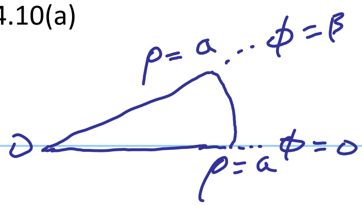
Homework Set 05, C. Greene solutions

3.22, 3.23, 4.6(a), 4.10(a)



3.22

2D problem (find the Dirichlet GF):

$$\nabla^2 G(\rho,\phi;\rho',\phi') = -4\pi \frac{\delta(\rho-\rho')}{\rho'} \delta(\phi-\phi')$$

Start from completeness relation based on ϕ-eigenfunctions that vanish at $\phi = 0$ and β, namely $P_n(\phi) = \left(\frac{2}{\beta}\right)^{1/2} \sin \frac{n\pi\phi}{\beta}$ = orthonormal + complete, $n = 1, 2, \ldots$

obeys $P_n'' + \frac{n^2\pi^2}{\beta^2} P_n = 0$

$\Rightarrow$ We know then that $\sum_{n=1}^{\infty} P_n(\phi) P_n(\phi') = \delta(\phi-\phi')$

Accordingly (using Method I, from the GF toolkit), our GF ansatz is:

$$G(\rho,\phi;\rho',\phi') = \sum_{n=1}^{\infty} P_n(\phi) P_n(\phi') g_n(\rho,\rho') \qquad (1)$$

$\Rightarrow g_n$ obeys:

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} g_n(\rho,\rho') + \frac{n^2\pi^2/\beta^2}{\rho^2} g_n(\rho,\rho') = -4\pi\, \delta(\rho-\rho')/\rho'$$

Let's check whether power law solutions obey the homogeneous diff. eq.

$-(\rho u')' + \frac{1}{\rho}\frac{n^2\pi^2}{\beta^2} u = 0$ and setting $u = \rho^s$

$\Rightarrow -s^2 \rho^{s-1} + \frac{n^2\pi^2}{\beta^2}\rho^{s-1} = 0$ so it works, with $s = \pm \frac{n\pi}{\beta}$!

The only solution regular at $\rho = 0$ is $u_1(\rho) = \left(\frac{\rho}{a}\right)^{n\pi/\beta}$ since $n>0$

for later convenience

The solution vanishing at $\rho = a$ is evidently

$$u_2(\rho) = \left(\frac{a}{\rho}\right)^{\frac{n\pi}{\beta}} - \left(\frac{\rho}{a}\right)^{n\pi/\beta}$$

and their Wronskian is

$$W(u_1,u_2) = u_1 u_2' - u_1' u_2 = \frac{n\pi}{\beta}(-1)\rho^{-1} - (+1)\frac{n\pi}{\beta}\rho^{-1} = \frac{-2n\pi}{\beta\rho}$$

and the Method 2 formula gives

$$g_n(\rho,\rho') = -\frac{u_1(\rho_<)\, u_2(\rho_>)}{\rho W} 4\pi = \left(\frac{\rho_<}{a}\right)^{n\pi/\beta}\left\{\left(\frac{a}{\rho_>}\right)^{\frac{n\pi}{\beta}} - \left(\frac{\rho}{a}\right)^{\frac{n\pi}{\beta}}\right\}\frac{2}{n}$$

plugging this into (1) gives

$$G(\rho,\phi;\rho',\phi') = \sum_{n=1}^{\infty} \left(\frac{\rho_<}{a}\right)^{n\pi/\beta}\left\{\left(\frac{a}{\rho_>}\right)^{\frac{n\pi}{\beta}} - \left(\frac{\rho}{a}\right)^{\frac{n\pi}{\beta}}\right\}\frac{2\beta}{n}\,\frac{2}{\beta}\sin\frac{n\pi\phi}{\beta}\sin\frac{n\pi\phi'}{\beta}$$

which is the result we're asked to prove.

3.23 Grounded cylindrical box $0 \le \rho \le a$, $0 \le z \le L$, $0 \le \phi \le 2\pi$

Note $\Phi(\vec{x}, \vec{x}')$ due to q at $\vec{x}'$ is the same as the Dirichlet GF if we set "$q \to 4\pi\epsilon_0$", so I will simply find G

For starters, consider 3 completeness relations, written for eigenfunctions that obey VANISHING BC's on their respective surfaces:

(i) $$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \delta(\phi-\phi')$$

(ii) $$\frac{2}{L} \sum_{k=1}^{\infty} \sin\frac{k\pi z}{L} \sin\frac{k\pi z'}{L} = \delta(z-z')$$

(iii) $$\sum_{n=1}^{\infty} \frac{J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(\frac{x_{\nu n}\rho'}{a}\right)}{(a^2/2) J_{\nu+1}^2(x_{\nu n})} = \frac{\delta(\rho-\rho')}{\rho'}$$

Method I - eigenfunctions in all coordinates: $\nabla^2 G = -4\pi \frac{\delta(\rho-\rho')}{\rho'} \delta(\phi-\phi')\delta(z-z')$

Ansatz:

$$G = \frac{1}{2\pi} \frac{2}{L} \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(e^{im(\phi-\phi')} \frac{\sin\frac{k\pi z}{L} \sin\frac{k\pi z'}{L} J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{(a^2/2) J_{m+1}^2(x_{mn})} \right) A_{mnk}$$

$$\nabla^2 G = \frac{-1}{\pi L} \sum_{m,k,n} (\quad) \left(\frac{k^2\pi^2}{L^2} + \frac{x_{mn}^2}{a^2}\right) A_{mnk}$$

and so we see that our ansatz works if we choose

$$A_{mnk} = \frac{4\pi}{\frac{k^2\pi^2}{L^2} + \frac{x_{mn}^2}{a^2}}$$

$$\Rightarrow G = \frac{4}{L} \sum_{m,k,n} \frac{e^{im(\phi-\phi')} \sin\frac{k\pi z}{L} \sin\frac{k\pi z'}{L} J_m\left(\frac{x_{mn}\rho}{a}\right) J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\left(\frac{a^2}{2}\right) J_{m+1}^2(x_{mn}) \left(\frac{k^2\pi^2}{L^2} + \frac{x_{mn}^2}{a^2}\right)}$$

This agrees with the 3rd form in 3.23 if we set $q \to 4\pi\epsilon_0$

Method II use completeness ONLY in ρ, ϕ and use two non-oscillatory solutions in z, one vanishes at $z=0$, $u_1(z) = \sinh\left(\frac{x_{mn}z}{a}\right)$ and the other vanishes at $z=L$, namely $u_2(z) = \sinh\left[\frac{x_{mn}}{a}(L-z)\right]$

and the 1D GF $g_{mn}(z,z')$ is found using the Wronskian formula,

$$g_{mn}(z,z') = -4\pi \frac{u_1(z_<)\,u_2(z_>)}{W(u_1,u_2)} \quad \text{and } W = u_1 u_2' - u_1' u_2 = -\frac{x_{mn}}{a}\sinh\left(\frac{x_{mn}L}{a}\right)$$

So this Method II GF is:

$$G = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi} \frac{J_m\left(\frac{x_{mn}\rho}{a}\right)J_m\left(\frac{x_{mn}\rho'}{a}\right)}{\frac{a^2}{2}J_{m+1}^2(x_{mn})} \frac{\sinh\frac{x_{mn}z_<}{a}\sinh\left[\frac{x_{mn}}{a}(L-z_>)\right]}{\frac{x_{mn}}{a}\sinh\left(\frac{x_{mn}L}{a}\right)} 4\pi$$

multiplying the above circled constants gives

$(\;\;) = \frac{4}{a\,x_{mn}}$ and this agrees with the 1^{st} form in J. 3.23

Method III use completeness only in ϕ and z (oscillatory solutions) and use the exponential-type Bessel solutions (modified Bessel Fcns) in ρ that obey correct B.C.'s in ρ:

$$u_1(\rho) = I_m\left(\frac{n\pi}{L}\rho\right), \quad u_2(\rho) = I_m\left(\frac{n\pi}{L}a\right)K_m\left(\frac{n\pi\rho}{L}\right) - I_m\left(\frac{n\pi\rho}{L}\right)K_m\left(\frac{n\pi a}{L}\right)$$

and we need $\rho W(u_1,u_2) = \rho(u_1u_2' - u_1'u_2) = \text{constant}$

Since $\rho W = \text{constant}$ I will evaluate it near $\rho\to\infty$

where

$$I_m(c\rho) \xrightarrow[c\rho\to\infty]{} \frac{e^{c\rho}}{\sqrt{2\pi}\,(c\rho)^{1/2}} \qquad K_m(c\rho) \xrightarrow[c\rho\to\infty]{} \frac{e^{-c\rho}}{(c\rho)^{1/2}}\left(\frac{\pi}{2}\right)^{1/2}$$

$$\Rightarrow \rho W(u_1,u_2) = I_m\left(\frac{n\pi}{L}a\right)\left(\frac{1}{2}\right)\left(\frac{-2c}{c}\right) = -I_m\left(\frac{n\pi a}{L}\right)$$

so the radial GF is

$$g_m(\rho,\rho') = \frac{4\pi}{I_m\left(\frac{n\pi a}{L}\right)}\, u_1(\rho_<)\,u_2(\rho_>)$$

and therefore:

$$G = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}}{2\pi}\,\frac{2}{L}\sin\frac{n\pi z}{L}\sin\frac{n\pi z'}{L}\,\frac{4\pi}{I_m\left(\frac{n\pi a}{L}\right)}\,I_m\left(\frac{n\pi\rho_<}{L}\right)\left(I_m\left(\frac{n\pi}{L}a\right)K_m\left(\frac{n\pi\rho_>}{L}\right) - I_m\left(\frac{n\pi\rho_>}{L}\right)K_m\left(\frac{n\pi a}{L}\right)\right)$$

and this agrees with expression #2 in 3.23 if we set $q\to 4\pi\epsilon_0$ which changes their Φ into G

4.6(a) The interaction energy between the electronic E-field and the nuclear charge distribution can be written as a multipole expansion (4.24), and the quadrupole term is:

$$W = -\frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_j(0)}{\partial x_i}$$

and owing to cylindrical or azimuthal symmetry (p.151, top), the off-diagonal elements of Q_{ij} vanish. As an example, consider $Q_{12} = Q_{xy} = \int \rho\,d\rho \int dz \int d\phi\; \rho_0(\rho,z)\,(\rho \sin\theta \cos\phi)$

$= 0$ and likewise $0 =$ all Q_{ij} for $i \neq j$

Now $Q \equiv \frac{1}{e} Q_{33}$ and since Q_{ij} = traceless

we have $Q_{33} + Q_{22} + Q_{11} = 0$, so $Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$

Moreover, since the fraction of the electron cloud actually INSIDE the nuclear volume is so tiny and basically negligible, we know that to an excellent approximation, $\nabla \cdot \vec{E} = 0$

or $0 = \frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}$, and by cylindrical symmetry,

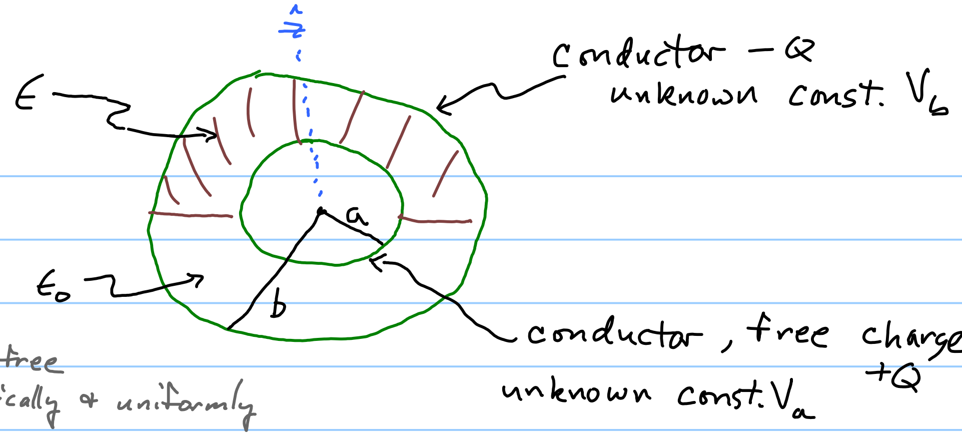
$$\frac{\partial E_1}{\partial x} = \frac{\partial E_2}{\partial y} = -\frac{1}{2}\frac{\partial E_3}{\partial z}$$

Hence, $W = -\frac{1}{6}\left\{\left(-\frac{1}{2}Q_{33}\right)\frac{\partial E_1}{\partial x_1}\Big|_0 + \left(-\frac{1}{2}Q_{33}\right)\frac{\partial E_2}{\partial x_2}\Big|_0 + Q_{33}\frac{\partial E_3}{\partial x_3}\Big|_0\right\}$

or $W = -\frac{1}{6}\left\{\frac{1}{4}Q_{33}\frac{\partial E_3}{\partial x_3}\Big|_0 + \frac{1}{4}Q_{33}\frac{\partial E_3}{\partial x_3}\Big|_0 + Q_{33}\frac{\partial E_3}{\partial x_3}\Big|_0\right\}$

and so $\boxed{W = -\frac{e}{4} Q \frac{\partial E_z}{\partial z}\Big|_0}$ as we're supposed to prove.

4.10(a)



preliminary thoughts

(1) we have azimuthal symmetry

(2) we cannot assume that the free charge is distributed symmetrically & uniformly over each sphere

(3) Each spherical surface in the dielectric can acquire some bound/polarization charge, but there can be no volume bound charge in the dielectric

(4) Therefore the electric potential $\Phi(r,\theta)$ obeys Laplace's equation in the dielectric top half, and in the non-dielectric bottom half.

(5) Since the conductors have constant voltages, θ-independent,
$\Rightarrow -\int_a^b \vec{E}\cdot d\vec{r} = V_b - V_a$ must be θ-independent, which in turn suggests that

$\vec{E}$ is probably purely radial and likely θ-independent. I will assume this for now, then check whether all BC's are satisfied.

Proof of θ-independence: Let $\Phi(r,\theta) = \sum_\ell (c_\ell r^\ell + d_\ell r^{-\ell-1}) P_\ell(\cos\theta)$ and this must equal a constant V_a at $r=a$ since it is a conductor, and another constant V_b at $r=b$ since that is also a conductor. So by "Fourier's trick" only $\ell=0$ can exist in $\Phi \Rightarrow \Phi(r,\theta) \to \Phi(r)$!

(6) In order for Φ to be θ-independent, we expect that it will be the TOTAL surface charge that is spherically-symmetric, rather than the free charge, i.e.
$$\sigma_{tot} = \begin{cases} \sigma_f^{top} + \sigma_b, & 0 \le \theta \le \pi/2 \\ \sigma_f^{bottom}, & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

Now, since σ_{tot} is the only charge inside any Gaussian sphere at $a<r<b$, and since $\vec{E}$ is spherically symmetric, we can find $\vec{E}(r)$ using Gauss's Law,

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{4\pi a^2}{\epsilon_0}\sigma_{tot} = E(r)\,4\pi r^2 \quad \Rightarrow \vec{E}(r) = \frac{a^2}{\epsilon_0 r^2}\hat{r}\,\sigma_{tot}$$

Next, relate σ_{tot} to σ_b using $\sigma_b = \vec{P}\cdot\hat{n}$, with $\hat{n}$ = outward normal, $= -\hat{r}$ at $r=a$

So at $r=a$, $\sigma_b = -\epsilon_0\chi_e \dfrac{a^2\sigma_{tot}}{\epsilon_0 a^2} = -\chi_e\sigma_{tot} \Rightarrow \boxed{\sigma_b = -\chi_e(\sigma_b + \sigma_f^{upper})}$ (*)

We also know that the total free charge at $r=a$ adds up to $+Q$, i.e. $2\pi a^2\sigma_f^{upper} + 2\pi a^2\sigma_f^{lower} = Q$ and $\boxed{\sigma_f^{lower} = \sigma_{tot} = \sigma_b + \sigma_f^{upper}}$ (**)

$\Rightarrow$ solve (*) for $\sigma_b = \dfrac{-\chi_e}{1+\chi_e}\sigma_f^{upper}$, plug into (**), gives $\sigma_f^{lower} = \sigma_f^{upper}\dfrac{1-\chi_e}{1+\chi_e}$

then $\sigma_f^{upper} + \dfrac{1-\chi_e}{1+\chi_e}\sigma_f^{upper} = \dfrac{Q}{2\pi a^2} \Rightarrow \sigma_f^{upper} = \dfrac{1+\chi_e}{2}\dfrac{Q}{2\pi a^2}$

and $\sigma_f^{lower} = \frac{1-\chi_e}{4\pi a^2} Q$ $\qquad \sigma_b = \frac{-\chi_e(1-\chi_e) Q}{4\pi a^2}$

and since $\sigma_{tot} = \frac{1-\chi_e}{4\pi a^2} Q$ on both upper + lower parts at $r=a$,

the E-field between the spheres works out by Gauss's Law to be:

$$\vec{E} = \frac{Q(1-\chi_e)}{4\pi\epsilon_0 r^2}\hat{r}$$

Note that we have satisfied all BC's.

In particular $D_\theta^{upper} = D_\theta^{lower}$ at $\theta = \frac{\pi}{2}$ is obeyed, since both vanish

Also $E_{\parallel}$ is continuous there since E_r is independent of θ everywhere!