Homework Set 07 C. Greene's Solutions

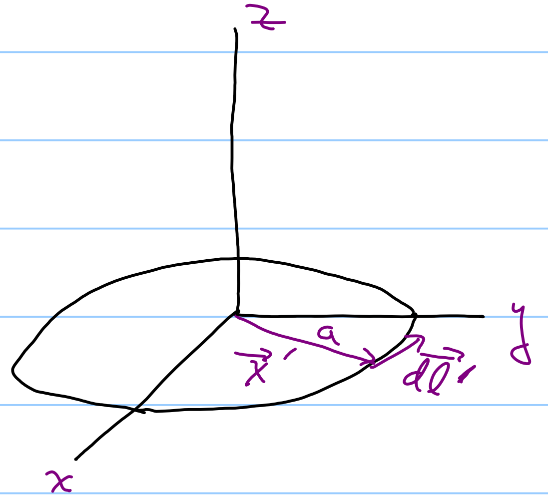
Problems 5.10, 5.16, 5.22, 5.30, 5.35 (a,b,c only)

#5.10 (a) Start from the cylindrical coordinate form of the free-space Green's function, derived in Eq. 3.148:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int dk\, e^{im(\phi-\phi')}\cos k(z-z')\, I_m(k\rho_<)K_m(k\rho_>)$$

and $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\oint \frac{I\,d\vec{l}\,'}{|\vec{x}-\vec{x}'|}$

$$\boxed{\begin{aligned} &, \vec{x}\,'(\phi') = a\hat{\rho}' \\ &= a(\hat{x}\cos\phi' + \hat{y}\sin\phi') \\ &\Rightarrow d\vec{l}\,' = d\vec{x}\,' \\ &= a\,d\phi'(-\hat{x}\sin\phi' + \hat{y}\cos\phi') \\ &= a\hat{\phi}'\,d\phi' \end{aligned}}$$



Then

$$\vec{A}(\vec{x}) = \frac{\mu_0 I}{4\pi} \times$$

$$a\sum_m \Big[\int_0^{2\pi} d\phi'(-\hat{x}\sin\phi' + \hat{y}\cos\phi')\,\frac{2}{\pi}\,e^{im(\phi-\phi')}$$

$$\times \int_0^{\infty}\cos kz\, I_m(k\rho_<)K_m(k\rho_>)\,dk$$

$$= \frac{\mu_0 I a}{4\pi}\,\frac{2}{\pi}\left(-2\pi\hat{x}\sin\phi + 2\pi\hat{y}\cos\phi\right)$$

(calculated in problem 5.8)

$$\times \int_0^{\infty}\cos kz\, I_1(k\rho_<)K_1(k\rho_>)\,dk$$

OR finally

$$\boxed{\begin{aligned} \vec{A}(\vec{x}) &= \frac{\mu_0 I a}{\pi}\,\hat{\phi}\int_0^{\infty}\cos kz\, I_1(k\rho_<)K_1(k\rho_>)\,dk \\ \text{where } \rho_< &= \min(a,\rho),\ \rho_> = \max(a,\rho) \end{aligned}}$$

(b) To derive A_ϕ in the requested form, we should use a different form for the free space Green's function. In fact the needed GF is given in Problem 3.16(b), but I expect you to derive that result if you use it in your solution.

Derivation: expand as oscillatory eigenfunctions in ϕ, ρ and use the Wronskian method to obtain the 1D GF in (z, z')

i.e. start from the form:

$$\frac{1}{|\vec{x}-\vec{x}'|} = G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^\infty k\,dk\, J_m(k\rho) J_m(k\rho') g_k(z,z')$$

This expression has been motivated by the completeness relation proved in problem 3.16(a), solved in our lecture notes this semester, pp. 99–100. $\nabla^2 G = -4\pi\delta(\vec{x}-\vec{x}')$

Next, apply ∇^2 and we get

$$-4\pi \frac{\delta(\rho-\rho')}{\rho} \delta(\phi-\phi')\delta(z-z')$$

$$= \sum_m \frac{e^{im(\phi-\phi')}}{2\pi} \int_0^\infty k\,dk\, J_m(k\rho) J_m(k\rho') \left(-k^2 + \frac{\partial^2}{\partial z^2}\right) g_k(z,z')$$

which will be satisfied if

$$\left(\frac{d^2}{dz^2} - k^2\right) g_k(z,z') = -4\pi\delta(z-z')$$

In class we solved this type of GF in general, obtaining

$$g_k(z,z') = -4\pi \frac{u_1(z_<)\,u_2(z_>)}{W(u_1,u_2)}$$

where the homogeneous solution regular at $z \to -\infty$ is

$$u_1(z) = e^{kz}$$

and the one regular at $z \to +\infty$ is e^{-kz}, since $k>0$

The wronskian is constant, equal to

$$W(u_1,u_2) = u_1 u_2' - u_1' u_2 = -2k$$

So $g_k(z,z') = \frac{2\pi}{k} e^{k(z_< - z_>)} = \frac{2\pi}{k} e^{-k|z-z'|}$

Finally, $$G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{\cancel{2\pi}} \int_0^\infty k\,dk\, J_m(k\rho) J_m(k\rho') \frac{\cancel{2\pi}}{k} e^{-k|z-z'|}$$

Now, as in (a), the ϕ'-integration selects only $m=\pm 1$

$$\Rightarrow \boxed{\vec{A}(\vec{x}) = \frac{\mu_0 I a}{4\pi}\left(\int_0^\infty dk\, J_1(k\rho) J_1(ka) e^{-k|z|}\right)(2\pi\hat{\phi})}$$

↳ agrees with the desired expression

(d) Evaluate the curl in cylindrical coordinates,

$$\vec{B} = \hat{\rho}\left(-\frac{\partial A_\phi}{\partial z}\right) + \hat{z}\,\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) \equiv B_\rho \hat{\rho} + B_z \hat{z}$$

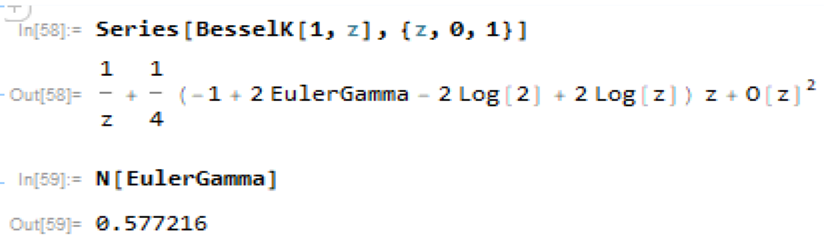
where from part (a) we get

$$\boxed{\begin{aligned} B_\rho &= \frac{\mu_0 I a}{\pi}\int_0^\infty k\,dk\, \sin kz\, I_1(k\rho_<) K_1(k\rho_>) \\ B_z &= \begin{cases} \frac{\mu_0 I a}{\pi}\int_0^\infty dk\, \cos kz\, \frac{1}{\rho}\frac{d}{d\rho}(\rho I_1(k\rho))\, K_1(ka), & \rho < a \\ \frac{\mu_0 I a}{\pi}\int_0^\infty dk\, \cos kz\, I_1(ka)\frac{1}{\rho}\frac{d}{d\rho}(\rho K_1(k\rho)), & \rho > a \end{cases} \end{aligned}}$$

On the z-axis, $\rho_< \to 0$ and $\rho_> \to a$, and $I_1(k\rho) \xrightarrow[k\rho\to 0]{} \left.\frac{k\rho}{2}\right|_{\rho=0} = 0$

while $K_1(k\rho) \xrightarrow[k\rho\to 0]{} \frac{1}{k\rho} + \frac{1}{4}\left[-1+2\gamma-2\ln 2+2\ln(k\rho)\right]k\rho + O(k\rho)^2$

where I used the statement in Mathematica:



```
In[58]:= Series[BesselK[1, z], {z, 0, 1}]

          1   1
Out[58]= - + - (-1 + 2 EulerGamma - 2 Log[2] + 2 Log[z]) z + O[z]^2
          z   4

In[59]:= N[EulerGamma]

Out[59]= 0.577216
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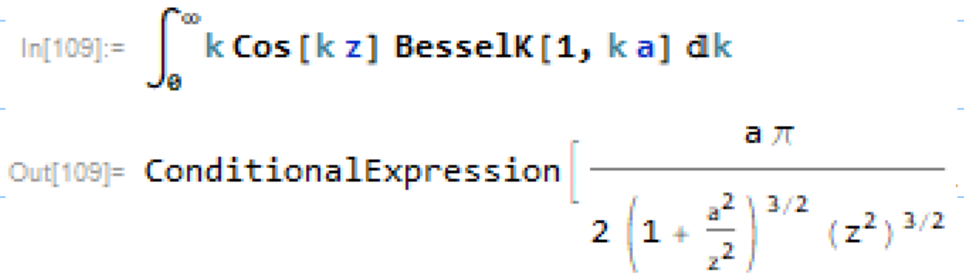
(Actually, we don't need the K_1 expansion since $\rho \to 0$ here.)

$$= \frac{1}{\rho}\frac{d}{d\rho}\left(\rho I_1(k\rho)\right) \xrightarrow[\rho\to 0]{} \frac{1}{\rho}\frac{d}{d\rho}\frac{k\rho^2}{2} = k$$

So on the axis, $\rho \to 0$ and $B_\rho \to 0$, and

$$B_z \xrightarrow[\rho\to 0]{} \frac{\mu_0 I a}{\pi}\int_0^\infty k\,dk\,\cos kz\,K_1(ka)$$



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In[109]:= ∫₀^∞ k Cos[k z] BesselK[1, k a] dk

                                          a π
Out[109]= ConditionalExpression[ ─────────────────────────────
                                  2 (1 + a²/z²)^(3/2) (z²)^(3/2)
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Or Finally $\vec{B} = B_z \hat{z}$ on the z-axis, where

$$\boxed{B_z = \frac{\mu_0 I a^2}{2\left(z^2+a^2\right)^{3/2}}}$$

in agreement with our in-class derivation using the Biot-Savart formula!

#5.16 (a,b) I will assume that $b >> a$ in both parts a,b of this problem, because it may not be analytically solvable, otherwise

Start from the B-field of a circular loop of radius a, evaluated at a distance $b > a$, namely

$$\vec{B}(\text{at } b, \text{due to } a) = \hat{r} B_r + \hat{\theta} B_\theta \quad \left(\begin{matrix}\text{see Eqs,} \\ 5.48, 5.49\end{matrix}\right)$$

where

$$B_\theta^{(a)}(r=b) = -\frac{\mu_0 I a^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \frac{1}{b^3} \left(\frac{a}{b}\right)^{2n} P^1_{2n+1}(\cos\theta)$$

while a loop of larger radius, $d > b$ produces a B-field at $r = b$ having tangential (θ) – component equal to:

$$B_\theta^{(d)} = +\frac{\mu_0 I d^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \left(\frac{2n+2}{2n+1}\right) \frac{1}{d^3} \left(\frac{b}{d}\right)^{2n} P^1_{2n+1}(\cos\theta)$$

Limiting forms, $b >> a$ and $d >> b$ are:

$$B_\theta^{(a)} \xrightarrow[b>>a]{} -\frac{\mu_0 I a^2}{4 b^3} P_1^1(\cos\theta), \quad B_\theta^{(d)} \xrightarrow[d>>b]{} \frac{\mu_0 I}{4d} (2) P_1^1(\cos\theta)$$

So these will cancel and give $B_{||} = B_\theta = 0$

if $\frac{2}{d} = \frac{a^2}{b^3}$, i.e. $\boxed{d = \frac{2b^3}{a^2}}$ ← answer to part (b)

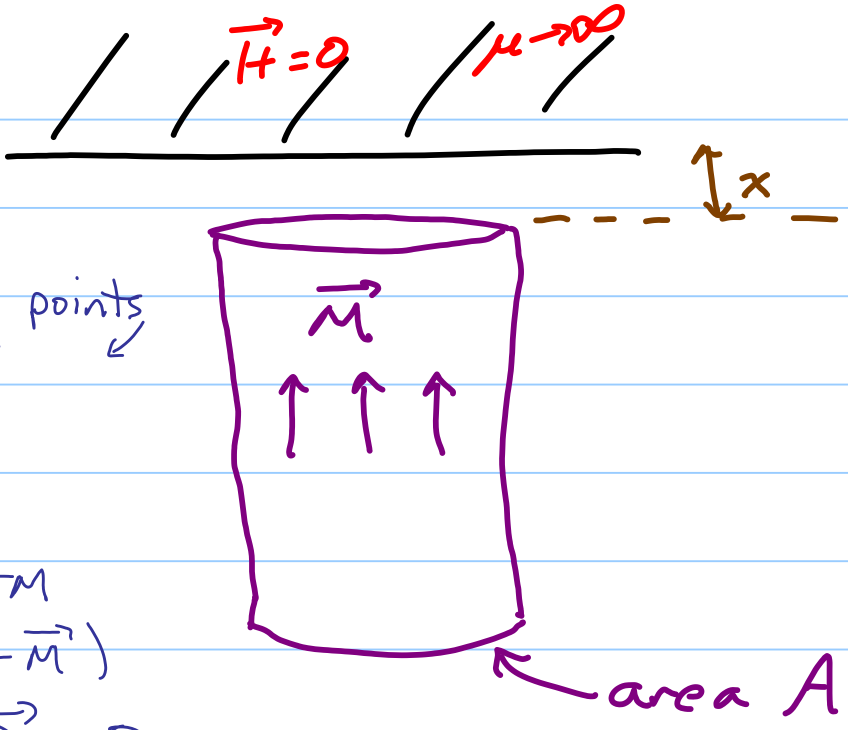
Now add the two fields to find $\vec{B}$ at $r \to 0$, $\left(\begin{matrix}\text{again use} \\ 5.48, 5.49\end{matrix}\right)$:

Radial: $B_r = B_r^{(d)} + B_r^{(a)} = \frac{\mu_0 I d}{2r} \frac{r}{d^2} \overset{''\cos\theta}{P_1(\cos\theta)} + \frac{\mu_0 I}{2a} \cos\theta$

Angular: $B_\theta = B_\theta^{(d)} + B_\theta^{(a)} = \frac{\mu_0 I}{4d} 2 \underset{''\sin\theta}{P_1^1(\cos\theta)} + \frac{\mu_0 I}{2a} \sin\theta$

So the presence of the iron increases all components of $\vec{B}$ by the factor $\boxed{\left(1 + \frac{a}{d}\right) = 1 + \frac{a^3}{2b^3}}$

5.22



Some preliminary points ↙

Here $\vec{J}_{\text{free}}=0$

so $\nabla\times\vec{H}=0$

and $\vec{H}=-\nabla\Phi_M$

and $\vec{B}=\mu_0(\vec{H}+\vec{M})$

Then, since $\nabla\cdot\vec{B}=0$,

$$\Rightarrow \nabla\cdot\vec{H}=-\nabla\cdot\vec{M} \quad \text{and} \quad \nabla^2\Phi_M=\nabla\cdot\vec{M}$$

which is an equation solved by 5.100,

$\rightarrow 0$ for $\vec{M}=\text{constant}$

$$\Phi_M(\vec{x})=-\frac{1}{4\pi}\cancel{\int\frac{\nabla'\cdot\vec{M}(\vec{x}')d^3x'}{|\vec{x}-\vec{x}'|}}+\frac{1}{4\pi}\oint\frac{\hat{n}'\cdot\vec{M}(\vec{x}')da'}{|\vec{x}-\vec{x}'|}$$

For the infinitely-permeable medium,

$$\vec{H}=\vec{B}/\mu \Rightarrow \vec{H}=0 \text{ in the medium}$$

So we have a close analogy with electrostatics, namely

$$\Phi\longleftrightarrow\Phi_M \qquad \frac{\sigma}{4\pi\epsilon_0}\longleftrightarrow\frac{\sigma_M}{4\pi}=\frac{\hat{n}\cdot\vec{M}}{4\pi}$$

$$\vec{E}\longleftrightarrow\vec{H}$$

magnetic "surface charge"

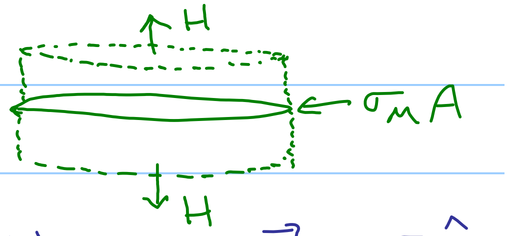
and $\mu\to\infty$ is "like" $\epsilon\to\infty$

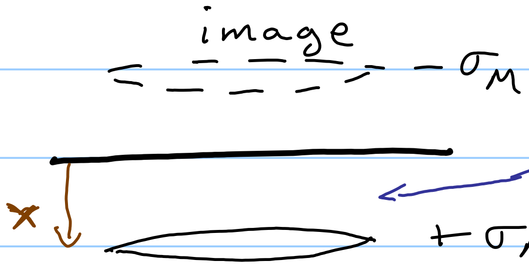
In order to make $\Phi_M = 0$ at the surface of the infinitely-permeable medium, we imagine that the top face of area A, $\sigma_M = M$ at a distance x below the surface has an "image surface magnetic change" $-\sigma_M = -M$ at a distance x above the surface.

Then, assuming that $x \ll \sqrt{A}$, we can neglect surface edge effects, just as we do in the usual parallel plate capacitor derivations.

We can use "Gauss's Law" to find $\vec{H}$ due to either disc, i.e. $\oint \vec{H} \cdot d\vec{a} = \sigma_M A = 2HA$

$$\Rightarrow H = \frac{\sigma_M}{2} \text{ due to 1}$$





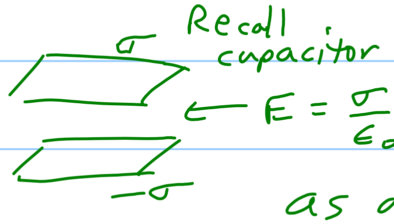
The total H-field here will be twice this, namely $\vec{H} = \sigma_M \hat{n}$

$+\sigma_M$ = real disc is attracted to its image

The energy stored between the real disc and the surface is $W = \frac{1}{2}\int \vec{B}\cdot\vec{H}\, d^3x$

$$= \frac{1}{2}\mu_0 \int H^2 d^3x \simeq \frac{\mu_0}{2}\sigma_M^2 A x$$

Then the force of attraction of the disc towards its image is $\boxed{F = -\frac{dW}{dx} = -\frac{A}{2}\mu_0 \sigma_M^2 = -\frac{A\mu_0 M^2}{2}}$



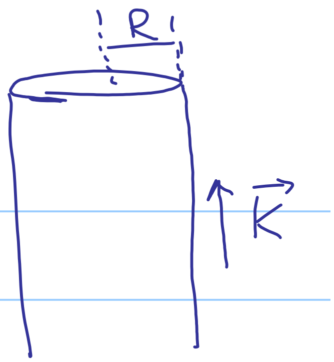
Recall capacitor

$\leftarrow E = \frac{\sigma}{\epsilon_0} \Rightarrow W = \frac{1}{2}\epsilon_0 E^2 A x \Rightarrow F = -\frac{1}{2}\epsilon_0 \frac{\sigma^2}{\epsilon_0^2} A$

as discussed on pp 42-43 for electrostatics

5.30

We are given that



(a) $$\vec{K}(\phi) = \frac{I\cos\phi}{2R}\hat{z}$$

Both inside and outside, $\vec{H} = -\vec{\nabla}\Phi_M$ and we have a BC:

$$\vec{H}^{out} - \vec{H}^{in} = \vec{K}\times\hat{\rho} \quad \text{at } \rho = R$$
$$= \frac{I\cos\phi}{2R}\hat{\phi}$$

$\Rightarrow$ Solve $\nabla^2\Phi_M^{in} = 0$, $\nabla^2\Phi_M^{out} = 0$

and since it is infinitely long and z-independent, we know that Φ_M cannot depend on z

$$\Rightarrow \frac{\partial^2\Phi_M}{\partial z^2} = 0$$

So we can use the 2D Laplace's equation, which from Chap 2 reads

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\Phi^{in}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi^{in}}{\partial\phi^2} = 0, \quad 0<\rho<R$$

$$\Rightarrow \Phi^{in}(\rho,\phi) = \sum_{m=-\infty}^{\infty}\rho^{|m|}e^{im\phi}A_m$$

and $$\Phi^{out}(\rho,\phi) = \sum_{m=-\infty}^{\infty}\rho^{-|m|}e^{im\phi}B_m + C_0\ln\rho$$

BCs $$\vec{B}^{in}\cdot\hat{\rho} = \vec{B}^{out}\cdot\hat{\rho} \Rightarrow -\mu_0\frac{\partial\Phi^{in}}{\partial\rho}\bigg|_{\rho=R} = -\mu_0\frac{\partial\Phi^{out}}{\partial\rho}\bigg|_{\rho=R}$$

and $$\vec{H}^{out}\cdot\hat{\phi} - \vec{H}^{in}\cdot\hat{\phi} = \frac{I\cos\phi}{2R} = -\frac{1}{R}\frac{\partial\Phi^{out}}{\partial\phi}\bigg|_{\rho=R} + \frac{1}{R}\frac{\partial\Phi^{in}}{\partial\phi}\bigg|_{\rho=R}$$

Now $\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2}$ $\Rightarrow$ only nonzero coefficients must be $A_{\pm 1}$, $B_{\pm 1}$

$$-\mu_0 A_1 e^{i\phi} - \mu_0 A_{-1} e^{-i\phi} = -\mu_0\left(-\frac{1}{R^2} e^{i\phi} B_1 - \frac{1}{R^2} e^{-i\phi} B_{-1}\right)$$

$$\Rightarrow A_1 = -\frac{1}{R^2} B_1 \ , \quad A_{-1} = -\frac{1}{R^2} B_{-1}$$

and
$$\left(-R^{-1} i e^{i\phi} B_1 + R^{-1} i e^{-i\phi} B_{-1}\right) + \left(R i e^{i\phi} A_1 - R i e^{-i\phi} A_{-1}\right) = \frac{I}{4}\left(e^{i\phi} + e^{-i\phi}\right)$$

$$\Rightarrow -\frac{B_1 i}{R} + A_1 i R = \frac{I}{4} \ ; \quad \frac{B_{-1} i}{R} - \frac{A_{-1} i}{R} = \frac{I}{4}$$

solutions are $A_1 = \frac{I}{8iR}$, $B_1 = -\frac{IR}{8i}$

$$A_{-1} = -\frac{I}{8iR} \ , \quad B_{-1} = \frac{IR}{8i}$$

So
$$\Phi_M^{in} = \frac{I}{8iR}\left(\rho e^{i\phi} - \rho e^{-i\phi}\right) = \frac{I}{4R}\,\underbrace{\rho\sin\phi}_{``y"}$$

and
$$\Phi_M^{out} = -\frac{IR}{8i\rho}\left(e^{i\phi} - e^{-i\phi}\right) = -\frac{IR}{4\rho}\sin\phi$$

↑ potential of a 2D dipole, see #5.15(a)

Then
$$\vec{H}^{in} = -\nabla\Phi_M^{in} = -\frac{I}{4R}\hat{y}$$

$$\boxed{\vec{B}^{in} = \mu_0 \vec{H}^{in} = -\frac{\mu_0 I}{4R}\hat{y}}$$ = constant field inside!

while $\vec{H}^{out} = -\nabla \Phi_M^{out} = R\frac{I}{4}\nabla\left(\frac{\rho\sin\phi}{\rho^2}\right)$

$$= R\frac{I}{4}\nabla\left(\frac{y}{x^2+y^2}\right) = \frac{I}{4}R\left(\frac{\hat{y}}{\rho^2} - \frac{2y^2\hat{y}}{\rho^4} - \frac{2xy\hat{x}}{\rho^4}\right)$$

But recall that $\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$$\Rightarrow \hat{x} = \frac{\hat{\rho} - \sin\phi\,\hat{y}}{\cos\phi}$$

which gives

$$\vec{H}^{out} = \frac{I}{4\rho^2}R\left(\hat{y}\,\frac{(\rho^2-2y^2)}{\rho^2} - \left(\frac{2\rho^2\sin\phi\cancel{\cos\phi}}{\rho^2}\right)\left(\frac{\hat{\rho} - \sin\phi\,\hat{y}}{\cancel{\cos\phi}}\right)\right)$$

or

$$\boxed{\begin{aligned}\vec{H}^{out} &= \frac{IR}{4\rho^2}\left[\hat{y}\,(1-4\sin^2\phi) - 2\sin\phi\,\hat{\rho}\right]\\ &= \vec{B}^{out}/\mu_0\end{aligned}}$$

(b) Energy / length inside and out is

$$\frac{W^{inside}}{l} = \frac{\frac{1}{2}\int_{in}\vec{B}\cdot\vec{H}\,d^3x}{l} = \frac{1}{2}\int_0^R \frac{\mu_0 I}{4R}\cdot\frac{I}{4R}\,\rho\,d\rho\int_0^{2\pi}d\phi$$

$$= \frac{\mu_0 I^2}{32R^2}\,\frac{R^2}{2}\,(2\pi) = \boxed{\frac{\mu_0 I^2}{32}\,\pi = \frac{W^{inside}}{l}}$$

while $\frac{W^{outside}}{l} = \frac{1}{2}\int_R^\infty \rho\,d\rho\int_0^{2\pi}d\phi\,\frac{\mu_0 R^2 I^2}{16\rho^4}\left[(1-2\sin^2\phi)^2 + (-\sin2\phi)^2\right]$

or $\frac{W^{outside}}{l} = \frac{\mu_0 I^2}{32}\,\frac{R^2}{2R^2}\cdot(2\pi) \leftarrow$ used Mathematica for angular integral and finally $$\boxed{\frac{W^{outside}}{\ell} = \frac{\mu_0 I^2 \pi}{32} = \frac{W^{inside}}{\ell}}$$

So half of the total energy $\frac{W^{total}}{\ell} = \frac{\mu_0 I^2 \pi}{16}$ is inside, half is outside!

(c) equating this to $\frac{W}{\ell} = \frac{1}{2} \frac{L}{\ell} I^2$ gives a self-inductance equal to $\boxed{L = \frac{\mu_0 \pi}{8}}$

5.35 (a) Recalling Jackson problem 5.13, a rotating sphere of charge produced a constant $\vec{B}$-field inside, where

For that case $\vec{B}_0^{(\#5.13)} = \frac{2\mu_0 \omega}{3} \sigma a \hat{z}$

For $\vec{K}^{(\#5.13)} = \sigma \omega a \sin\theta \hat{\phi}$

Clearly we can use the same formula here, except $\mu_0 \to \mu$, $\Rightarrow$ calling $\vec{K} = K_0 \hat{\phi} \sin\theta$,

inside,

$$B_0 \hat{z} = \vec{B} = \frac{2}{3} \mu K_0 \hat{z} \Rightarrow K_0 = \frac{3 a^2 B_0}{2\mu}$$

giving

$$\boxed{\vec{K} = \frac{3 B_0}{2\mu} \sin\theta \hat{\phi}}$$

The $\vec{H}$-field outside is produced only by these free currents $\vec{K}$, so $\nabla \times \frac{\vec{B}}{\mu} = \vec{J} = \vec{K} \delta(r-a)$

$$\vec{A}^{out} = \mu \int da' \frac{\vec{K}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \mu \left(\frac{3 B_0}{2\mu} \right) a^2 \int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} d\phi' \sin\theta'$$

$$\times \sum_{lm} \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} Y_{lm}^*(\hat{x}') Y_{lm}(\hat{x}) (\hat{y} \cos\phi' - \hat{x} \sin\phi')$$

$\Uparrow$ only $l=1$, $m=\pm 1$ survive the integral

use $\sin\theta' \cos\phi' = \left(\frac{8\pi}{3}\right)^{1/2} (Y_{1,-1} - Y_{1,1}) \frac{1}{2}$

and $\sin\theta' \sin\phi' = \left(\frac{8\pi}{3}\right)^{1/2} (Y_{1,-1} + Y_{1,1}) \frac{1}{2i}$

$$= \frac{3a^2 B_0}{2} \frac{4\pi}{3} \frac{r_<}{r_>^2} \left(\frac{2\pi}{3}\right)^{1/2} \left[\hat{y} (Y_{1,-1} - Y_{1,1}) - \hat{x} (Y_{1,-1} + Y_{1,1})/i \right]$$

$$\boxed{\vec{A}_{\text{in or out}} = \frac{3a^2 B_0}{2} \frac{r_<}{r_>^2} \sin\theta \hat{\phi}}$$

aside: $\frac{4\pi}{3} \left[Y_{1,1}^*(\hat{x}') Y_{1,1}(\hat{x}) + Y_{1,-1}^*(\hat{x}') Y_{1,-1}(\hat{x}) \right] = \frac{1}{2} \sin\theta' e^{-i\phi'} \sin\theta e^{i\phi} + \text{c.c.}$

(b) Preliminaries

The Cartesian components of $\vec{A}$ are:

$$\vec{A}(\vec{x}) = [\hat{x}(-\sin\phi \sin\theta) + \hat{y}(\sin\theta \cos\phi)] \frac{r_<}{r_>^2} \frac{B_0 a^2}{2}$$

and according to Eq. 5.160, the Cartesian components of $\vec{A}$ obey

$$\nabla^2 A_i = \mu\sigma \frac{\partial A_i}{\partial t}$$

A Laplace transform-type solution starts from the separable particular solutions,

$$A_i(\vec{x},t) \equiv F_i(\vec{x})\, T(t)$$, which we plug in and divide by FT:

$$\Rightarrow \frac{\nabla^2 F_i(\vec{x})}{F_i(\vec{x})} = \mu\sigma \frac{T'(t)}{T(t)} = -K^2 = \text{constant}$$

$\Rightarrow$ separated equations are:

$$T'(t) = -\frac{K^2}{\mu\sigma} T(t) \Rightarrow T(t) = e^{-\frac{K^2}{\mu\sigma}t}$$

and $F_i(\vec{x})$ obeys $\quad \nabla^2 F_i(\vec{x}) + K^2 F_i(\vec{x}) = 0$

whose spherical separated solutions in free space are $\quad F_i(\vec{x}) = Y_{lm}(\hat{x})\, j_l(Kr)$

So the general solution, where there are no boundaries, must be

$$A_i(\vec{x},t) = \sum_{lm} \int_0^\infty dK \; C^{(i)}_{lm}(K) Y_{lm}(\hat{x}) j_l(Kr) e^{-\frac{K^2}{\mu\sigma}t}$$

Here, for instance, we demand that our initial vector potential obeys

$$A_x(\vec{x},0) = -\sin\theta \sin\phi \frac{r_<}{r_>^2} \frac{B_0 a^2}{2}$$

$$= \sum_{lm} \int_0^\infty C^{(1)}_{lm}(K) Y_{lm}(\hat{x}) j_l(Kr)\, dK$$

and now Fourier's trick determines $C^{(1)}_{lm}(K)$, which vanishes unless $(l,m) = (1,\pm 1)$:

So using $\frac{2}{\pi}\int_0^\infty j_l(Kr)\, j_l(K'r) r^2 dr = \frac{\delta(k-k')}{k^2}$, we obtain

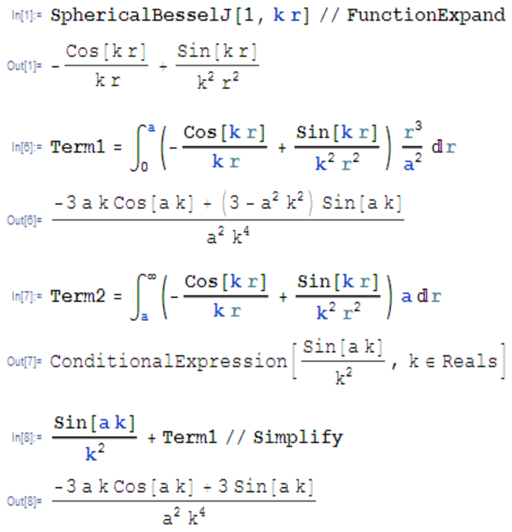
$$\frac{\pi}{2} \frac{C^{(1)}_{lm}(K)}{K^2} = \frac{-B_0 a^2}{2} \left(\int d\Omega\, Y^*_{lm}(\hat{x}) \sin\theta \sin\phi \right)$$

$$\times \left(\int_0^\infty j_l(Kr) \frac{r_<}{r_>^2} r^2 dr \right)$$

The radial integral here is

$$(\quad) = \int_0^a j_1(Kr) \frac{r^3}{a^2} dr + \int_a^\infty j_1(Kr)\, a\, dr$$

This is a simple integral to evaluate, so I'll use Mathematica:



In[1]:= `SphericalBesselJ[1, k r] // FunctionExpand`

Out[1]= $-\frac{\text{Cos}[k\, r]}{k\, r}+\frac{\text{Sin}[k\, r]}{k^2 r^2}$

In[6]:= Term1 = $\int_0^a\left(-\frac{\text{Cos}[k\, r]}{k\, r}+\frac{\text{Sin}[k\, r]}{k^2 r^2}\right)\frac{r^3}{a^2}\,dr$

Out[6]= $\frac{-3\, a\, k\, \text{Cos}[a\, k]+\left(3-a^2 k^2\right)\text{Sin}[a\, k]}{a^2 k^4}$

In[7]:= Term2 = $\int_a^\infty\left(-\frac{\text{Cos}[k\, r]}{k\, r}+\frac{\text{Sin}[k\, r]}{k^2 r^2}\right)a\,dr$

Out[7]= ConditionalExpression$\left[\frac{\text{Sin}[a\, k]}{k^2},\ k\in\text{Reals}\right]$

In[8]:= $\frac{\text{Sin}[a\, k]}{k^2}$ + Term1 // Simplify

Out[8]= $\frac{-3\, a\, k\, \text{Cos}[a\, k]+3\, \text{Sin}[a\, k]}{a^2 k^4}$

$$(\ \ ) = \frac{3}{k^2}\, j_1(ka)$$

And similarly the angular integrals are

$$(\ \ )_{\substack{l=1\\ m=1}} = i\sqrt{\frac{2\pi}{3}} \quad , \quad (\ \ )_{\substack{l=1\\ m=-1}} = i\sqrt{\frac{2\pi}{3}}$$

Thus $$C^{(1)}_{1\pm1}(k) = -\frac{B_0 a^2}{\pi}\, k^2\, \frac{3}{k^2}\, j_1(ka)\, i\sqrt{\frac{2\pi}{3}}$$

Therefore
$$\vec{A}(\vec{x},0) = \frac{2}{\pi}\,\frac{3B_0 a^2}{2}\sin\theta\,\hat{\phi}\int_0^\infty j_1(ka)\,j_1(kr)\,dk$$
and after a variable change to $k' = ka$,
and insertion of the time dependence $e^{-(k^2/\sigma\mu)t}$, the solution is
$$\boxed{\vec{A}(\vec{x},t) = \sin\theta\,\hat{\phi}\,\frac{3B_0 a}{\pi}\int_0^\infty j_1(k')\,j_1\!\left(\frac{k' r}{a}\right)e^{-\frac{k'^2 t}{\mu\sigma a^2}}\,dk'}$$

To find $\vec{B}$ at the sphere center, expand $\vec{A}$ about $r=0$ using $j_1\left(\frac{k'r}{a}\right) \simeq \frac{k'r}{3a} + O(r^3)$

From Mathematica:

$$\int_0^\infty j_1(k')\, k' e^{-Qk'^2}\, dk' = -\left(\frac{\pi}{4Q}\right)^{1/2} e^{-\frac{1}{4Q}} + \frac{\pi}{2}\mathrm{Erf}\left(\frac{1}{2\sqrt{Q}}\right)$$

where here $Q = t/\mu\sigma a^2$

Set $\nu \equiv 1/\mu\sigma a^2$

Thus $$\vec{A}(\vec{x},t) \xrightarrow[r\simeq 0]{} \overbrace{r\sin\theta\,\hat{\phi}}^{\rho\hat{\phi}}\, \frac{B_0}{\pi}\left\{-\left(\frac{\pi}{4\nu t}\right)^{1/2} e^{-\frac{1}{4\nu t}} + \frac{\pi}{2}\mathrm{Erf}\left(\frac{1}{2\sqrt{\nu t}}\right)\right\}$$

and taking the curl of $\rho\hat{\phi}$ in cylindrical coordinates,

$$\vec{B} = \hat{z}\overset{=2}{\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho^2\right)} B_0\left\{-(4\pi\nu t)^{-1/2} e^{-\frac{1}{4\nu t}} + \frac{1}{2}\mathrm{Erf}\left(\frac{1}{2\sqrt{\nu t}}\right)\right\}$$

or $$\boxed{\vec{B}(0,t) = B_0\hat{z}\left\{-(\pi\nu t)^{-1/2} e^{-\frac{1}{4\nu t}} + \mathrm{Erf}\left(\frac{1}{\sqrt{4\nu t}}\right)\right\}}$$

5.35(c) Let's find the energy using the form

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\, d^3x$$

and $\vec{J} = \sigma\vec{E} = -\sigma\frac{\partial\vec{A}}{\partial t}$

$$\Rightarrow W = \frac{-\sigma}{2}\int \frac{\partial\vec{A}}{\partial t}\cdot\vec{A}\, d^3x = \frac{-\sigma}{4}\frac{d}{dt}\int A^2 d^3x$$

$$= -\frac{\sigma}{4}\frac{d}{dt}\left(\frac{3B_0 a}{\pi}\right)^2 \int d^3x \left[\int_0^\infty e^{-\nu t k^2} j_1(k)\, j_1\left(\frac{kr}{a}\right) dk\right]$$

$$\times\left(\int_0^\infty e^{-\nu t k'^2} j_1(k')\, j_1\left(\frac{k'r}{a}\right) dk'\right)\sin^2\theta$$

or $$W(t) = -\frac{\sigma}{4}\frac{d}{dt}\left(\frac{3B_oa}{\pi}\right)^2 2\pi\int_0^\infty dK\int_0^\infty dK'\, j_1(K)j_1(K')$$

$$\times\, e^{-\nu t K'^2-\nu t K^2}\left(\int_{-1}^{1}dx\,(1-x^2)\right)^{=4/3}$$

$$\times\left[\int_0^\infty r^2dr\, j_1\left(\frac{Kr}{a}\right)j_1\left(\frac{K'r}{a}\right)\right]_{=\frac{\pi}{2}\delta\left(\frac{K}{a}-\frac{K'}{a}\right)}$$

So $$W(t) = -\frac{2\pi\sigma}{4}\frac{d}{dt}\left(\frac{3B_oa}{\pi}\right)^2\frac{\pi a^3}{2}\,\frac{4}{3}\int_0^\infty dK\,\frac{j_1^2(K)\,e^{-2\nu tK^2}}{K^2}$$

where again $\nu \equiv \frac{1}{\mu\sigma a^2}$

So finally, $$\boxed{W(t) = \frac{6B_o^2a^3}{\mu}\int_0^\infty dK\, j_1^2(K)\,e^{-2\nu tK^2}}$$

At long time t, only very small K will contribute, so let's use $j_1(K)\simeq K/3$

$$\Rightarrow W(t)\xrightarrow[t\to\infty]{}\frac{6B_o^2a^3}{9\mu}\underbrace{\int_0^\infty K^2e^{-2\nu tK^2}dK}_{=\frac{\pi^{1/2}}{8(\nu t)^{3/2}\sqrt{2}}}$$

using Mathematica

or finally $$\boxed{W(t)\xrightarrow[\text{large } t]{}\frac{(2\pi)^{1/2}B_o^2a^3}{24\mu(\nu t)^{3/2}}}$$