} & 0 \\ z & 0 & 0 & +\frac{1}{2} \end{array}$$

and $$\hat{n}\cdot\overleftrightarrow{T}\Big|_{\substack{x\\ \text{surface}}} = \epsilon_0\,\frac{V^2}{d^2}\left(-\frac{1}{2}\right)\underbrace{(\hat{n}\cdot\hat{x})}_{\substack{\shortparallel\\ +1}}\hat{x}$$ ('−' sign means to the right)

while $$\hat{n}\cdot\overleftrightarrow{T}\Big|_{x=0 \text{ surface}} = \epsilon\,\frac{V^2}{d^2}\left(-\frac{1}{2}\,\underbrace{\hat{n}\cdot\hat{x}}_{\substack{\shortparallel\\ -1}}\right)\hat{x} = \underset{\substack{\text{to the}\\ \text{left}}}{\frac{\epsilon V^2}{2d^2}\,\hat{x}}$$

Thus the net electrical force in the x-direction is $$\vec{F} = (\epsilon-\epsilon_0)\frac{V^2}{2d^2}\,(\underset{\nwarrow \text{ area}}{Ld})\,\hat{x}$$ which is in the $+\hat{x}$-direction

So the E-field pushes the slab <u>INTO</u> the capacitor

Compare this result with our derivation based on energy conservation:

$$\underset{\substack{\uparrow\\ \text{change in}\\ \text{energy stored}\\ \text{in the fields}}}{d\left(\tfrac{1}{2}CV^2\right)} = \overset{''\,\vec{F}^{us}\cdot\vec{dx}}{dW(\text{done by us})} + \overset{''\,VdQ = V^2 dC}{dW(\text{done by battery})}$$

But here

$$C(x) = \frac{\epsilon_0 L(L-x)}{d} + \epsilon\,\frac{Lx}{d}$$

$$= \frac{\epsilon_0 L^2}{d} + (\epsilon-\epsilon_0)\frac{Lx}{d}$$

$$\Rightarrow dC = dx\,\frac{dC}{dx} = dx\,(\epsilon-\epsilon_0)\frac{L}{d}$$

$$\Rightarrow \frac{1}{2}V^2\frac{dC}{dx}\,dx = \vec{F}^{us}\cdot\vec{dx} + V^2\frac{dC}{dx}\,dx$$

or $$\vec{F}^{us}\cdot\vec{dx} = -\frac{1}{2}V^2\frac{dC}{dx}\,dx < 0$$

or $\vec{F}^{us} = -\frac{1}{2}V^2\frac{dC}{dx}\hat{x}$ = force WE apply to the right to keep the slab from accelerating

which must be equal and opposite to the electrical force on the slab, namely

$$\vec{F}^{elec} = \frac{1}{2}V^2\frac{dC}{dx} = (\epsilon - \epsilon_0)\frac{V^2}{2d}L\,\hat{x}$$

which checks!

Sec. 6.9 Poynting's theorem for harmonic fields, impedance + admittance

Consider now a class of solutions for which ALL fields and sources have the same, monochromatic time-dependence,

i.e. $\vec{E}(\vec{x},t) = \mathrm{Re}\left[\vec{E}(\vec{x})e^{-i\omega t}\right] = \frac{1}{2}\left[\vec{E}(\vec{x})e^{-i\omega t} + \vec{E}^*(\vec{x})e^{i\omega t}\right]$

(provided ω = real)

⇒ Harmonic sources + fields are especially relevant for AC circuit theory.

Now, we will often want to calculate the time-average of a product, such as:

$$\vec{J}(\vec{x},t)\cdot\vec{E}(\vec{x},t) = \frac{1}{4}\left[\vec{J}(\vec{x})e^{-i\omega t} + \vec{J}^*(\vec{x})e^{i\omega t}\right]\cdot\left[\vec{E}(\vec{x})e^{-i\omega t} + \vec{E}(\vec{x})e^{i\omega t}\right]$$

$$= \frac{1}{4}\Big\{ \vec{J}^*(\vec{x})\cdot\vec{E}(\vec{x}) + \vec{J}(\vec{x})\cdot\vec{E}^*(\vec{x}) + \vec{J}\cdot\vec{E}\, e^{-2i\omega t} + \vec{J}^*\cdot\vec{E}^*\, e^{2i\omega t} \Big]$$

$$= \frac{1}{2}\,\mathrm{Re}\Big[\vec{J}^*(\vec{x})\cdot\vec{E}(\vec{x}) + \underbrace{\vec{J}(\vec{x})\cdot\vec{E}(\vec{x})\, e^{-2i\omega t}}_{\to\, 0 \text{ when time-averaged}} \Big]$$

Conclusion: The time-average of ANY quadratic quantity for harmonic fields and sources is

$$\big\langle \vec{A}(\vec{x},t)\cdot\vec{B}(\vec{x},t) \big\rangle = \frac{1}{2}\,\mathrm{Re}\big[\vec{A}^*(\vec{x})\cdot\vec{B}(\vec{x})\big]$$

assuming $\vec{A}(\vec{x},t) = \mathrm{Re}\big[\vec{A}(\vec{x})\,e^{-i\omega t}\big]$

$\vec{B}(\vec{x},t) = \mathrm{Re}\big[\vec{B}(\vec{x})\,e^{-i\omega t}\big]$, $\omega = \text{real}$

Of course, Maxwell's equations for harmonic sources + fields are purely SPATIAL PDEs:

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{E} - i\omega\vec{B} = 0$$

$$\nabla\cdot\vec{D} = \rho \qquad \nabla\times\vec{H} + i\omega\vec{D} = \vec{J}$$

One particularly important quantity of interest is the TIME-AVERAGED RATE at which the fields perform work, within a volume V:

$\Rightarrow$ Consider

$$\frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}\, d^3x = \frac{1}{2}\int_V \vec{E}\cdot\left(\nabla\times\vec{H}^* - i\omega\vec{D}^*\right) d^3x$$

(insert identity: $\vec{a}\cdot\nabla\times\vec{b} = -\nabla\cdot(\vec{a}\times\vec{b}) + \vec{b}\cdot\nabla\times\vec{a}$)

$$= \frac{1}{2}\int_V \left[-\nabla\cdot(\vec{E}\times\vec{H}^*) - i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)\right] d^3x$$

Now define the following, for harmonic fields only,

$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$ = complex Poynting vector

$w_e = \frac{1}{4}\vec{E}\cdot\vec{D}^*$ = complex electric energy density

$w_m = \frac{1}{4}\vec{B}\cdot\vec{H}^*$ = Complex magnetic energy density

$$\Rightarrow \boxed{\frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}\, d^3x + 2i\omega\int_V (w_e - w_m)\, d^3x + \oint_S \vec{S}\cdot\hat{n}\, da = 0}$$

= Poynting's theorem for harmonically time-varying fields.

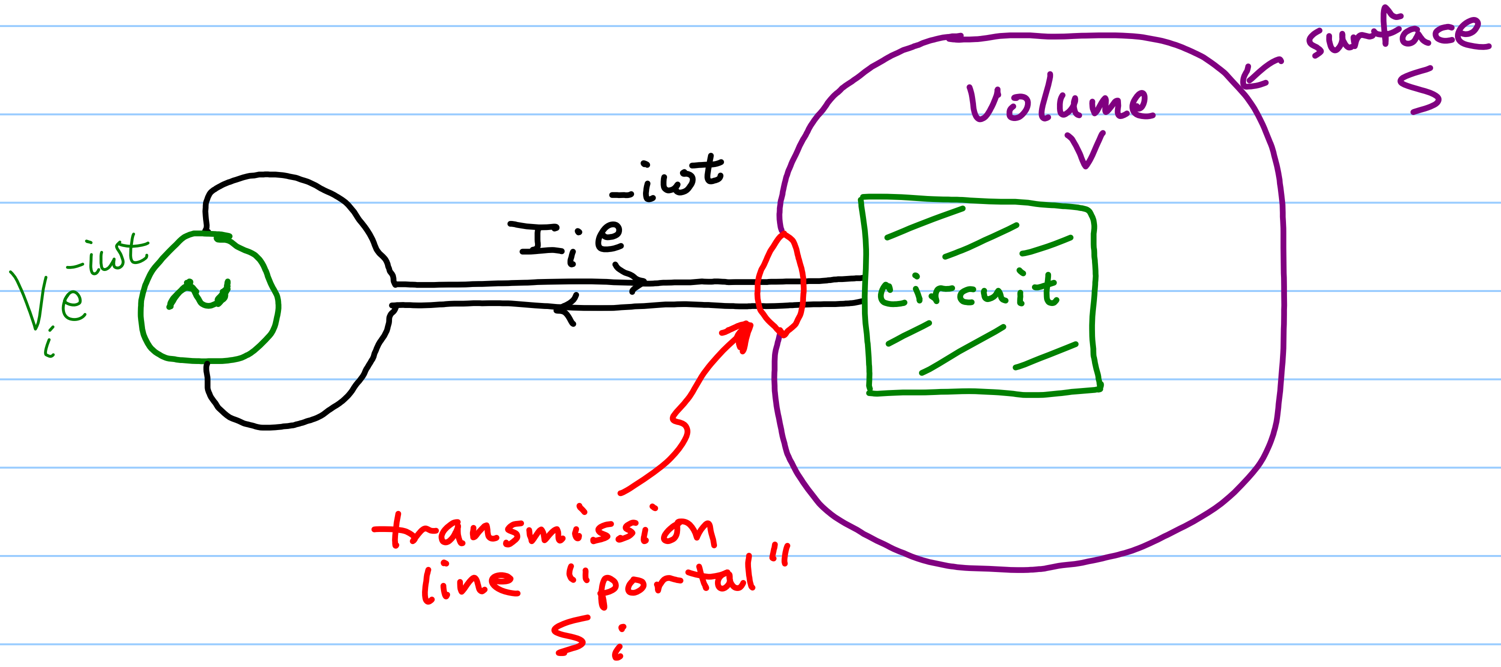
And observe that for real w_e, w_m (e.g. w_e, w_m are real if ϵ, μ = real) then the real part of this equation reads:

$$\boxed{\int_V \frac{1}{2}\mathrm{Re}(\vec{J}^*\cdot\vec{E})\, d^3x = -\oint_S \vec{S}\cdot\hat{n}\, da}$$

↑ time-averaged rate that EM-fields do work within V

↑ time-averaged flow of power IV through surface S, into vol. V.

Next, consider the application of these ideas to a circuit inside a finite volume V, specifically a 2-terminal linear network, whose power is brought in by a transmission line through a small portion S_i of the surface S.



If we neglect losses in power transmission, then the average power generated at the source must equal the power transmitted through S_i,

$$\Rightarrow \quad \frac{1}{2} I_i^* V_i = -\oint_{S_i} \vec{S}\cdot\hat{n}\,da$$

However, we established above that for this total volume V,

$$\frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 2i\omega\int_V (w_e - w_m)\,d^3x + \overset{\int_{S_i}\vec{S}\cdot\hat{n}\,da+\int_{S-S_i}\vec{S}\cdot\hat{n}\,da}{\overset{''}{\oint_S \vec{S}\cdot\hat{n}\,da}} = 0$$

$$\boxed{\Rightarrow \frac{1}{2} I_i^* V_i = -\int_{S_i} \vec{S}\cdot\hat{n}\, da = \frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}\, d^3x + 2i\omega\int_V (W_e - W_m)\, d^3x + \int_{S-S_i} \vec{S}\cdot\hat{n}\, da} \quad (*)$$

Next, define $V_i = I_i Z$,

where $Z = R - iX$ = impedance, with $\begin{cases} R = \text{resistance} \\ X = \text{reactance} \end{cases}$

$$\Rightarrow \frac{1}{2} I_i^* V_i = \frac{1}{2}|I_i|^2 Z$$

and so, by comparison with $(*)$ above, we identify:

$$R = \frac{1}{|I_i|^2}\left\{ \mathrm{Re}\int_V \vec{J}^*\cdot\vec{E}\, d^3x + 2\underset{\text{assume this to be real}}{\int_{S-S_i} \vec{S}\cdot\hat{n}\, da} + 4\omega\, \mathrm{Im}\int_V (W_m - W_e)\, d^3x \right\}$$

and

$$X = \frac{1}{|I_i|^2}\left\{ -\mathrm{Im}\int_V \vec{J}^*\cdot\vec{E}\, d^3x + 4\omega\, \mathrm{Re}\int_V (W_m - W_e)\, d^3x \right\}$$

We can take the surface S to be quite far away, so only radiation contributions survive in the Poynting vector term, and we identify:

$$\boxed{R_{rad} = \frac{2}{|I_i|^2}\int_{S-S_i} \vec{S}\cdot\hat{n}\, da} = \text{Radiation resistance,}$$

which represents losses due to escaping radiation i.e. from the coupling of the circuit to the radiation field

– This R_{rad} is usually negligible at low frequencies.

The $\vec{J}^* \cdot \vec{E}$ term can be simplified further for ohmic resistance, using $\vec{J} = \sigma \vec{E}$, where σ = conductivity

$$\Rightarrow R = \frac{1}{|I_i|^2} \int_V \sigma |\vec{E}|^2 d^3x = \text{ohmic resistance}$$

and

$$X = \frac{4\omega}{|I_i|^2} \int_V (W_m - W_e)\, d^3x \equiv X_L + X_c$$

inductive reactance   capacitive reactance

which connects to ordinary circuit theory as follows

$$W_m = \int_V w_m d^3x = \frac{1}{2} L I^2 \xrightarrow[\text{time averaged}]{} \frac{1}{4} L |I_i|^2$$

$$\Rightarrow \boxed{X_L = \omega L}$$

and

$$W_e = \int w_e d^3x = \frac{Q^2}{2C} \xrightarrow[\text{time ave.}]{} \frac{1}{4} \frac{|I_i|^2}{\omega C}$$

$$\Rightarrow \boxed{X_c = \frac{-1}{\omega C}}$$

and the connection to LRC circuit theory is complete.

Sec. 6.8 Poynting's theorem in linear dispersive media with losses

The previous derivation of Poynting's Theorem assumed that ϵ, μ are both REAL and FREQUENCY-INDEPENDENT with $\vec{D} = \epsilon \vec{E}$, $\vec{B} = \mu \vec{H}$

Next, we consider effects of Losses and of DISPERSION (= frequency-dependent ϵ, μ).

Begin with a Fourier analysis of the fields, i.e.

$$\vec{E}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega \, \vec{E}(\vec{x},\omega) e^{-i\omega t}$$

$$\vec{D}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega \, \vec{D}(\vec{x},\omega) e^{-i\omega t}, \text{ etc.}$$

We will sometimes encounter a complex, frequency-dependent susceptibility $\epsilon(\omega)$,

$$\Rightarrow \quad \vec{D}(\vec{x},\omega) = \epsilon(\omega) \vec{E}(\vec{x},\omega)$$

$$\text{and} \quad \vec{B}(\vec{x},\omega) = \mu(\omega) \vec{H}(\vec{x},\omega)$$

Some relations must hold for the t-dependent fields to be real, i.e. $\vec{E}^*(\vec{x},t) = \vec{E}(\vec{x},t)$, namely $\vec{E}(\vec{x},-\omega) = \vec{E}^*(\vec{x},\omega)$, $\epsilon(-\omega) = \epsilon^*(\omega)$ and $\vec{D}(\vec{x},-\omega) = \vec{D}^*(\vec{x},\omega)$, etc.

Next consider one of the terms that arose in our proof of Poynting's Theorem, e.g. Eq. 6.105,

$$\vec{E}(\vec{x},t)\cdot\frac{\partial\vec{D}(\vec{x},t)}{\partial t} = \left(\int d\omega'\,\vec{E}^*(\vec{x},\omega)e^{i\omega t}\right)\cdot\left(\int d\omega\left[-i\omega\epsilon(\omega)\vec{E}(\vec{x},\omega)e^{-i\omega t}\right)\right.$$

$$= \tfrac{1}{2}(\quad)(\quad) + \text{c.c.} \quad \text{since it is } \underline{\text{real}}$$

$$= \frac{1}{2}\int d\omega\int d\omega'\,\vec{E}^*(\vec{x},\omega')\cdot\vec{E}(\vec{x},\omega)\left(-i\omega\epsilon(\omega)\right)e^{-i(\omega-\omega')t}$$

$$+\frac{1}{2}\int d\omega\int d\omega'\,\underset{\vec{E}(\vec{x},\omega)}{\underset{\shortparallel}{\vec{E}^*(\vec{x},-\omega)}}\cdot\underset{\vec{E}^*(\vec{x},\omega')}{\underset{\shortparallel}{\vec{E}(\vec{x},-\omega')}}\left(i\omega'\epsilon^*(\omega')\right)e^{-i(\omega-\omega')t}$$

dummy integration variables have been switched here $\omega\leftrightarrow\omega'$

and therefore

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{1}{2}\int d\omega\int d\omega'\,\vec{E}^*(\vec{x},\omega')\cdot\vec{E}(\vec{x},\omega)\left[-i\omega\epsilon(\omega)+i\omega'\epsilon^*(\omega')\right]e^{-i(\omega-\omega')t}$$

Suppose now that the frequency-dependence of $\epsilon(\omega)$ is weak and smooth, so we can Taylor-expand:

$$\Rightarrow \quad \omega'\epsilon^*(\omega') \simeq \omega\epsilon^*(\omega) + (\omega'-\omega)\frac{d}{d\omega}\left(\omega\epsilon^*(\omega)\right)+\ldots$$

$$\Rightarrow \left(-i\omega\epsilon(\omega)+i\omega'\epsilon^*(\omega')\right) \simeq -i\omega\epsilon(\omega)+i\omega\epsilon^*(\omega)$$
$$-i(\omega-\omega')\frac{d}{d\omega}\left(\omega\epsilon^*(\omega)\right)$$
$$= 2\omega\,\mathrm{Im}\,\epsilon(\omega) - i(\omega-\omega')\frac{d}{d\omega}\left(\omega\epsilon^*(\omega)\right)$$

and in this approximation

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \int d\omega\int d\omega'\,\vec{E}^*(\omega')\cdot\vec{E}(\omega)\,\omega\,\mathrm{Im}\,\epsilon(\omega)\,e^{-i(\omega-\omega')t}$$
$$+\frac{\partial}{\partial t}\frac{1}{2}\iint d\omega\,d\omega'\,\vec{E}^*(\omega')\cdot\vec{E}(\omega)\frac{d}{d\omega}\left(\omega\epsilon^*(\omega)\right)e^{-i(\omega-\omega')t}$$

This term represents the conversion of electrical energy into heat and/or radiation, as in resonant absorption/reemission

Often we have nearly monochromatic radiation fields that can be approximated as

$$\vec{E}(\vec{x},t) = \vec{E}_0(\vec{x},t)\cos(\omega_0 t + \alpha)$$

$$\vec{B}(\vec{x},t) = \vec{B}_0(\vec{x},t)\cos(\omega_0 t + \beta)$$

where $\vec{E}_0(\vec{x},t)$, $\vec{B}_0(\vec{x},t)$ are slowly varying envelope functions

If $e^{i\omega t}$ oscillates very fast compared to $\vec{E}_0, \vec{B}_0$, we can approximately take them outside the Fourier transformation integral, i.e. write

$$\vec{E}(\vec{x},t_0,\omega) = \frac{1}{2\pi}\int_{t_0 - N\frac{2\pi}{\omega_0}}^{t_0 + N\frac{2\pi}{\omega_0}} \vec{E}_0(\vec{x},t_0)\cos(\omega_0 t + \alpha)e^{i\omega t}\,dt \quad , N >> 1$$

and changing variables: $t' = t - t_0$, $dt' = dt$

$$\Rightarrow \vec{E}(\vec{x},t_0,\omega) = \vec{E}_0(\vec{x},t_0)\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\omega_0 t' + \omega_0 t_0 + \alpha)e^{i\omega t' + i\omega t_0}\,dt'$$

$$= \vec{E}_0(\vec{x},t_0)\left(\frac{1}{2}\delta(\omega-\omega_0)e^{-i\alpha} + \frac{1}{2}\delta(\omega+\omega_0)e^{i\alpha}\right)$$

And now plugging this into our above result,

$$\Rightarrow \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{1}{4}\int d\omega\int d\omega'\,\vec{E}^*(\vec{x},t_0)\cdot\vec{E}(\vec{x},t_0)\left(e^{+i\alpha}\delta(\omega'-\omega_0) + e^{-i\alpha}\delta(\omega'+\omega_0)\right)$$
$$\times\,\omega\,\mathrm{Im}\,\epsilon(\omega)\,e^{-i(\omega-\omega')t_0}\left(e^{-i\alpha}\delta(\omega-\omega_0) + e^{i\alpha}\delta(\omega+\omega_0)\right)$$
$$+\frac{\partial}{\partial t_0}\frac{1}{8}\int d\omega\int d\omega'\,\vec{E}^*(\vec{x},t_0)\cdot\vec{E}(\vec{x},t_0)\frac{d}{d\omega}(\omega\,\epsilon^*(\omega))\,e^{-i(\omega-\omega')t_0}$$
$$\times\left\{e^{i\alpha}\delta(\omega'-\omega_0) + e^{-i\alpha}\delta(\omega'+\omega_0)\right\}\left\{e^{-i\alpha}\delta(\omega-\omega_0) + e^{i\alpha}\delta(\omega+\omega_0)\right\}$$

$$\text{So } \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} = \frac{1}{4}\vec{E}^*(\vec{x},t_0)\cdot\vec{E}(\vec{x},t_0)\,\omega_0\,\mathrm{Im}\,\epsilon(\omega_0)$$

$$+\frac{1}{4}\vec{E}^*(\vec{x},t_0)\cdot\vec{E}(\vec{x},t_0)(-\omega_0)\,\mathrm{Im}\,\epsilon(-\omega_0)$$

+ cross terms $\propto \exp(\pm 2i\omega_0 t_0) \rightarrow$ average to 0

$$+\frac{1}{8}\frac{\partial}{\partial t_0}\left\{\vec{E}^*(\vec{x},t_0)\cdot\vec{E}(\vec{x},t_0)\frac{d}{d\omega_0}\left(\omega_0\epsilon^*(\omega_0)\right)\right.$$

$$\left.+\vec{E}^*(\vec{x},t_0)\cdot\vec{E}(\vec{x},t_0)\left[\frac{-d}{d\omega_0}\left(-\omega_0\epsilon^*(-\omega_0)\right)\right]\right\}$$

+ cross terms $\rightarrow$ average to 0

Let's now relabel $t_0 \longrightarrow t$

average over $\Delta t = \frac{2\pi}{\omega_0}$

$$\Rightarrow \left\langle \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right\rangle = \omega_0\,\mathrm{Im}\,\epsilon(\omega_0)\left\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\right\rangle$$

$$+\omega_0\,\mathrm{Im}\,\mu(\omega_0)\left\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\right\rangle$$

$$+\frac{\partial}{\partial t}u_{\mathrm{eff}}$$

where the effective electromagnetic energy density is

$$u_{\mathrm{eff}} = \frac{1}{2}\mathrm{Re}\left(\frac{d}{d\omega}(\omega\epsilon)\right)\left\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\right\rangle$$

$$+\frac{1}{2}\mathrm{Re}\left(\frac{d}{d\omega}(\omega\mu)\right)\left\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\right\rangle$$

and our extended version of Poynting's theorem is:

$$\frac{\partial u_{\mathrm{eff}}}{\partial t} + \nabla\cdot\vec{S} = -\vec{J}\cdot\vec{E} - \omega_0\,\mathrm{Im}\,\epsilon(\omega_0)\langle E^2\rangle$$

$$-\omega_0\,\mathrm{Im}\,\mu(\omega_0)\langle H^2\rangle$$

ohmic loss

these terms represent absorptive dissipation in the medium.

Aside: dissipation can be described in terms of the conductivity σ OR through the imaginary parts of ϵ,μ, but don't double-count!

Sec. 6.10 Transformation properties of $\vec{E}, \vec{B}, \Phi, \vec{A}, \rho, \vec{J}, \ldots$

Observe - For the equations interrelating various quantities like $\vec{J}, \vec{B}, \frac{\partial \vec{E}}{\partial t}$, etc to make sense, the terms in an equation must ALL behave the same under the operations of

- coordinate rotations
- coordinate inversions
- time-reversal

A) Rotations $\Rightarrow$ These are orthogonal transformations that replace the original coordinates x_β by transformed coordinates x'_α

$$\Rightarrow \boxed{x'_\alpha = \sum_\beta a_{\alpha\beta} x_\beta} \qquad (*)$$

where orthogonality is expressed by

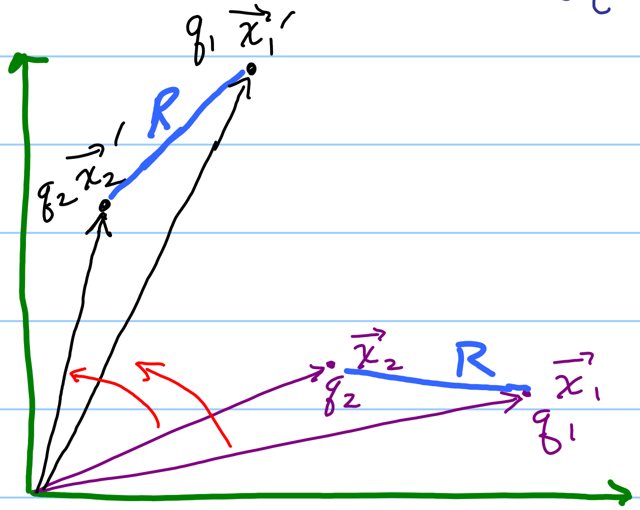
$$\underline{a}^T \underline{a} = \underline{I}, \text{ or } \sum_\beta (a^T)_{\alpha\beta} a_{\beta\gamma} = \delta_{\alpha\gamma}$$

which ensures that $\vec{x}'^2 = \vec{x}^2$

And if $\det(\underline{a}) = +1$, the rotation can be replaced by a sequence of infinitesimal steps and is a "PROPER ROTATION" while if $\det(\underline{a}) = -1$, it is an "IMPROPER ROTATION" which requires a reflection plus a rotation.

- Positions $\vec{x}_i$, Velocities $\vec{v}_i$, momenta $\vec{p}_i$ all transform according to $(*)$ and these are VECTORS = tensors of rank 1
- Scalar products like $\vec{x}_1 \cdot \vec{x}_2$ or $\vec{x}_1 \cdot \vec{v}_2$ are invariant under rotations and are called SCALARS = tensors of rank 0
- The Maxwell Stress Tensor is an example of a rank 2 tensor, i.e.

$$\overleftrightarrow{T}'_{\alpha\beta} = \sum_{\gamma\delta} a_{\alpha\gamma} \overleftrightarrow{T}_{\gamma\delta} (\underline{a}^T)_{\delta\beta}$$



e.g. an active rotation of all physical vectors through the same angle preserves scalars such as $\dfrac{1}{|\vec{x}_1 - \vec{x}_2|}$

Behavior of differential operators under rotation:

$\nabla \Phi$ = vector, assuming Φ = scalar

$\nabla \circ \vec{V}$ = scalar, if $\vec{V}$ = vector

$\nabla \circ \overleftrightarrow{T}$ = vector, if $\overleftrightarrow{T}$ = 2nd-rank tensor

$\nabla^2 \Phi$ = scalar

$\nabla^2 \vec{A}$ = vector, etc....

B) Spatial inversions and reflections

→ INVERSION (I) changes the sign of all 3 coordinates of a position vector, i.e.

$$\vec{x} = x\hat{x} + y\hat{y} + z\hat{z}$$

and $$I\,\vec{x} = -x\hat{x} - y\hat{y} - z\hat{z}$$

REFLECTION (R) reflects through one chosen plane, i.e.

$$R_y\,\vec{x} = x\hat{x} - y\hat{y} + z\hat{z}$$

⇒ Neither R nor I can be accomplished by simple (ie. PROPER) rotations.

Polar vectors (or vectors) – these transform like (*) under proper rotations, i.e. $V'_\alpha = \sum_\beta a_{\alpha\beta} V_\beta$

AND they change sign under inversion, i.e. $\vec{V}' = I\,\vec{V} = -\vec{V}$

Axial vectors (or **pseudovectors**) – these rotate the same way (*) but they are SYMMETRIC under inversion

i.e. $\vec{L}' = I\vec{L} = +\vec{L}$

e.g. a cross product of 2 polar vectors,

$$I(\vec{x} \times \vec{p}) = (-\vec{x}) \times (-\vec{p}) = \vec{x} \times \vec{p}$$

Note that pseudovectors and pseudoscalars imply a "TWIST" somewhere in the physics!

Pseudoscalars are ODD under inversion:

$$I[\vec{x}_1 \cdot (\vec{x}_2 \times \vec{p}_2)] = [(-\vec{x}_1) \cdot (-\vec{x}_2) \times (-\vec{p}_2)]$$
$$= -\vec{x}_1 \cdot (\vec{x}_2 \times \vec{p}_2)$$

c) TIME REVERSAL — the classical microscopic laws of physics are invariant if the direction of time is reversed, i.e. if time runs backwards, $\Rightarrow T: dt = -dt'$

Electromagnetic entities

Entity	Proper rotations	Inversions	Time Reversal	Overall
charge	scalar	even	even	true scalar
E-field ($\nabla \cdot \vec{E} = \rho/\epsilon_0$)	vector	odd	even	polar vector
B-field ($\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$)	vector	even	odd	pseudovector
Current $\vec{J} = \rho\vec{v}$	vector	odd	odd	polar vector

see Table 6.1, p 271 for more

Question: What are the most general relations among $\vec{E}, \vec{B}$ and their sources, consistent with

$$\begin{cases} \text{rotational invariance} \\ \text{inversions} \\ \text{time-reversal invariance} \end{cases} ?$$

Answer: suppose the relationships are linear in $\vec{E}, \vec{B}$ and first derivatives w.r.t. $\vec{x}, t$ and linear in the sources

Notation: call R = rotations
I = inversions
T = time-reversal

Behavior under

R	I	T	entities
vector	odd	even	$\vec{E}$
vector	odd	odd	$\boxed{\dfrac{\partial \vec{E}}{\partial t}, \ \nabla \times \vec{B}, \ \vec{J}}$
pseudovector	even	odd	$\vec{B}$
pseudovector	even	even	$\boxed{\dfrac{\partial \vec{B}}{\partial t}, \ \nabla \times \vec{E}}$ ($\vec{J}_m = \rho_m \vec{v}$ magnetic monopole current density if ever found)
scalar	even	even	$\boxed{\nabla \cdot \vec{E}, \ \rho}$
pseudoscalar	odd	odd	$\boxed{\nabla \cdot \vec{B}}$ ← (and ρ_m = magnetic monopole charge if ever found)

This table gives deeper insight into why Maxwell's equations are what they are!

Electromagnetic waves (Chap. 7)

7.1 Plane waves in a linear, lossless, nonconducting medium, with no sources ($\rho = 0$, $J = 0$)

$$\Rightarrow \quad \nabla \cdot \vec{E} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{D} = \epsilon \vec{E}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \qquad \vec{B} = \mu \vec{H}$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{B} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\left(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}\right) \underset{\substack{\uparrow \\ \text{or } \vec{B}}}{\vec{E}}(\vec{x}, t) = 0$$

We know that this 3D wave equation has plane wave solutions, e.g. traveling along $+x$ with wavenumber $+k$: $\exp(ikx - i\omega t)$

$\Rightarrow$ plugging this in gives

$$(-k^2 + \mu\epsilon\omega^2) e^{i(kx - \omega t)} = 0$$

which is satisfied provided $k = \omega\sqrt{\mu\epsilon}$

(we have assumed that μ, ϵ are homogeneous)

Definition The phase velocity is the velocity at which the wave maxima/minima travel

Recall that $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$

where $n = \left(\frac{\mu}{\mu_0}\frac{\epsilon}{\epsilon_0}\right)^{1/2}$ is the index of refraction and $n = n(\omega)$ typically

The general 3D plane wave solution for any component looks like

$$u(\vec{x}, t) = u_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

where $\hat{k}$ = phase propagation direction, $\vec{k} = k\hat{k}$

v = phase velocity $= \frac{\omega}{k}$

and the solutions for $\vec{E}, \vec{B}$ are:

$$\vec{E}(\vec{x}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{B}(\vec{x}, t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

where the amplitudes are constrained by Maxwell's equations to be: $\nabla\cdot\vec{E} = 0 = i\vec{k}\cdot\vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$\nabla\cdot\vec{B} = 0 = i\vec{k}\cdot\vec{B}_0 e^{i(\quad)}$ $\Rightarrow \boxed{\vec{k}\cdot\vec{E}_0 = 0}$ (1)

$\Rightarrow \boxed{\vec{k}\cdot\vec{B}_0 = 0}$ (2)

$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow i\vec{k}\times\vec{E}_0 = -(-i\omega)\vec{B}_0$, or $\vec{k}\times\vec{E}_0 = \omega\vec{B}_0$

$$\Rightarrow \quad \hat{k}\times\vec{E}_0 = \frac{\omega}{k}\vec{B}_0 \Rightarrow \boxed{\hat{k}\times\vec{E}_0 = \frac{1}{\sqrt{\mu\epsilon}}\vec{B}_0 = \frac{c}{n}\vec{B}_0} \quad (3)$$

Next, $\nabla\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} \Rightarrow i\vec{k}\times\vec{B}_0 = \mu\epsilon(-i\omega)\vec{E}_0$

$$\Rightarrow \quad \vec{k}\times\vec{B}_0 = -\mu\epsilon\omega\vec{E}_0$$

or $\hat{k}\times\vec{B}_0 = -\mu\epsilon\frac{\omega}{k}\vec{E}_0 = -\frac{\mu\epsilon}{\sqrt{\mu\epsilon}}\vec{E}_0$

$$\Rightarrow \quad \boxed{\hat{k}\times\vec{B}_0 = -\sqrt{\mu\epsilon}\,\vec{E}_0} \quad (4)$$

$$\hat{k}\times(\hat{k}\times\vec{B}_0) = \hat{k}(\cancelto{0}{\hat{k}\cdot\vec{B}_0}) - \vec{B}_0(\cancelto{1}{\hat{k}\cdot\hat{k}}) = -\vec{B}_0$$

$$= -\sqrt{\mu\epsilon}\,\hat{k}\times\vec{E}_0$$

$\Rightarrow$ (3) and (4) are equivalent

Monochromatic case:

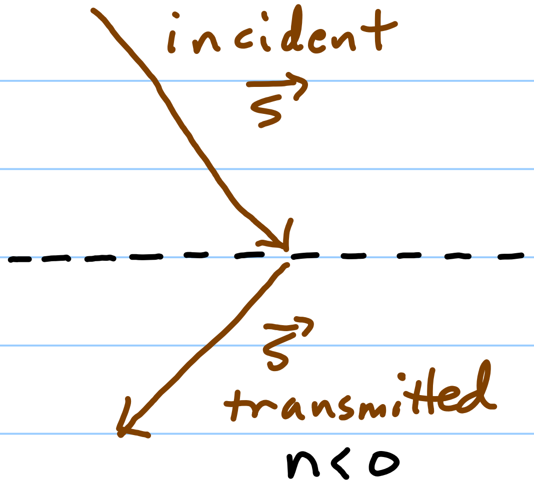
a) time-averaged Poynting vector is $\langle\vec{S}\rangle = \frac{1}{2}\vec{E}\times\vec{H}^*$

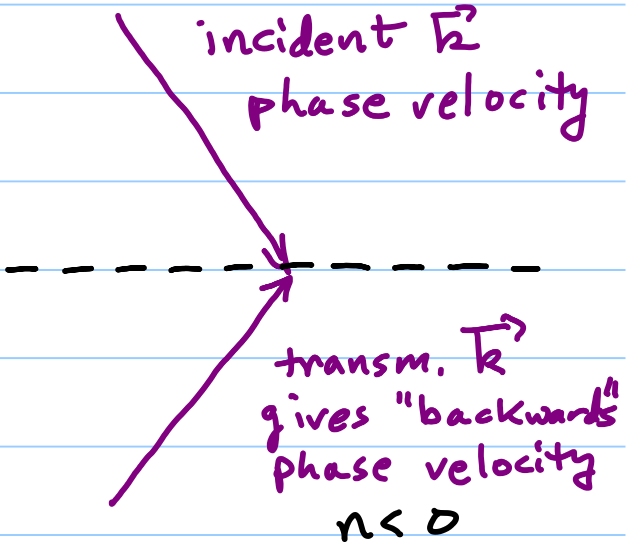
or

$$\langle\vec{S}\rangle = \frac{1}{2}\vec{E}_0\times\left(\frac{\sqrt{\mu\epsilon}\,\hat{k}\times\vec{E}_0^*}{\mu}\right) = \boxed{\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|\vec{E}_0|^2\,\hat{k} = \langle\vec{S}\rangle}$$

Aside: in a "normal material" where $n = c\sqrt{\mu\epsilon} > 0$, the phase velocity is PARALLEL to the direction of energy flow. But in certain "metamaterials", with $n < 0$, the energy flow can be OPPOSITE to the phase velocity!

Refs: J. Pendry, Physics World, Sept. 2001, p.47
V. G. Veselago, Sov. Phys. Usp. 10, 509 (1968)
P. Markos + C.M. Soukoulis, Opt. Exp. 11, 649 (2003)





b) time-averaged energy density:

$$\langle u \rangle = \frac{1}{4}\left(\epsilon \vec{E}\cdot\vec{E}^* + \frac{1}{\mu}\vec{B}\cdot\vec{B}^*\right)$$

$$= \frac{1}{4}\epsilon|E_0|^2 + \frac{1}{4\mu}(\mu\epsilon)|E_0|^2$$

or $$\boxed{\langle u \rangle = \frac{\epsilon}{2}|E_0|^2}$$

and observe that $\langle \vec{S} \rangle = \langle u \rangle \vec{v}$

where $\vec{v} = \frac{c}{n}\hat{k} = \frac{\hat{k}}{\sqrt{\mu\epsilon}}$

i.e. the energy flow can be expressed as an energy density multiplied by the wave phase velocity.

Polarization – Next consider the different types of monochromatic plane waves that can arise:

start from $\vec{E}(\vec{x},t) = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

and recall that it is implied to take $\mathrm{Re}[\vec{E}(\vec{x},t)]$ in any physical calculation.

We consider different possible cases (a) – (c):

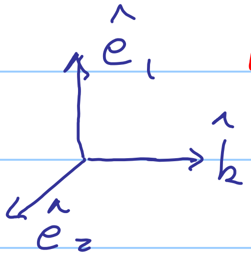
(a) $\vec{E}_0$ is real, (aside from an overall possible complex factor, i.e. $\vec{E}_0 = \vec{E}_0^{\,real} \, e^{i\delta}$, $\delta = \mathrm{real}$)

⇒ $\vec{E}$ is always directed along (±) a fixed axis, $\vec{E}_0^{\,real}$

i.e. $\mathrm{Re}[\vec{E}(\vec{x},t)] = \vec{E}_0^{\,real} \cos(\vec{k}\cdot\vec{x} - \omega t + \delta)$

⇒ This is called LINEAR POLARIZATION

(b) Circular polarization Set this up using a right-handed coordinate system formed by 3 vectors $(\hat{e}_1, \hat{e}_2, \hat{k})$



i.e. $\hat{e}_1 \times \hat{e}_2 = \hat{k}$

$\hat{e}_1 \cdot \hat{e}_2 = 0$

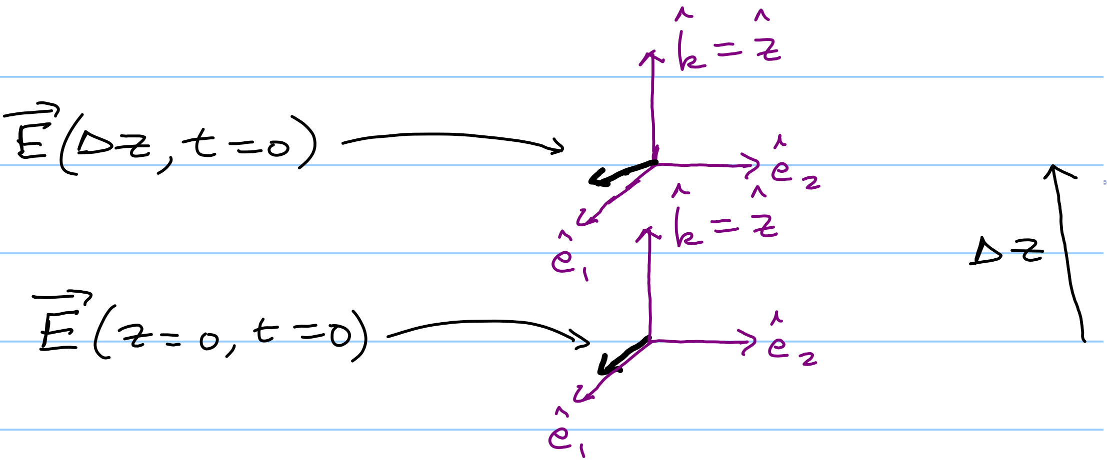
etc.

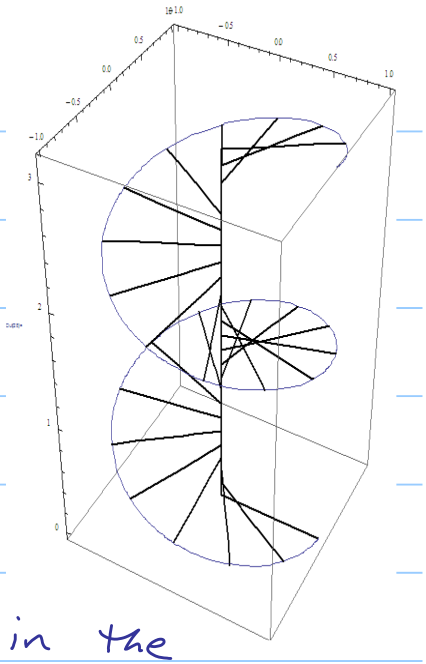
Now consider a complex wave like:

$$\vec{E}_0 = E_0 \left(\frac{\hat{e}_1 + i\hat{e}_2}{\sqrt{2}} \right) \quad \text{where } E_0 = \text{real}$$

$$\Rightarrow \mathrm{Re}\, \vec{E}(\vec{x},t) = \frac{E_0}{\sqrt{2}} \left\{ \hat{e}_1 \cos(\vec{k}\cdot\vec{x} - \omega t) - \hat{e}_2 \sin(\vec{k}\cdot\vec{x} - \omega t) \right\}$$

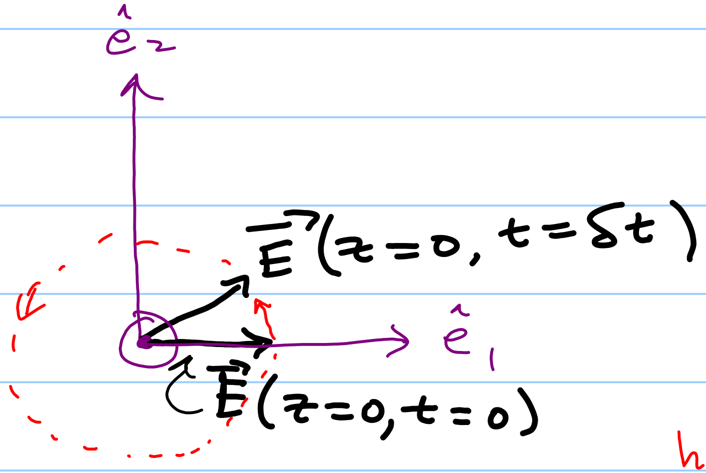
To visualize this wave, look at it in space for a Fixed time, e.g. $t = 0$





$\Rightarrow$ At a fixed time, as z advances in the direction of propagation, as we look <u>towards</u> the propagation direction, the field rotates CCW (a "left-handed rotation sense") as z advances

<u>OR</u> Looking versus time, for a fixed position, e.g. $z=0$, the $\vec{E}$-vector looks as follows:



This state has <u>positive</u> <u>helicity</u> $h = +1$ and in <u>most</u> literature it is called

LEFT-CIRCULARLY POLARIZED

(b') For $\vec{E}_0 = E_0 \left(\frac{\hat{e}_1 - i\hat{e}_2}{\sqrt{2}}\right)$, the $\vec{E}$-field rotates in the opposite sense

$\Rightarrow$ NEGATIVE HELICITY STATE $(h=-1)$

or Right-circularly polarized

(c)

More generally we have $\vec{E} = E_1 \hat{e}_1 + E_2 \hat{e}_2$ which can be recast in terms of circularly-polarized unit vectors,

$$\hat{e}_{\pm 1} = \mp \frac{1}{\sqrt{2}} (\hat{e}_1 \pm i \hat{e}_2)$$ ← note difference from Jackson Eq. 7.22

(and when using these to describe an arbitrary 3D vector, can also include $\hat{k} \equiv \hat{e}_0$, and recall that $\hat{e}_\mu^* \cdot e_\nu = \delta_{\mu\nu}$)

Here, our general plane-wave $\vec{E}$ can be written as

$$\vec{E} = \sum_{\mu=\pm 1} (\vec{E} \cdot \hat{e}_\mu^*) \hat{e}_\mu \equiv E_+ \hat{e}_{+1} + E_- \hat{e}_{-1}$$

i.e. explicitly, $E_+ = \vec{E} \cdot \hat{e}_{+1}^* = (\hat{e}_1 \cdot \hat{e}_{+1}^*) E_1 + (\hat{e}_2 \cdot \hat{e}_{+1}^*) E_2$

$$\Rightarrow \quad E_+ = \frac{-E_1 + iE_2}{\sqrt{2}}, \quad E_- = \frac{E_1 + iE_2}{\sqrt{2}}$$

Hence, if $E_+ = \pm E_-$ => linear polarization along $\hat{e}_1$ or $\hat{e}_2$

or if $E_+ = \pm i E_-$ => linear polarization along $\frac{\hat{e}_1 \pm \hat{e}_2}{\sqrt{2}}$

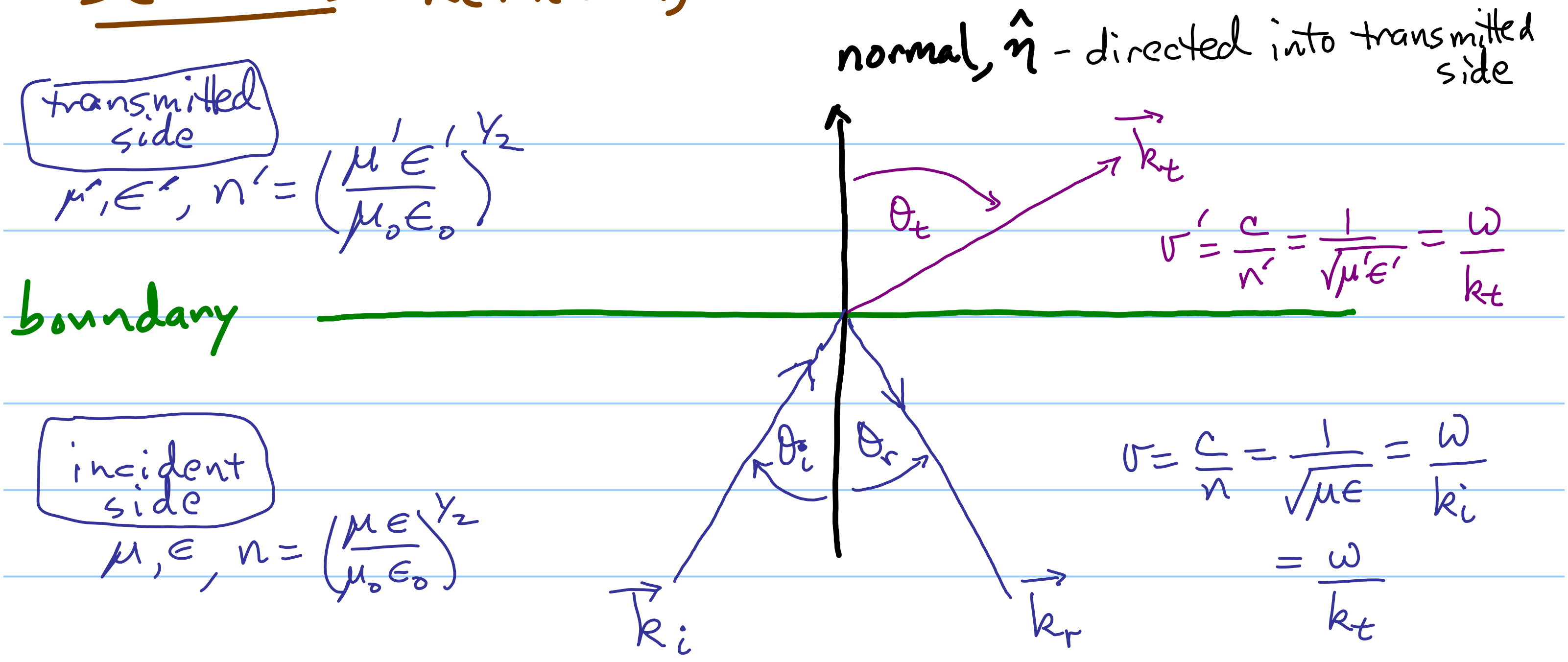
or if E_+ or E_- exists alone, => circular polarization, $h = \pm 1$

and other possibilities that are neither linear nor circular are generally called ELLIPTICALLY-polarized.

=> Read on your own about the general characterization of light polarization in terms of 4 Stokes parameters (sec. 7.2)

Aside Some light beams have a "mixed" polarization state => one cannot simply describe it as an electric + magnetic field obeying Maxwell's equations. e.g. unpolarized light is an incoherent statistical mixture of equal parts $\hat{e}_1$- and $\hat{e}_2$- polarized light. In QM statistical mixtures of this type are best described using density matrices instead of a single wavefunction (state vector).

Sec. 7.3 Reflection, refraction at an interface



$\mu', \epsilon', n' = \left(\frac{\mu'\epsilon'}{\mu_0\epsilon_0}\right)^{1/2}$

$\mu, \epsilon, n = \left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2}$

$v' = \frac{c}{n'} = \frac{1}{\sqrt{\mu'\epsilon'}} = \frac{\omega}{k_t}$

$v = \frac{c}{n} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{\omega}{k_i}$
$= \frac{\omega}{k_t}$

The 3 arrows here correspond to 3 plane waves:

<u>Incident</u>

$$\vec{E}_i = \vec{E}_{oi}\, e^{i(\vec{k}_i \cdot \vec{x} - \omega t)}$$

$$\vec{B}_i = \sqrt{\mu\epsilon}\; \hat{k}_i \times \vec{E}_i = \frac{n}{c}\, \hat{k}_i \times \vec{E}_i$$

<u>Reflected</u>

$$\vec{E}_r = \vec{E}_{or}\, e^{i(\vec{k}_r \cdot \vec{x} - \omega t)}$$

$$\vec{B}_r = \frac{n}{c}\, \hat{k}_r \times \vec{E}_r$$

<u>Transmitted</u>

$$\vec{E}_t = \vec{E}_{ot}\, e^{i(\vec{k}_t \cdot \vec{x} - \omega t)}$$

$$\vec{B}_t = \frac{n'}{c}\, \hat{k}_t \times \vec{E}_t$$

and of course, here $k_i = k_r = \frac{n\omega}{c}$

while $k_t = \frac{n'\omega}{c} = \frac{n'}{n} k_i = \frac{n'}{n} k_r$

- One must have the <u>same</u> ω in all 3 plane waves in order to satisfy BC's at <u>all</u> <u>times</u>.

- Also, assuming both media are homogeneous, <u>symmetry</u> demands that all 3 wavevectors,

$$\vec{k}_i\ ,\ \vec{k}_r\ ,\ \vec{k}_t \quad (+\text{ the normal } \hat{n})$$

must lie in a <u>single</u> <u>plane</u>, taken here to be the $z=0$ plane

- In order for the imposed boundary conditions to hold along the entire planar interface, we must also have:

$$e^{i\vec{k}_i\cdot\vec{x}}\Big|_{z=0} = e^{i\vec{k}_r\cdot\vec{x}}\Big|_{z=0} = e^{i\vec{k}_t\cdot\vec{x}}\Big|_{z=0}$$

$$\Rightarrow \vec{k}_i\cdot\vec{x}\Big|_{z=0} = \vec{k}_r\cdot\vec{x}\Big|_{z=0} = \vec{k}_t\cdot\vec{x}\Big|_{z=0}$$

$$\Rightarrow \quad k_i\sin\theta_i = k_r\sin\theta_r = k_t\sin\theta_t \quad \left(\begin{array}{c}k\text{-components}\\ \underline{\text{along}}\text{ surface}\\ \text{are equal}\end{array}\right)$$

$$\Rightarrow \boxed{\theta_i = \theta_r} \quad \text{angle of incidence} = \text{angle of reflection} \quad \left(\begin{array}{c}\text{since}\\ k_i = k_r\end{array}\right)$$

and since $k_i = \dfrac{n\omega}{c}$, $k_t = n'\dfrac{\omega}{c}$

$$\Rightarrow \boxed{n\sin\theta_i = n'\sin\theta_t} \leftarrow \text{Snell's law of refraction}$$

Next, match all the BCs at interface, namely: $\vec{D}\cdot\hat{\eta}$ and $\vec{B}\cdot\hat{n}$ continuous,

and $\vec{E}\times\hat{\eta}$ and $\vec{H}\times\hat{\eta}$ continuous

(since no free surface charges or surface currents are assumed present)

Therefore, we see that:

$$\left(\epsilon(\vec{E}_{oi}+\vec{E}_{or})-\epsilon'\vec{E}_{ot}\right)\cdot\hat{\eta}=0$$

$$\left(\frac{n}{c}\hat{k}_i\times\vec{E}_{oi}+\frac{n}{c}\hat{k}_r\times\vec{E}_{or}-\frac{n'}{c}\hat{k}_t\times\vec{E}_{ot}\right)\cdot\hat{\eta}=0$$

$$\left(\vec{E}_{oi}+\vec{E}_{or}-\vec{E}_{ot}\right)\times\hat{\eta}=0$$

$$\left(\frac{1}{\mu}\frac{n}{c}\hat{k}_i\times\vec{E}_{oi}+\frac{1}{\mu}\frac{n}{c}\hat{k}_r\times\vec{E}_{or}-\frac{1}{\mu'}\frac{n'}{c}\hat{k}_t\times\vec{E}_{ot}\right)\times\hat{\eta}=0$$

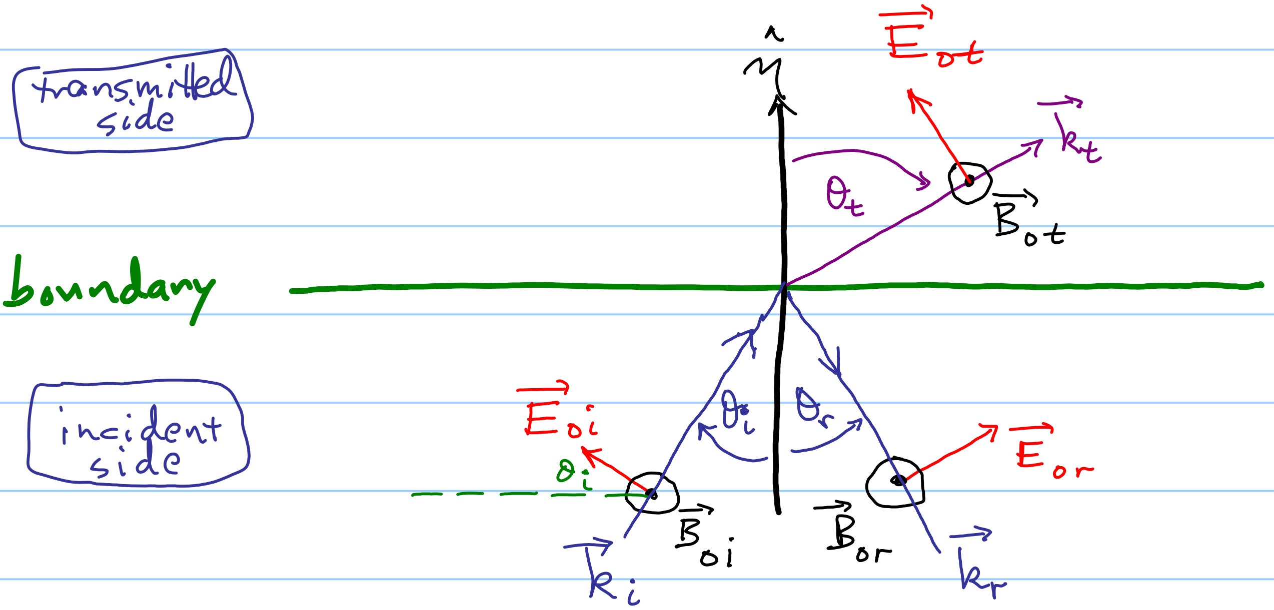
To apply these BCs, it is simplest to consider 2 separate cases:

(i) The incident linearly-polarized wave has $\vec{E}_{oi}$ IN the plane of incidence

(ii) The incident linearly-pol. wave has $\vec{E}_{oi}$ PERPENDICULAR to the plane of incidence

Of course, if the incident light has general elliptical polarization, it will just be a linear combination of these 2 cases.

Case (i) $\vec{E}_{0i}$ lies IN the plane of incidence "POI"



Now $\vec{D}\cdot\hat{n}$ = continuous $\Rightarrow$ $\epsilon E_{0i}\sin\theta_i + \epsilon E_{0r}\sin\theta_i = \underbrace{\epsilon' E_{0t}\sin\theta_t}$

$E_{0t}\,\frac{\epsilon' n \sin\theta_i}{n'}$

and $\vec{E}\times\hat{n}$ = continuous,

$$\Rightarrow \boxed{E_{0i}\cos\theta_i - E_{0r}\cos\theta_i = E_{0t}\cos\theta_t}$$

and $\vec{H}\times\hat{n}$ = continuous $\Rightarrow$ $\frac{1}{\mu}(B_{0i}+B_{0r}) = \frac{1}{\mu'}B_{0t}$

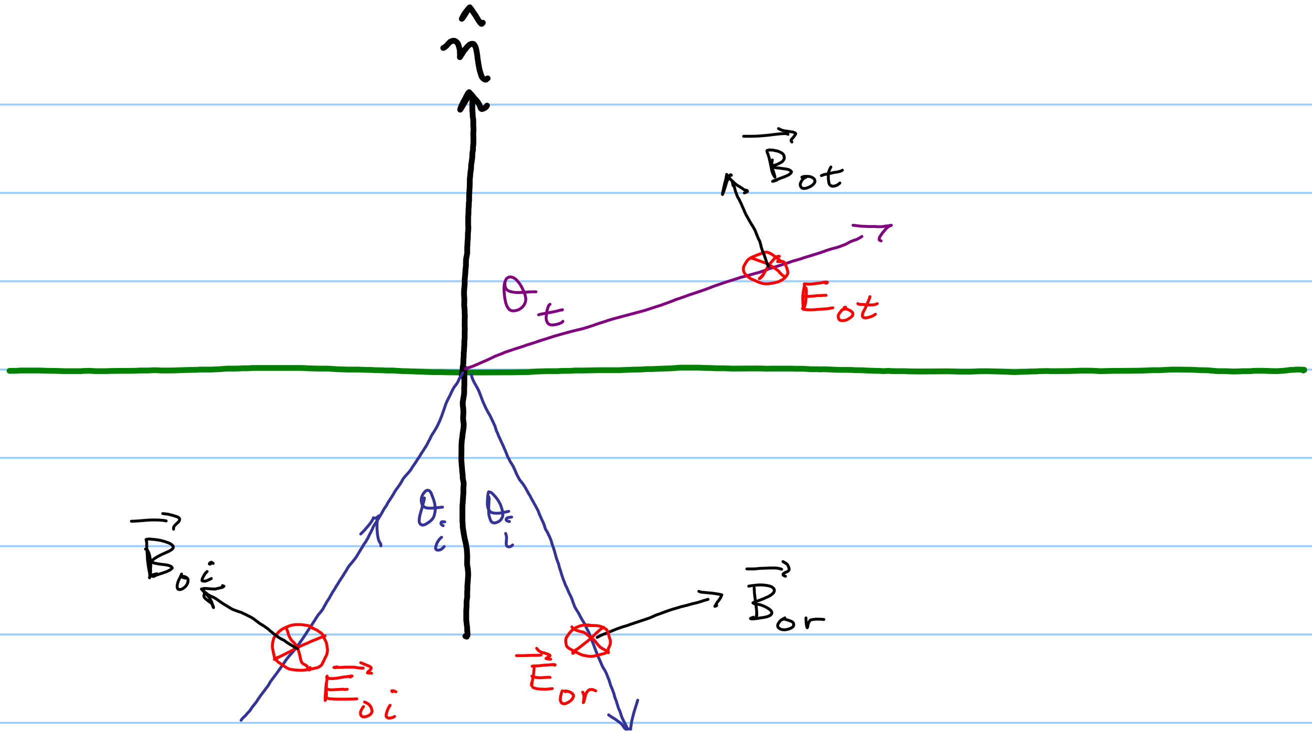
$$\Rightarrow \boxed{\frac{n}{\mu}(E_{0i}+E_{0r}) = \frac{n'}{\mu'}E_{0t}}$$

solution of these 2 equations, 2 unknowns, for an assumed known E_{0i}, gives:

$$\boxed{E_{0r} = E_{0i}\,\frac{n'\mu\cos\theta_i - n\mu'\cos\theta_t}{n'\mu\cos\theta_i + n\mu'\cos\theta_t} \qquad E_{0t} = E_{0i}\,\frac{2n\mu'\cos\theta_i}{n'\mu\cos\theta_i + n\mu'\cos\theta_t}}$$

for $\vec{E}_{0i}$ IN plane of incidence

Case (ii) $\vec{E}_{oi} \perp$ to P.O.I.



Continuity of $\vec{E} \times \hat{n} \Rightarrow$ $\boxed{E_{oi} + E_{or} = E_{ot}}$

Continuity of $\vec{H} \times \hat{n} \Rightarrow \frac{1}{\mu}\left(B_{oi}\cos\theta_i - B_{or}\cos\theta_i\right) = \frac{B_{ot}}{\mu'}\cos\theta_t$

or $\boxed{\frac{n}{\mu}\left(E_{oi} - E_{or}\right)\cos\theta_i = \frac{n'}{\mu'}E_{ot}\cos\theta_t}$

Solution is:

$$\boxed{\begin{aligned} E_{or} &= E_{oi}\,\frac{n\mu'\cos\theta_i - n'\mu\cos\theta_t}{n\mu'\cos\theta_i + n'\mu\cos\theta_t} \\ E_{ot} &= E_{oi}\,\frac{2n\mu'\cos\theta_i}{n\mu'\cos\theta_i + n'\mu\cos\theta_t} \end{aligned}}$$

$\boxed{\vec{E}_{oi} \perp \text{ to P.O.I.}}$

and we could eliminate θ_t using Snell's law, if desired: $\cos\theta_t = \sqrt{1 - \frac{n^2}{n'^2}\sin^2\theta_i}$

Specialized cases For optical frequencies it is usually accurate to set $\frac{\mu'}{\mu} \simeq 1$, and this gives a simple result for the case of normal incidence, i.e. $\theta_i = 0$:

$\vec{E}_{oi} \parallel$ to P.O.I.

$$\frac{E_{or}}{E_{oi}} = \frac{n'-n}{n'+n}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2n}{n'+n}$$

$\vec{E}_{oi} \perp$ to P.O.I.

$$\frac{E_{or}}{E_{oi}} = \frac{n-n'}{n+n'}$$

$$\frac{E_{ot}}{E_{oi}} = \frac{2n}{n'+n}$$

These results for E_{or} appear to have a sign difference, a consequence of our choices of coordinate axes – not an actual difference!

Brewster's Angle – At a certain incidence angle, θ_B there is no reflected wave for the $\vec{E}$-field component $\parallel$ to the P.O.I. To work out the value of θ_B, just set $E_{or} = 0$ above, i.e. (again take $\frac{\mu'}{\mu} \simeq 1$)

$$\frac{E_{or}}{E_{oi}} \propto \frac{\frac{n'}{n}\cos\theta_i - \sqrt{1 - \frac{n^2}{n'^2}\sin^2\theta_i}}{(\text{denominator})} \xrightarrow[\theta_i \to \theta_B]{} 0,$$

and we can solve for θ_B,

$$\left(\frac{n'}{n}\right)^2 \cos^2\theta_B = 1 - \frac{n^2}{n'^2}\sin^2\theta_B$$

$$= \cos^2\theta_B + \sin^2\theta_B\left(1 - \frac{n^2}{n'^2}\right)$$

$$\Rightarrow \tan^2\theta_B = \frac{\left(\frac{n'}{n}\right)^2 - 1}{1 - \frac{n^2}{n'^2}}$$

which simplifies to:

$$\boxed{\tan\theta_B = \frac{n'}{n}} = \text{Brewster angle}$$

$\Rightarrow$ At the Brewster angle, all reflected light is polarized $\underline{\perp}$ to the P.O.I., i.e. $\parallel$ to the surface itself.

e.g. – for a water surface,

$$\frac{n'}{n} \simeq 1.35 \Rightarrow \theta_B(\text{water}) \simeq 53°$$

while for glass, $\frac{n'}{n} \simeq 1.5 \Rightarrow \theta_B(\text{glass}) \simeq 56°$

<u>Next</u>: Total internal reflection

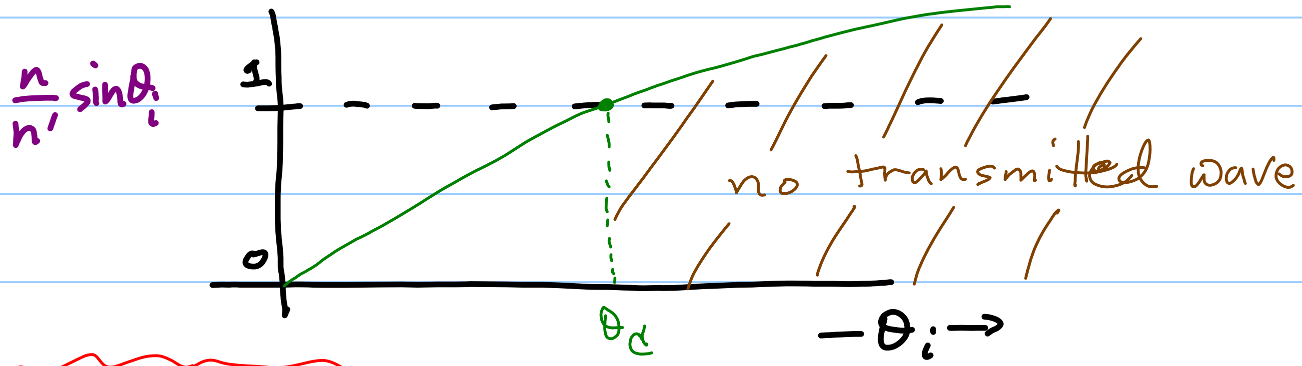
$\Rightarrow$ Suppose light is incident from a medium of <u>higher</u> refractive index, $n > n'$

e.g. water $\longrightarrow$ air

Then since $\sin\theta_t = \frac{n}{n'}\sin\theta_i$,

or $\theta_t = \sin^{-1}\left(\frac{n}{n'}\sin\theta_i\right)$

$\Rightarrow$ beyond a certain angle $\theta_i > \theta_c$, clearly $\frac{n}{n'} \sin\theta_i > 1$ so it has no real arcsin,



$$\Rightarrow \boxed{\theta_c = \sin^{-1}\left(\frac{n'}{n}\right) \equiv \text{critical angle}}$$

$\Rightarrow$ no transmitted light and the reflection is <u>total</u>

We can explore the nature of the transmitted wave at $\theta_i > \theta_c$, by analytically continuing our previous derived solution, which simply means inserting $k_t \rightarrow$ imaginary!

To see this, recall that Snell's law was derived from $\vec{k}_i \cdot \vec{x}\Big|_{z=0} = \vec{k}_t \cdot \vec{x}\Big|_{z=0}$

$$\Rightarrow k_i \sin\theta_i = k_t \sin\theta_t$$

$$\text{or} \quad \frac{n\omega}{c}\sin\theta_i = \frac{n'\omega}{c}\sin\theta_t$$

i.e. the component of $\vec{k}$ $\parallel$ to interface is continuous, $\Rightarrow k_{ix} = k_{tx}$

So when $\frac{n}{n'}\sin\theta_i > 1$, for $\theta_i > \theta_c$,

$\Rightarrow \sin\theta_t > 1$, whereby $\cos^2\theta_t = 1 - \sin^2\theta_t < 0$

$\Rightarrow \cos\theta_t =$ purely imaginary

But the <u>normal</u> component of the transmitted wave vector $\vec{k}_t$ is $k_{tz} = k_t \cos\theta_t \rightarrow$ imaginary

$$\equiv iK$$

where $K = \sqrt{\frac{n^2}{n'^2}\sin^2\theta_i - 1}$

Then the transmitted wave looks like

$$\vec{E}_t = \vec{E}_{ot} e^{i\vec{k}_t \cdot \vec{x}} = \vec{E}_{ot} e^{i(k_{tz}z + k_{tx}x)}$$

$$\Rightarrow \boxed{\vec{E}_t = \vec{E}_{ot} e^{-Kz} e^{ik_t \sin\theta}}$$

and we note this exponential damping of the transmitted wave in the direction normal to the surface.

Let's calculate the time-averaged energy flow through the surface:

$$\langle \vec{S}\cdot\hat{z}\rangle = \frac{1}{2}\mathrm{Re}\left(\hat{z}\cdot\vec{E}_t \times \vec{H}^*\right)$$

where $\vec{H}_t = \frac{(\mu'\epsilon')^{1/2}\hat{k}_t \times \vec{E}_t}{\mu'} = \frac{c}{n'\omega}\frac{(\mu'\epsilon')^{1/2}}{\mu'}\vec{k}_t \times \vec{E}_t$

$$= \frac{1}{\mu'\omega}\vec{k}_t \times \vec{E}_t$$

Thus $\langle \vec{S}\cdot\hat{z}\rangle = \frac{1}{2}\,\mathrm{Re}\left[\hat{z}\cdot\vec{E}_t \times (\vec{k}_t \times \vec{E}_t)^*\right]\frac{1}{\mu'\omega}$

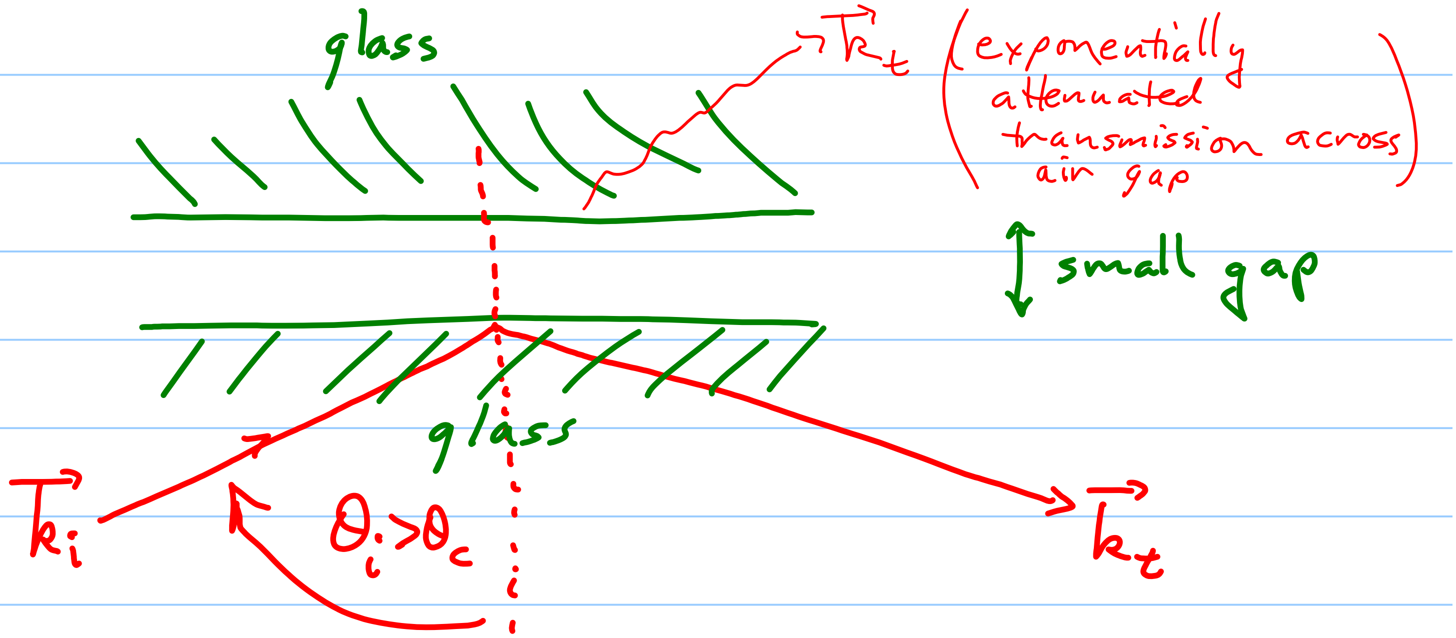
$$= \frac{1}{2\mu'\omega}\,\mathrm{Re}\left(\hat{z}\cdot\vec{k}_t\,|\vec{E}_t|^2\right)$$

$$= 0 \quad \text{since} \quad \mathrm{Re}(k_{tz}) = 0\ !$$

$\Rightarrow$ ALL ENERGY is Reflected!

On your own, read about:

- Evanescent waves
- light tunneling across a total internal reflection gap



Sec. 7.5 Frequency dispersion

Ⓐ We can understand much using a simple classical model for $\epsilon(\omega)$

Assumptions

(i) material has low density, with $\mu \simeq \mu_0$

(ii) each e^- is bound harmonically, freq. ω_0, and is subjected to an E-field,

$$\vec{E}(\vec{x},t) = \vec{E}_0 e^{-i\omega t}$$

This assumes that $e^{i\vec{k}\cdot\vec{x}} \simeq$ constant over the distance scale where the e^- moves.

$$\Rightarrow \quad m\left(\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right) = -e\vec{E}_0 e^{-i\omega t}$$

phenomenological damping term

$\Rightarrow$ Look for a solution at the driving frequency, namely

$$\boxed{\vec{x}(t) = \vec{x}_0 e^{-i\omega t}}$$

plug in

$$\Rightarrow \quad m\cancel{e^{-i\omega t}}\left(-\omega^2 - i\gamma\omega + \omega_0^2\right)\vec{x}_0 = -eE_0\cancel{e^{-i\omega t}}$$

hence $$\vec{x}_0 = -\frac{e}{m}\,\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}\,\vec{E}_0$$

and the induced electric dipole moment is

$$\vec{p} = -e\vec{x} = \frac{e^2}{m} \frac{1}{\omega_o^2 - \omega^2 - i\omega\gamma} \vec{E}$$

- In this model, imagine each e^- behaving as a classical damped oscillator of frequency ω_o

- Suppose there are N $\frac{\text{molecules}}{\text{unit volume}}$, each with Z electrons with different frequencies ω_j and damping constants γ_j.

- Call f_j the number of those Z electrons whose frequency is ω_j, i.e. $\boxed{\sum_j f_j = Z}$

$\Rightarrow$ The POLARIZATION is $\vec{P} = N\vec{p}$

or

$$\vec{P} = \frac{Ne^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}\right)\vec{E} = \epsilon_o \chi_e \vec{E}$$

and therefore

$$\boxed{\frac{\epsilon(\omega)}{\epsilon_o} + 1 = 1 + \chi_e = 1 + \frac{Ne^2}{\epsilon_o m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}}$$

Quantum Derivation

See Fano & Rau,
Atomic collisions & spectra,
p. 29

Let the applied electric field be

$$\vec{E} = \hat{x} E_0 \left(e^{-i\omega t} + e^{i\omega t}\right)$$

$\Rightarrow$ the potential energy is $V = -exE_0\left(e^{-i\omega t} + e^{i\omega t}\right)$

1st-order time-dependent perturbation theory states that

energy of j^{th} atomic level

$$\Psi = \Psi^{\text{ground}} + \sum_j \Psi_j \, e^{-i\epsilon_j t/\hbar} \, c_j(t)$$

where $c_j(t) = \dfrac{eE_0}{\hbar} \langle \Psi_j | x | \Psi^{\text{ground}} \rangle$

$$\times \left[\frac{e^{i(\omega + \omega_j - \frac{1}{2} i\gamma_j)t}}{\omega_j + \omega - \frac{1}{2} i\gamma_j} + \frac{e^{-i(\omega - \omega_j + \frac{i\gamma_j}{2})t}}{\omega_j - \omega - \frac{1}{2} i\gamma_j} \right]$$

where γ_j = decay rate of excited state Ψ_j

and $\dfrac{\epsilon_j - \epsilon_{\text{ground}}}{\hbar} = \omega_j - \frac{1}{2} i\gamma_j$

simplified notation: call $\Psi^{\text{ground}} \longrightarrow |0\rangle$

$\Psi_j \longrightarrow |j\rangle$

Then the mean dipole moment is:

$$\langle p \rangle = e\langle x \rangle = e \sum_j \left\{ \langle 0|x|j\rangle c_j(t) e^{-i(\omega_j - \frac{i\gamma_j}{2})t} + c_j^*(t) e^{i(\omega_j + \frac{i\gamma_j}{2})t} \langle j|x|0\rangle \right\}$$

$$\Rightarrow \langle p \rangle = \frac{e^2 E_0}{\hbar} \sum_j \langle 0|x|j\rangle \left(\frac{e^{i\omega t}}{\omega_j^2 - \omega^2 + i\omega\gamma_j} + \frac{e^{-i\omega t}}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right)$$

$$\equiv E_0\, \alpha(\omega) e^{-i\omega t} + E_0\, \alpha^*(\omega) e^{i\omega t}$$

and the polarizability is

$$\boxed{\alpha(\omega) = \frac{e^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}}$$

where the quantal oscillator strength is

$$\boxed{f_j = \frac{2m\omega_j}{\hbar} |\langle j|x|0\rangle|^2}$$

which is dimensionless

Aside: can prove in nonrelativistic QM that

$$\sum_j f_j = Z,$$

which is the Thomas-Reiche-Kuhn sum rule.

- see Fano + Cooper, Rev. Mod. Phys. 40, 441 (1968).

Ⓑ Anomalous Dispersion

Recall, from Chap. 4, $\chi_e = \frac{\alpha N}{1 - \frac{1}{3}\alpha N} \simeq \alpha N$

And typically $\gamma_j \ll \omega_j$, so damping is not very important except when ω is near resonance, $\omega_j^2 \simeq \omega^2$

First look at two limits:

Low ω

$$\frac{\epsilon(\omega)}{\epsilon_0} \xrightarrow[\omega \to 0]{} 1 + \frac{e^2 N}{\epsilon_0 m} \sum_{j>0} \frac{f_j}{\omega_j^2} + \underbrace{\frac{i e^2 N}{\epsilon_0 m} \frac{f_0}{\omega(\gamma_0 - i\omega)}}$$

this term is present if an $\omega_0 = 0$ resonant frequency exists

High ω

$$\frac{\epsilon(\omega)}{\epsilon_0} \xrightarrow[\substack{\omega \to \infty \\ \omega \gg \text{all } \omega_j}]{} 1 - \frac{N e^2 Z}{m \epsilon_0 \omega^2} \equiv 1 - \frac{\omega_p^2}{\omega^2}$$

where $\omega_p^2 \equiv \frac{N Z e^2}{\epsilon_0 m}$ is called the PLASMA FREQUENCY (squared)

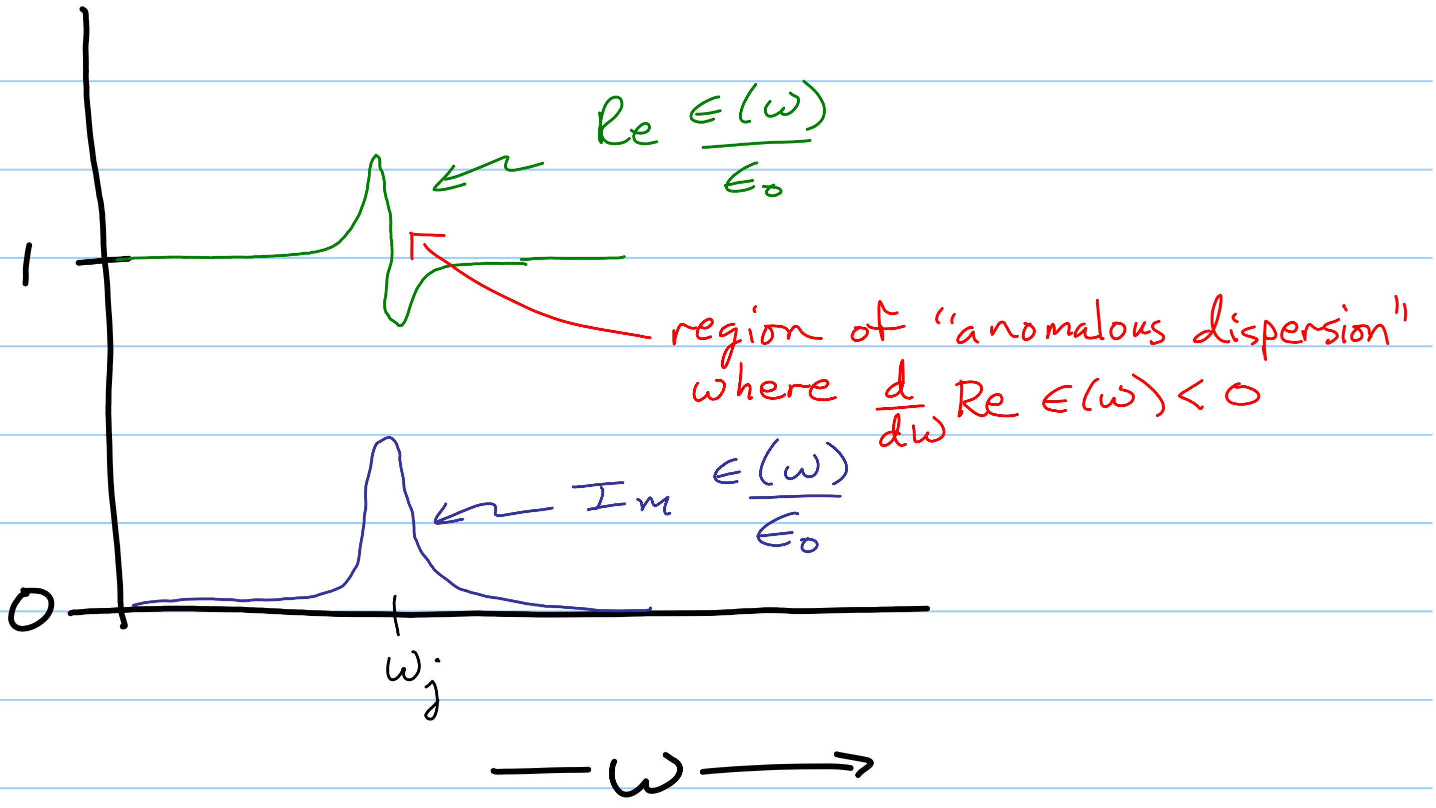
Intermediate ω near resonant ω_j:

$\Rightarrow$ for $\omega \simeq \omega_j$,

$$\alpha(\omega) \simeq \frac{e^2}{m} \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$$

$$\Rightarrow \text{Re}\, \alpha(\omega) \simeq \frac{e^2}{m} f_j \frac{\omega_j^2 - \omega^2}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$

$$\text{Im}\, \alpha(\omega) = \frac{e^2}{m} f_j \frac{\gamma_j \omega}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$



On your own, read Jackson, Sec. 7.5(E) pp 314-316, on the importance of the WATER WINDOW, where water has virtually NO absorption between $\nu \simeq 4 \times 10^{14}$ Hz $- 8 \times 10^{14}$ Hz in the visible region!

Sec. 7.5 Wavepackets, dispersion, group velocity

Recall that the monochromatic plane waves discussed up to now are a mathematical abstraction, a δ-function in $\vec{k}$-space and in ω $\Rightarrow$ they extend over all space and exist for all time

Of course, any EM waves occurring in nature will have finite extent + duration

$\Rightarrow$ these require a superposition of frequencies

Moreover, in general $\epsilon(\omega)$ is complex, which has the following implications:

(1) The phase velocity depends on ω
$\Longrightarrow$ different frequency components in a wavepacket disperse, i.e. drift out of phase

(2) Energy flows at the group velocity, rather than the phase velocity c/n.

(3) For $\text{Im}\,\epsilon(\omega) \neq 0$ there is also dissipation

Sec 7.8

For starters, consider the 1D wave eqn,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

This equation has monochromatic plane wave solutions, $u(x,t) = u_0 e^{i(kx-\omega t)}$, where $\omega = kv$, or in general $\frac{\omega(k)}{k} = v(k)$ = phase velocity

Assume for now that μ, ϵ are real

$\Rightarrow$ k, $\omega(k)$ are real

$\Rightarrow$ DISSIPATION is ignored for now

Next, form a general wavepacket solution by superposing monochromatic plane waves, i.e.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx - i\omega(k)t}\, dk$$

In general, to determine $A(k)$, we must specify BOTH $u(x,0)$ AND $\frac{\partial u}{\partial t}(x,t)\Big|_{t=0} = u_t(x,0)$

because this wave equation is 2^{nd}-order in t;

(this is in contrast to the Schrödinger equation, first-order in t, which only requires knowledge of $\Psi(x,0)$ to determine $\Psi(x,t)$ at all later t.)

Then for any real wavepacket $u(x,t)$ we must have:

$$\Rightarrow \quad u(x,0) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk + c.c.$$

and

$$u_t(x,0) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-i\omega(k)\right) A(k)\, e^{ikx}\, dk + c.c.$$

Next apply "Fourier's Trick" to find $A(k)$,

$$\Rightarrow \text{set} \quad C_{k'} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-ik'x}\, dx = \frac{A(k') + A^*(-k')}{2}$$

and
$$D_{k'} \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(x,0)\,e^{-ik'x}\,dx$$
$$= -\frac{i}{2}\left(\omega(k')A(k') - \omega^*(-k')A^*(-k')\right)$$
$$= \frac{1}{2}i\omega(k')\left(-A(k') + A^*(-k')\right),$$

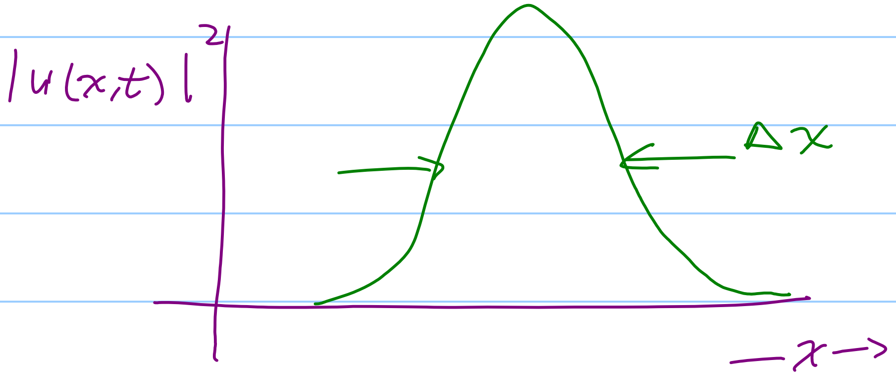
(since frequencies here are real and direction-independent)

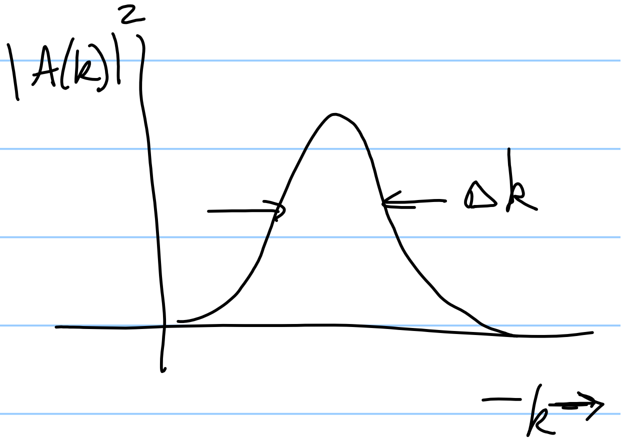
These 2 equations can be solved, giving

$$C_k + \frac{D_k}{i\omega(k)} = A^*(-k) \qquad C_k - \frac{D_k}{i\omega(k)} = A(k)$$

$$\Rightarrow \boxed{A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\left[u(x,0) + \frac{i}{\omega(k)}u_t(x,0)\right]dx}$$

These waves obey an uncertainty relation, both in time and in space, e.g.





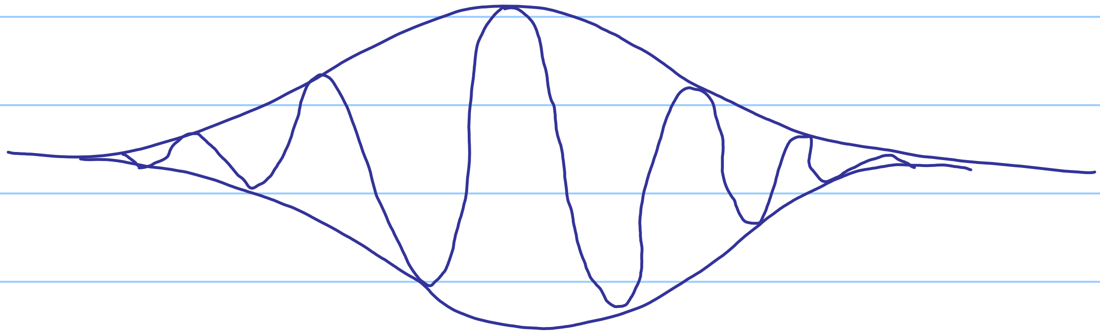
Then from Fourier analysis, $\Delta k \Delta x \geq \frac{1}{2}$ with equality only for a Gaussian pulse

To proceed farther, consider an $A(k)$ that is peaked around some value, say, k_0, and Taylor expand $\omega(k)$ about k_0:

$$\omega(k) \simeq \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0}(k-k_0) + \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0}(k-k_0)^2 + \cdots$$

Then the following model can be solved exactly: suppose that the $t=0$ solution is

$$u(x,0) = e^{-x^2/\gamma^2}\cos(k_0 x),$$ and neglect $u_t(x,0) \simeq 0$



(This actually corresponds to 2 different pulses moving in the ± x-directions)

Then $$A(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-ikx} e^{-x^2/\gamma^2} \cos(k_0 x)$$

$$= \frac{1}{2\sqrt{2\pi}} \int dx \left\{ e^{-i(k-k_0)x - x^2/\gamma^2} + e^{-i(k+k_0)x - \frac{x^2}{\gamma^2}} \right\}$$

To do this integral, complete the square, i.e. set $$\frac{x^2}{\gamma^2} + i(k-k_0)x = \frac{1}{\gamma^2}\left[x + \frac{i(k-k_0)\gamma^2}{2}\right]^2 + \frac{(k-k_0)^2\gamma^2}{4}$$

and change variables to $x' = x + \frac{i(k-k_0)\gamma^2}{2}$
whereby

$$A(k) = \frac{1}{2\sqrt{2\pi}}\, e^{-(k-k_0)^2\frac{\gamma^2}{4}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x'^2}{\gamma^2}}\, dx'}_{\simeq \gamma\sqrt{\pi}}$$

$$+ (k \to -k_0)$$

Now plug into the time-dependent wavepacket integral, with $\omega(k) \simeq \omega_0 + v_g(k-k_0) + \frac{1}{2}\omega_0 a^2 (k-k_0)^2$

with $v_g \equiv \frac{d\omega}{dk}\Big|_{k_0}$, $a^2 \equiv \frac{1}{\omega_0}\frac{d^2\omega}{dk^2}\Big|_{k_0}$

⇒ the final wavepacket is

$$\boxed{u(x,t) = \frac{e^{ik_0x - i\omega_0 t}\,\gamma}{4}\, \frac{e^{-(x-v_g t)^2/(\gamma^2 + 4ia^2\omega_0 t)}}{\sqrt{\gamma^2 + 4ia^2\omega_0 t}}}$$

and $|u(x,t)|^2 = \frac{\gamma^2}{16}\, \frac{e^{-(x-v_g t)^2\left(\frac{2\gamma^2}{\gamma^4 + 16a^4\omega_0^2 t^2}\right)}}{(\gamma^4 + 16a^4 t^2 \omega_0^2)^{1/2}}$

And based on this expression, we conclude that:

① In media where μ, ϵ, and n are constants, independent of ω, k, the phase and group velocities are <u>equal</u> since

$$\omega = \frac{kc}{n} \Rightarrow v_g = \frac{d\omega}{dk} = \frac{c}{n} = v_{phase}$$

② In media where $n \to n(\omega)$, $k = \frac{n(\omega)\omega}{c}$

$$\Rightarrow \boxed{v_g = \frac{1}{\left.\frac{dk}{d\omega}\right|_{\omega_0}} = \frac{c}{n + \omega \frac{dn}{d\omega}}}$$

<u>NORMAL DISPERSION</u> occurs when $n \geq 1$ and $\frac{dn}{d\omega} > 0$

$$\Rightarrow v_g < c$$

<u>ANOMALOUS DISPERSION</u> has $\frac{dn}{d\omega} < 0$ so that $v_g > c$ could arise. This apparent violation of special relativity (causality) is traceable to a failure of our Taylor approximation to $\omega(k)$.

③ The pulse width spreads in general, as time increases.

When the above approximations are valid, under conditions of normal dispersion, the energy propagates at speed $v_g \leq c$.

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