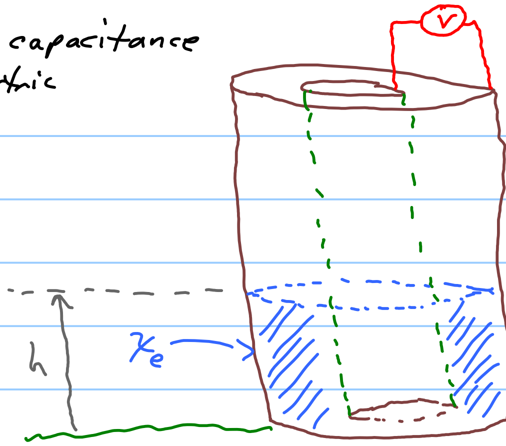
HW Set 06 – C. Greene's solutions

Problem 4.13 Our first step is to calculate the capacitance of a coaxial cable filled with dielectric



Similar to our reasoning in #4.10, we expect the potential between the cylinders to be independent of z (and of course ϕ-independent, too)

$\Rightarrow \Phi = \Phi(\rho)$ obeys $\nabla^2 \Phi(\rho) = 0$ between conductors

For simplicity, let's say that the inner conductor is held at $\Phi = V$, the outer one at $\Phi = 0$

Then the most general form for Φ between them is

$\Phi(\rho) = A + B \ln \rho$, $a < \rho < b$ or concretely, $\Phi(\rho) = \dfrac{V \ln \frac{\rho}{b}}{\ln \frac{a}{b}}$

Then $\vec{E}(\rho) = -\nabla \Phi = -\hat{\rho} \dfrac{V}{\ln \frac{a}{b}} \dfrac{1}{\rho}$

We can now integrate to find the energy of a length L part of the dielectric cable, when no liquid is inside ($h=0$): $W^{\text{no}\,\chi_e} = L \int \frac{\epsilon_0}{2} E^2 \, 2\pi\rho \, d\rho$

$$= \frac{2\pi\epsilon_0 L}{2} \frac{V^2}{\ln^2 \frac{b}{a}} \ln\frac{b}{a} = \frac{1}{2}\left(\frac{2\pi\epsilon_0 L}{\ln \frac{b}{a}}\right) V^2$$

equating this result to $\frac{1}{2} C^{\text{no}\,\chi_e} V^2$, we find $C^{\text{no}\,\chi_e} = \dfrac{2\pi\epsilon_0 L}{\ln b/a}$

Now, when $\chi_e = \frac{\epsilon - \epsilon_0}{\epsilon_0}$ is nonzero (and >0) up to height h, the energy is instead:

$$W^{\chi_e \neq 0} = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, d^3x = \frac{1}{2} \frac{2\pi\epsilon_0 (L-h)}{\ln b/a} V^2 + \frac{1}{2} \frac{2\pi\epsilon h}{\ln b/a} V^2 \equiv \frac{C(h) V^2}{2}$$

where $C(h) = \dfrac{2\pi\epsilon_0 (L + \chi_e h)}{\ln b/a}$

Next how much energy is needed to move the liquid column from h to $h+dh$?

that is $dW^{\text{grav}} = m(h) g \, dh = \rho_0 \pi (b^2 - a^2) h \, g \, dh$, for mass density ρ_0

And now the full energy change formula is $dW^{\text{field}} + dW^{\text{gravity}} - dW^{\text{battery}} = 0$

or $-\frac{1}{2} dC \, V^2 + \rho_0 \pi (b^2 - a^2) g h \, dh = 0$

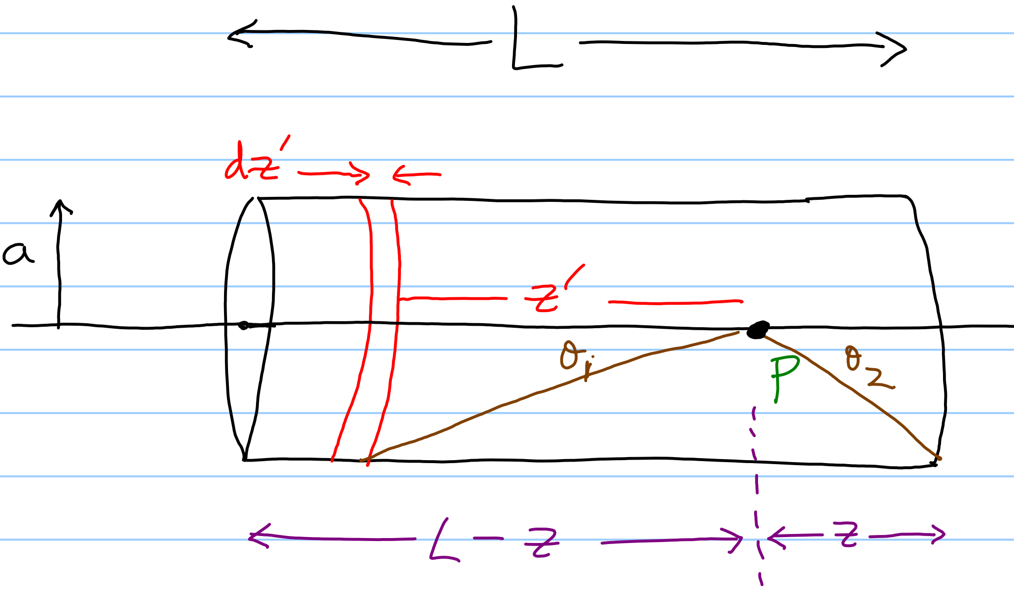
and $dC = \dfrac{2\pi\epsilon_0}{\ln b/a} \chi_e \, dh$ $\Rightarrow -\frac{1}{2}\left(\dfrac{2\pi\epsilon_0 \chi_e}{\ln b/a}\right) V^2 + \rho_0 \pi (b^2 - a^2) g h = 0$

which we solve for χ_e, giving $\boxed{\chi_e = \dfrac{\rho_0 g h (b^2 - a^2) \ln b/a}{\epsilon_0 V^2}}$

Interpretation: Think of $\rho_0 \pi (b^2 - a^2) g h = \dfrac{dW^{\text{grav}}}{dh} = -\left(\vec{F}^{\text{grav}}\right)_h$

and think of $\dfrac{d}{dh}\left(W^{\text{field}} - W^{\text{battery}}\right) = -\frac{1}{2}\dfrac{dC}{dh} V^2$ as the constant force $-\left(F^{\text{pulling}}\right)_h$ that attracts the dielectric up the column. Thus the fluid is pulled up until those forces are equal.

5.3



Setup: The B-field of a single circular current loop carrying current dI, at a distance on axis, a distance z' from the center of the loop, is

$$dB_z = \frac{\mu_0 a^2 dI}{2(z'^2 + a^2)^{3/2}}, \text{ as derived in class}$$

and here $dI = N I dz'$

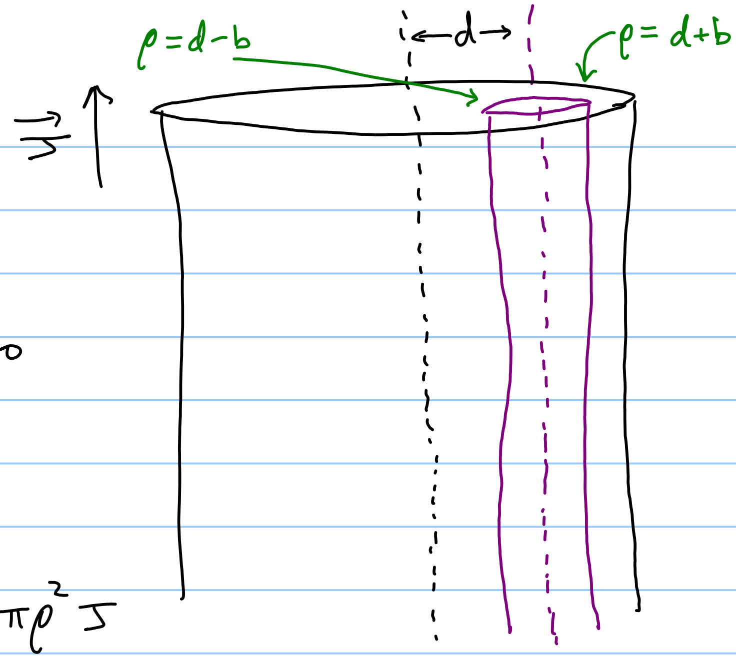
$$\Rightarrow B_z = \int_{z-L}^{z} \frac{\mu_0 N I a^2 dz'}{2(z'^2 + a^2)^{3/2}}$$

used Mathematica

$$= \frac{\mu_0 N I \cancel{a^2}}{2} \frac{1}{\cancel{a^2}} \left[\frac{L-z}{[a^2 + (L-z)^2]^{1/2}} + \frac{z}{(a^2+z^2)^{1/2}} \right]$$

or $\boxed{\vec{B} = \hat{z} \frac{\mu_0 N I}{2} (\cos\theta_1 + \cos\theta_2)}$

#5.6



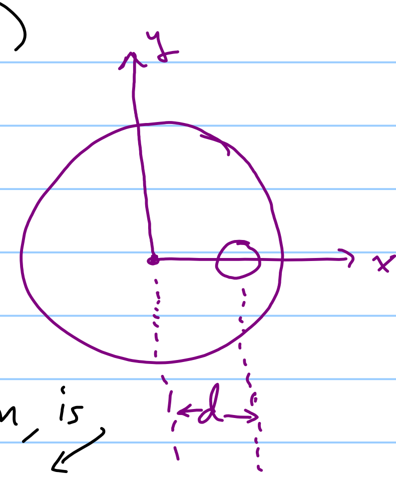
Solve this by superposing the fields due to $+\vec{J}$ in the big cylinder, $\vec{B}_1$, and $\vec{B}_2$ due to $-\vec{J}$ in the little cylinder.

Ampere's Law gives $\vec{B}_1$ simply as $B_1(\rho)(2\pi\rho) = \mu_0 \pi \rho^2 J$

$$\Rightarrow \vec{B}_1 = \frac{\mu_0 J}{2} \rho \hat{\phi} \quad \text{inside } (\rho < a)$$

$$= \frac{\mu_0 J}{2} \rho (-\hat{x} \sin\phi + \hat{y} \cos\phi)$$

$$= \frac{\mu_0 J}{2} (-y\hat{x} + x\hat{y})$$



Next, $\vec{B}_2$ due to current density $-\vec{J}$, i.e. oppositely-directed in the hole region, is

$$\vec{B}_2 = -\frac{\mu_0 J}{2} \rho' \hat{\phi}'$$ with ρ', $\hat{\phi}'$ measured from the center of the hole, or shifting to the origin of the bigger cylinder,

$$\vec{B}_2 = -\frac{\mu_0 J}{2} (-y\hat{x} + (x-d)\hat{y})$$ ← inside the hole region

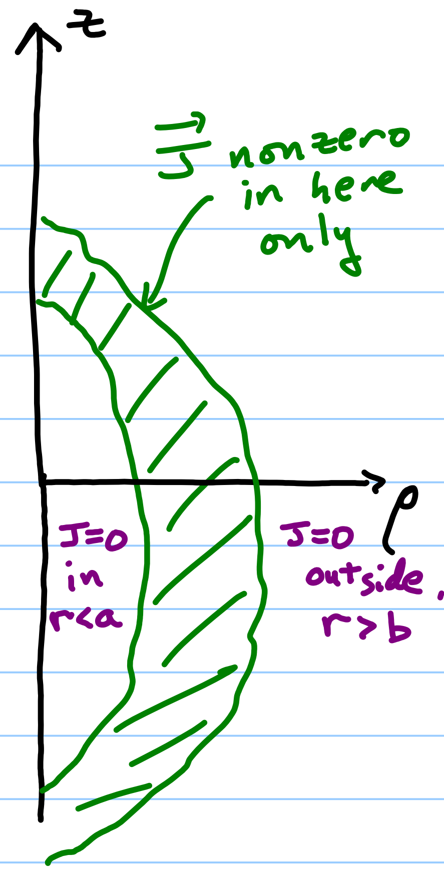
since $\vec{x}' = \vec{x} - \vec{d} = \vec{x} - d\hat{x}$

$\Rightarrow$ Total B-field inside the hole is

$$\vec{B} = \vec{B}_1 + \vec{B}_2 = \frac{\mu_0 J}{2} d \hat{y}$$

#5.8 $\vec{J} = J(r,\theta)\,\hat{\phi}$

Current flows only in the region $a<r<b$, where r = spherical radial coordinate



(a,b) Start from

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}\,')\,d^3x'}{|\vec{x}-\vec{x}\,'|}$$

Then for $r<a$, use:

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \sum_{L,M} \frac{4\pi}{2L+1}\frac{r^L}{r'^{L+1}} Y^*_{LM}(\theta',\phi')\,Y_{LM}(\theta,\phi)$$

and $\hat{\phi}' = -\hat{x}\sin\phi' + \hat{y}\cos\phi'$

$\Rightarrow$ only terms with $M=\pm 1$ will survive the ϕ'-integral.

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{L=1}^{\infty}\int_{-1}^{1} d(\cos\theta')\int_a^b \frac{r'^2dr'}{r'^{L+1}}\left(\frac{4\pi}{2L+1}\right) r^L P_{L1}(\cos\theta')P_{L1}(\cos\theta)$$

$$\times\ \frac{2L+1}{4\pi}\,\frac{(L-1)!}{(L+1)!}\left(\int_0^{2\pi} d\phi'\left[e^{-i\phi'}(-\hat{x}\sin\phi'+\hat{y}\cos\phi')e^{i\phi} + e^{i\phi'}(-\hat{x}\sin\phi'+\hat{y}\cos\phi')e^{-i\phi}\right](-1)\right)$$

used Jackson 3.53

used 3.54

$$\parallel$$

$$-\hat{x}(-i\pi e^{i\phi}) + \hat{y}\,\pi e^{i\phi} - \hat{x}(i\pi e^{-i\phi}) + \hat{y}\,\pi e^{-i\phi} = -2\pi\hat{x}\sin\phi + 2\pi\hat{y}\cos\phi = 2\pi\hat{\phi}$$

$$\Rightarrow \vec{A}(\vec{x}) = -\hat{\phi}\,\frac{\mu_0}{4\pi}\sum_L r^L P_{L1}(\cos\theta)\,\frac{2\pi}{L(L+1)}\int_{-1}^{1} d(\cos\theta')\int_a^b \frac{r'^2dr'}{r'^{L+1}} J(r',\theta')\,P_{L1}(\cos\theta')$$

which can be written as:

$$\boxed{\vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\hat{\phi}\sum_{l=1}^{\infty} m_l r^l P_{l,1}(\cos\theta), \quad r<a \qquad \text{where} \quad m_l = \frac{-1}{l(l+1)}\int d^3x' \, r'^{-l-1} P_{l,1}(\cos\theta') J(r',\theta')}$$

and for $r>b$ the calculation works similarly, only now $r_< \to r'$, $r_> \to r$, giving

$$\boxed{\vec{A}(\vec{x}) = -\frac{\mu_0}{4\pi}\hat{\phi}\sum_{l=1}^{\infty} \mu_l r^{-l-1} P_{l,1}(\cos\theta), \quad r>b \qquad \text{where} \quad \mu_l \equiv -\frac{1}{l(l+1)}\int d^3x' \, r'^{l} P_{l,1}(\cos\theta') J(r',\theta')}$$