Homework Set 04 - C. Greene's Solutions

3.5 Hollow sphere, radius a, we're given that $\Phi = V(\theta,\phi)$ on the surface

(a) This form looks like it fits the form from the Dirichlet Green's function method, i.e.

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \cancel{\rho(\vec{x}')} G_D(\vec{x},\vec{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}da' \quad (1)$$

0 here

where $G_D = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'|\vec{x}-\frac{a^2}{x'^2}\vec{x}'|}$ (see J. Eq 2.16, still true here) (2)

and essentially the GF inside was derived in my solution to problem 2.2, namely for an observation point $P=(r,\theta,\phi)$, source (r',θ',ϕ')

$$G_D = \frac{1}{[r^2+r'^2-2rr'\cos\gamma]^{1/2}} - \frac{a/r'}{[r^2+(\frac{a^2}{r'})^2-2r(\frac{a^2}{r'})\cos\gamma]^{1/2}} \quad (3)$$

and $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

we need
$$\left.\frac{\partial G_D}{\partial r'}\right|_{r'=a} = \left.\frac{-\frac{1}{2}(2r'-2r\cos\gamma)}{[r^2+r'^2-2rr'\cos\gamma]^{3/2}}\right|_{r'=a} + \left.\frac{\frac{1}{2}a(2r'r^2-2ra^2\cos\gamma)}{[r'^2r^2+(a^2)^2-2r(a^2r')\cos\gamma]^{3/2}}\right|_{r'=a}$$

$$= \frac{-(a-r\cos\gamma)}{(r^2+a^2-2ar\cos\gamma)^{3/2}} + \frac{a(ar^2-a^2r\cos\gamma)}{[a^2r^2+a^4-2ra^3\cos\gamma]^{3/2}}$$

or
$$\left.\frac{\partial G_D}{\partial r'}\right|_{r'=a} = \frac{r^2-a^2}{a(r^2+a^2-2ar\cos\gamma)^{3/2}} \quad (3)$$

and plugging this into (1), with $da' = a^2 d\Omega' = a^2 d(\cos\theta')d\phi'$ gives

$$\boxed{\Phi(\vec{x}) = \frac{a(a^2-r^2)}{4\pi}\int d\Omega' \frac{V(\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}}$$

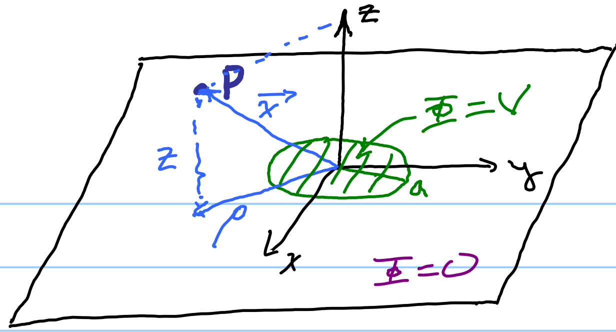
(b) This solution starts from the general solution of Laplace's eqn. in spherical coord's, namely Eq. 3.61, with $B_{lm}=0$ of course since our volume includes the point $r=0$

i.e. $\Phi(r,\theta,\phi) = \sum_{lm} A_{lm} r^l Y_{lm}(\theta,\phi) \xrightarrow[r\to a]{} V(\theta,\phi)$ ← now multiply by $Y^*_{l'm'}$ + $\int d\Omega$

then use orthonormality ("Fourier's trick") to obtain

$$A_{lm} = a^{-l}\int Y^*_{lm}(\theta,\phi)V(\theta,\phi)d\Omega \quad \text{where } d\Omega = d(\cos\theta)d\phi$$

Problem 3.12



Solution We have azimuthal symmetry, so only the $e^{im\phi}$ solutions with $m=0$ are relevant

We are only solving above the plane, i.e. $0 \le z \le \infty$ and we must demand that the solutions in z decay to 0 at $z=\infty$

(a) Hence, start from cylindrical Laplace's eqn solns:

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{-kz} J_m(k\rho) e^{im\phi} A_m(k)$$

Note, no $\begin{Bmatrix} N_m \\ Y_m \end{Bmatrix}$ because they are irregular @ $\rho = 0$

$$\xrightarrow[\text{only}]{m=0} \boxed{\int_0^\infty dk\, e^{-kz} J_0(k\rho) A_0(k) = \Phi(\rho,\phi,z) \xrightarrow{z\to 0} V(1-\Theta(\rho-a))} \quad (*)$$

and now use the Bessel function orthonormality integral (Fourier's trick)

$$\int_0^\infty \rho\, d\rho\, J_m(k\rho) J_m(k'\rho) = \frac{\delta(k-k')}{k'} \qquad \text{Eq. 3.108}$$

i.e. apply $\int_0^\infty \rho J_0(k'\rho)(\cdots\cdots)d\rho$ to both sides of Eq. $(*)$:

$$= V\int_0^a \rho J_0(k'\rho)\,d\rho = \int_0^\infty dk \underbrace{\left[\int_0^\infty \rho\, d\rho\, J_0(k'\rho) J_0(k\rho)\right]}_{=\ \delta(k-k')/k'} A_0(k)$$

$$\Rightarrow A_0(k) = kV\int_0^a \rho J_0(k\rho)\,d\rho = VaJ_1(ka)\,,$$ doing the integral with Mathematica

or $$\boxed{\Phi(\rho,\phi,z) = Va\int_0^\infty dk\, e^{-kz} J_0(k\rho) J_1(ka)}$$ ← answer to (a)

(b) At height z above the disk center ($\rho=0$), use $J_0(0)=1$

$$\Rightarrow \boxed{\Phi(\rho=0,\phi,z) = Va\int_0^\infty dk\, e^{-kz} J_1(ka) = Va\left(\frac{1}{a} - \frac{z}{a\sqrt{a^2+z^2}}\right) = V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)}$$ (via Mathematica)

solution to (b)

(c) Instead at $\rho=a$ and height z, we have to evaluate

$$\Phi(\rho=a,\phi,z) = Va\left(\int_0^\infty dk\, e^{-kz} J_0(ka) J_1(ka)\right) = \frac{V}{2} - \frac{V}{\pi} K\left(-\frac{4a^2}{z^2}\right)$$

$$\int_0^\infty dk\, e^{-kz} J_0(ka) J_1(ka) = \overbrace{\frac{1}{2a} - \frac{1}{a\pi}K\left(-\frac{4a^2}{z^2}\right)}^{\text{using Mathematica}}$$

K: elliptic integral 1st kind

Grader note: This answer is fine if a student stops here

This appears to differ from the answer in the book which we can rewrite in the Mathematica notation as

$$\Phi^{\text{Jackson}} = \frac{V}{2}\left[1 - \frac{1}{\pi d}\,\frac{dz}{(z^2+4a^2)^{1/2}}\,\underbrace{K\left(\frac{4a^2}{z^2+4a^2}\right)}\right]$$

I rewrote the argument of K in Mathematica's notation, whereas the Jackson text uses the notation of Gradsteyn & Ryzhik

and there is an identity which can be found in the NIST DLMF: $K(-m) = \frac{1}{\sqrt{1+m}} K\left(\frac{m}{1+m}\right)$

and this transforms the result from the previous page into the desired final expression Φ^{Jackson}

(also note $K^{\text{Mathematica}}(k^2) = K^{GR}(k)$)

Problem 3.26 Note to grader: r and r' may be interchanged in some students' solutions, because I urged students to do that, in order to be consistent with my notation rather than J's.

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Start from a form for the Neumann GF as written, between two concentric spherical surfaces: $\hat{x}\cdot\hat{x}'$

used the ADDITION THEOREM HERE

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty} g_l(r,r') P_l(\cos\gamma) = \sum_{lm} g_l(r,r') \frac{4\pi}{2l+1} Y^*_{lm}(\hat{x}') Y_{lm}(\hat{x})$$

$\Rightarrow \nabla^2 G(\vec{x},\vec{x}') = -4\pi \frac{\delta(r-r')}{r'^2}\left(\sum_{lm} Y^*_{lm}(\hat{x}')Y_{lm}(\hat{x})\right)$, so g_l obeys $\boxed{\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} g_l - \frac{l(l+1)}{r^2} g_l = -(2l+1)\frac{\delta(r-r')}{r'^2}}$

Assume: $g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + f_l(r,r')$ is stated, but $f_l(r,r')$ not given

(a) Solve for $g_l(r,r')$ with $l>0$ only for now, and we require $\left.\frac{\partial g_l(r,r')}{\partial r}\right|_{r=a} = 0 = \left.\frac{\partial g_l(r,r')}{\partial r}\right|_{r=b}$ for $l>0$ only

I will use Method II from the lecture notes

The homogeneous differential equation to solve is

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial u(r)}{\partial r} - \frac{l(l+1)}{r^2} u(r) = 0$$

or recasting in Sturm–Liouville form (multiply by r^2) gives

$(r^2 u')' - l(l+1)u = 0$ and we make $u_1'(a) = 0 = u_2'(b)$

$u = r^l, r^{-l-1}$

e.g. could choose $u_1(r) = \frac{r^l}{a^l}\frac{a}{l} - a^{l+1} r^{-l-1}\left(\frac{a}{-l-1}\right)$ designed to make $\left.\frac{du_1}{dr}\right|_{r=a} = 0$

alternatively write $u_1 = r^l + B r^{-l-1}$

set $u_1'(a) = l a^{l-1} + B(-l-1) a^{-l-2} = 0 \Rightarrow B = \frac{l}{l+1} a^{2l+1}$

i.e. $u_1 = r^l + \frac{l}{l+1}\frac{a^{2l+1}}{r^{l+1}}$, and similarly $u_2(r) = r^l + \frac{l}{l+1}\frac{b^{2l+1}}{r^{l+1}}$

Now from the general solution for the Sturm–Liouville GF we have for $l>0$

$$g_l(r,r') = -\frac{u_1(r_<)\, u_2(r_>)\,(2l+1)}{r'^2 \left.W(u_1,u_2)\right|_{r'}}$$

(see Notes p. 77) with $\beta = \frac{1}{2l+1}$

let's compute the Wronskian at $r=a$ where $u_1'(a) = 0$ (since $r'^2 W = $ const.)

i.e. $a^2 W(u_1,u_2)\big|_{r'=a} = a^2(u_1 u_2' - \cancel{u_1' u_2})_{r'=a} = a^2\left(r'^l + \frac{l}{l+1}a^l\right)\left(r'^{l-1} l - l\frac{b^{2l+1}}{r'^{l+2}}\right)\Big|_{r'=a}$

gives $r'^2 W = \frac{l(2l+1)}{l+1}\left(a^{2l+1} - b^{2l+1}\right)$

so our $l>0$ radial GF is:

$$g_l(r,r') = \left(r_<^l + \frac{l}{l+1}\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(r_>^l + \frac{l}{l+1}\frac{b^{2l+1}}{r_>^{l+1}}\right)(-1)\,\frac{l+1}{l}\,\frac{1}{a^{2l+1}-b^{2l+1}}$$

$$\boxed{g_l = \left[r_<^l r_>^l + \frac{l}{l+1}\left(a^{2l+1}\frac{r_>^l}{r_<^{l+1}} + b^{2l+1}\frac{r_<^l}{r_>^{l+1}}\right) + \frac{l^2}{(l+1)^2}(ab)^{2l+1}\frac{1}{(r_< r_>)^{l+1}}\right](-1)\,\frac{l+1}{l}\,\frac{1}{a^{2l+1}-b^{2l+1}}} \quad (*)$$

Note $r_<^l r_>^l = (rr')^l$

Let's separate out the term involving $\frac{r_<^l}{r_>^{l+1}}$ to see if it matches J's expected answer:

i.e. consider $\left(b^{2l+1}\frac{r_<^l}{r_>^{l+1}}\right)\frac{1}{b^{2l+1}-a^{2l+1}} + \underbrace{\frac{r_<^l}{r_>^{l+1}}\left(1 - \frac{b^{2l+1}-a^{2l+1}}{b^{2l+1}-a^{2l+1}}\right)}_{\text{added 0 here!}}$ cancels

$$= \frac{r_<^l}{r_>^{l+1}} + a^{2l+1}\frac{r_<^l}{r_>^{l+1}}\frac{1}{b^{2l+1}-a^{2l+1}}$$

Plugging this identity into Eq. (*) now gives the desired final form, for $l>0$:

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{(b^{2l+1}-a^{2l+1})}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right]$$

(b) The $l=0$ radial GF obeys $\left.\frac{\partial G(\vec{x},\vec{x}')}{\partial n}\right|_{\vec{x}\text{ on }S} = \frac{-4\pi}{S}$ $(**)$, $S = 4\pi(a^2+b^2)$

and $\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}g_0(r,r') = -\delta(r-r')$

and two homogeneous solutions are $1, \frac{1}{r}$

so we'll set $g_0(r,r') = \begin{cases} A + \frac{B}{r}, & a<r<r' \\ C + \frac{D}{r}, & r'<r<b \end{cases}$ where A, B, C, D may depend on r'

Continuity: $\boxed{A + \frac{B}{r'} = C + \frac{D}{r'}}$ (1) ; Deriv. Discontinuity: $\left. r'^2\frac{\partial}{\partial r}g_0(r,r')\right|_{r=r'-}^{r=r'+} = -1$

$$\Rightarrow r'^2\left(-\frac{D}{r'^2} + \frac{B}{r'^2}\right) = -1 \quad \Rightarrow \boxed{D - B = 1} \ (2)$$

And imposing $(**)$ above means that $\left.-\frac{\partial g_0(r,r')}{\partial r}\right|_{r=a} = -(a^2+b^2)^{-1} = +\frac{B}{a^2}$

and moreover, $\left.\frac{\partial g_0(r,r')}{\partial r}\right|_{r=b} = -(a^2+b^2)^{-1} = -\frac{D}{b^2}$; $\Rightarrow \boxed{D = \frac{b^2}{a^2+b^2} ;\ B = \frac{-a^2}{a^2+b^2}}$

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check: $D - B = \frac{b^2+a^2}{a^2+b^2} = 1$, agreeing with (2) above

And Eq. (1) becomes $A - \frac{a^2}{r'(a^2+b^2)} = C + \frac{b^2}{r'(a^2+b^2)}$, or $A - C = \frac{1}{r'}$ or $\boxed{A = C + \frac{1}{r'}}$

In the problem we are asked to demonstrate the following, after interchanging $r \leftrightarrow r'$ according to my notation for GF's:

$$g_0^J(r,r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r} + f(r')$$

But my radial GF is: $g_0(r,r') = \begin{cases} C + \frac{1}{r'} - \frac{a^2}{a^2+b^2}\frac{1}{r}, & a < r < r' \\ C + \frac{b^2}{a^2+b^2}\frac{1}{r}, & r' < r < b \end{cases}$

while in this notation

$$g_0^J(r,r') = \begin{cases} \frac{1}{r'} - \frac{a^2}{a^2+b^2}\frac{1}{r} + f(r'), & a < r < r' \\ \underbrace{\frac{1}{r} - \frac{a^2}{a^2+b^2}\frac{1}{r}}_{= \frac{b^2}{a^2+b^2}\frac{1}{r}} + f(r') \end{cases}$$

Hence, our GF agrees with J's provided we identify $C(r') = f(r')$, and if we now choose $C = f(r') = \frac{-a^2}{a^2+b^2}\frac{1}{r'}$, then $g_0(r,r')$ is symmetric under $r \leftrightarrow r'$, which is easier to see if we rewrite $\frac{1}{r} + \frac{1}{r'} = \frac{r+r'}{rr'}$ and then finally $\boxed{g_0(r,r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{r+r'}{rr'}}$

And finally, look at Eq. 1.46: $\Phi(\mathbf{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_N(\mathbf{x},\mathbf{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}G_N\, da' \quad (1.46)$

considering now only the $l=0$ terms in G, we have, if $g_0 \to g_0 + \Delta f(r')$,

$$\underset{\substack{l=0\\ \text{part only}}}{\Phi(\vec{x})} = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\left(g_0(r,r') + \Delta f(r')\right)d^3x'$$
$$+ \frac{1}{4\pi}\int d\Omega'\, a^2\left(-\frac{\partial\Phi}{\partial r'}\right)_a\left(g_0(r,a) + \Delta f(a)\right)$$
$$+ \frac{1}{4\pi}\int d\Omega'\, b^2\left(\frac{\partial\Phi}{\partial r'}\right)_b\left(g_0(r,b) + \Delta f(b)\right)$$

And we see that all of the terms involving Δf will only change Φ by a constant, which is irrelevant. In this sense I appear to disagree with Jackson, because Φ does change if I change $f(r')$, but the $\vec{E}$-field does not change.

Problem 4.2 Let $\rho_{eff}(\vec{x}') = -\vec{p}\cdot\nabla'\delta(\vec{x}'-\vec{x}_0)$. Then the potential is

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho_{eff}(\vec{x}')}{4\pi\epsilon_0|\vec{x}-\vec{x}'|} = -\vec{p}\cdot\int d^3x' \frac{\nabla'\delta(\vec{x}'-\vec{x}_0)}{|\vec{x}-\vec{x}'|}$$

Recall that in 1D: $f(x)\frac{d}{dx}\delta(x-a) = -\frac{df(x)}{dx}\delta(x-a)$ which is easily proved using integration by parts, e.g. $\int_a^b dx\, f(x)\delta'(x-a) = -\int_a^b f'(x)\delta(x-a)dx = -f'(a)$ (note: surface terms vanish)

Applying this in all 3 dimensions gives

$$\Phi(\vec{x}) = \vec{p}\cdot\int\frac{d^3x'}{4\pi\epsilon_0}\delta(\vec{x}'-\vec{x}_0)\nabla'\frac{1}{|\vec{x}-\vec{x}'|} = \frac{\vec{p}}{4\pi\epsilon_0}\cdot\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\Bigg|_{\vec{x}'=\vec{x}_0}$$

or $\boxed{\Phi(\vec{x}) = \frac{\vec{p}\cdot(\vec{x}-\vec{x}_0)}{4\pi\epsilon_0|\vec{x}-\vec{x}_0|^3}}$, in agreement with Eq. 4.10 ✓