Homework Set 03 J. 2.17, 2.23, 3.1, 3.2

C. Greene's Solutions

2.17 (a) Start from the requested integral, free to set $z=0$:

$$I = \int_{-Z}^{Z} \frac{dz'}{(X^2+Y^2+z'^2)^{1/2}} = \ln\left[\frac{Z+\sqrt{X^2+Y^2+Z^2}}{-Z+\sqrt{X^2+Y^2+Z^2}}\right]$$

integral evaluated using mathematica

Next take the limit as $Z\to\infty$. First do a binomial expansion of the numerator + denominator inside the ln:

so $$I = \ln\left[\frac{1+\left(1+\frac{X^2+Y^2}{Z^2}\right)^{1/2}}{-1+\left(1+\frac{X^2+Y^2}{Z^2}\right)^{1/2}}\right] \xrightarrow[Z\to\infty]{} \ln\left[\frac{1+1+\frac{1}{2}\frac{X^2+Y^2}{Z^2}+O(z^{-4})}{-1+1+\frac{1}{2}\frac{X^2+Y^2}{Z^2}}\right]$$

setting $X\equiv x-x'$, $Y=y-y'$

$$= \ln\left[\frac{4+\frac{X^2+Y^2}{Z^2}}{\frac{X^2+Y^2}{Z^2}}\right] = \ln\left[\frac{4Z^2+X^2+Y^2}{X^2+Y^2}\right]$$

$$= \ln 4Z^2 + \ln\left[1+\frac{X^2+Y^2}{4Z^2}\right] - \ln(X^2+Y^2)$$

used $\ln(1+x)\simeq x$ at $x\ll 1$

$$\xrightarrow[Z\to\infty]{} \underbrace{\ln 4Z^2}_{\text{irrelevant constant at } Z\to\infty \text{ so ignore it}} + \underbrace{\frac{X^2+Y^2}{4Z^2}}_{\text{vanishes at } Z\to\infty} - \ln(X^2+Y^2)$$

So our 2D Green's function is $G = -\ln\left[(x-x')^2+(y-y')^2\right]$

2.17(b) The equation obeyed by the 2D GF is:

$$-\nabla^2_{\vec{x}}\, G(\vec{x},\vec{x}') = 4\pi\,\delta(\vec{x}-\vec{x}')$$

We will try the ansatz $G(\vec{x},\vec{x}') = \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')}}{2\pi}\, g_m(\rho,\rho')$

$$\Rightarrow -\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right\} G(\rho,\phi;\rho',\phi') = -\sum_m \frac{e^{im(\phi-\phi')}}{2\pi}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\right\} g_m(\rho,\rho')$$

$$= 4\pi\,\frac{\delta(\rho-\rho')}{\rho}\,\delta(\phi-\phi')$$; now recall that $\sum_m \frac{e^{im(\phi-\phi')}}{2\pi} = \delta(\phi-\phi')$

so we demand $\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\right\} g_m(\rho,\rho') = -4\pi\,\delta(\rho-\rho')/\rho$

2.17(c) We expect the radial GF g_m to have the form:

$$g_m(\rho,\rho') = C\, u_1(\rho_<)\, u_2(\rho_>)$$

where u_1 and u_2 are solutions of the homogeneous diff. eqn, i.e. $\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{du}{d\rho} - \frac{m^2}{\rho^2}u = 0$, solved by a power law, namely

$$u(\rho) = \rho^{\pm|m|}$$

and the solution regular at $\rho\to 0$ is $u_1(\rho) = \rho^{|m|}$ while the one regular at $\rho\to\infty$ is

$$u_2(\rho) = \rho^{-|m|}$$

except for $m=0$ where

$$u_1(\rho) = 1,\ u_2(\rho) = \ln\rho$$

are the 2 linearly-indep solutions, see 2.69 2.70

Now demand the derivative discontinuity condition, i.e. integrating $\int\rho d\rho$ from $\rho'-\epsilon$ to $\rho'+\epsilon$ the final eqn on p. 01 gives

$$\rho'\frac{\partial g_m(\rho,\rho')}{\partial\rho}\Big|_{\rho=\rho'+\epsilon} - \rho'\frac{\partial g_m}{\partial\rho}\Big|_{\rho=\rho'-\epsilon} = -4\pi$$

$$\Rightarrow -\frac{4\pi}{\rho'} = C u_1(\rho')u_2'(\rho') - C u_1'(\rho')u_2(\rho')$$

$$\Rightarrow C(\rho') = \frac{-4\pi}{\rho' W(u_1,u_2)|_{\rho'}}$$

and for $m=0$ $\rho' W(u_1,u_2) = \rho'\left(\frac{1}{\rho'}\right) = 1$

while for $m\neq 0$

$$\rho' W(u_1,u_2) = -2|m|$$

Therefore

$$g_0(\rho,\rho') = -4\pi\ln\rho_>,\quad m=0$$

$$g_m(\rho,\rho') = \frac{2\pi}{|m|}\rho_<^{|m|}\rho_>^{-|m|},\quad m\neq 0$$

Aside: how to find the second solution for $m=0$.
The problem is that for $m=0$ the two solutions $u_1=\rho^0$, $u_2=\rho^{-0}$ are linearly-dependent, so we need a different $u_2(\rho)$. One way is to take the limit $u_2(\rho) = \lim_{m\to 0}\frac{\rho^m - \rho^{-m}}{m}$ and apply de l'Hospital's rule, $\Rightarrow u_2 = \frac{d}{dm}(\rho^m - \rho^{-m}) = \frac{d}{dm}(e^{m\ln\rho} - e^{-m\ln\rho})$

$$= (\ln\rho - (-\ln\rho))e^0 = 2\ln\rho$$

and finally,

$$G(\rho,\phi;\rho',\phi') = -\frac{4\pi}{2\pi}\ln\rho_> + \sum_{\text{all } m\neq 0} e^{im(\phi-\phi')}\frac{2\pi}{|m|}\rho_<^{|m|}\rho_>^{-|m|}\frac{1}{2\pi}$$

$$= -\ln\rho_>^2 + \sum_{m=1}^{\infty}\left(e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right)\rho_<^{|m|}\rho_>^{-|m|}\frac{1}{|m|}$$

or $$\boxed{G = -\ln\rho_>^2 + \sum_{m=1}^{\infty}\frac{2}{m}\cos m(\phi-\phi')\,\rho_<^{m}\rho_>^{-m}}$$ as we were asked to show!

2.23 Hollow cube from $x,y,z = 0,0,0$ to (a,a,a)
The 2 walls at $z=0$, $z=a$ are held at constant $\Phi = V$
and the other 4 sides are held at $\Phi = 0$

So the final potential will be the sum of 2 potentials,
namely Φ_1 for $\Phi_1(z=0) = V$ and $\Phi_1(z=a) = 0$
and Φ_2 for $\Phi_2(z=0) = 0$ and $\Phi_2(z=a) = V$

For Φ_2 the solution inside is $\Phi_2(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm}^{(2)} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \sinh \gamma_{mn} z$
where $\gamma_{nm} = \frac{\pi}{a}\sqrt{n^2+m^2}$

and by "Fourier's trick", $A_{nm}^{(2)} = \frac{4V}{a^2} \frac{1}{\sinh(\gamma_{nm} a)} \int_0^a dx \int_0^a dy \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$

For Φ_1 the solution is $\Phi_1(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm}^{(1)} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \sinh[\gamma_{nm}(a-z)]$

where $A_{nm}^{(1)} = \frac{4V}{a^2 \sinh \gamma_{nm} a} \int_0^a dx \int_0^a dy \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} = A_{nm}^{(2)}$

$$= \frac{4V}{a^2 \sinh(\pi\sqrt{n^2+m^2})} \left\{ -\frac{a}{n\pi} \cos\frac{n\pi x}{a} \Big|_0^a \right\} \left\{ -\frac{a}{m\pi} \cos\frac{m\pi x}{a} \Big|_0^a \right\}$$

$$= \frac{4V}{\pi^2 \sinh(\pi\sqrt{n^2+m^2})} \frac{1}{mn} \left(1-(-1)^n\right)\left(1-(-1)^m\right)$$

or $A_{nm} = \frac{16V}{\pi^2 nm \sinh(\pi\sqrt{n^2+m^2})}$ for n,m both odd
and 0 otherwise

Then the final solution is

$$\Phi = \Phi_1 + \Phi_2 = \frac{16V}{\pi^2} \sum_{n=1}^{\infty \, \text{odd}} \sum_{m=1}^{\infty \, \text{odd}} \frac{\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \left(\sinh \gamma_{mn}(a-z) + \sinh \gamma_{mn} z\right)}{nm \sinh[\pi(n^2+m^2)^{1/2}]}$$

(b) the numerical calculation was done in Mathematica (see next page)
and converges to $\Phi(\frac{a}{2},\frac{a}{2},\frac{a}{2}) = \frac{1}{3} = \frac{2}{6}$ with a few terms

(c) The surface charge density at $z=a$ is found using $\hat{n}_{in} \cdot (\vec{E}_{in} - \vec{E}_{out}) = \frac{\sigma}{\epsilon_0}$ and $E_{out} = 0$ is in the conductor
So $\sigma = \epsilon_0 \frac{\partial \Phi}{\partial z}\Big|_{z=a}$

$\underbrace{\text{cond. } E=0}$
E_{inside} $\uparrow \hat{n}_{inside}$

and $\hat{n}_{in} = -\hat{z}$

or $$\sigma = \frac{\epsilon_0 16V}{\pi^2} \sum_{n,m=1}^{\infty \, \text{odd}} \frac{\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \gamma_{nm} [\cosh \gamma_{nm} a - \overbrace{1}^{\cosh(0)}]}{nm \sinh[\pi(n^2+m^2)^{1/2}]}$$

03

Numerical Calculation for Problem 2.23 in Jackson

In[1]:= summand[n_, m_] :=

$$\frac{16}{n\,m\,\pi^2}\,\frac{\text{Sin}\left[\frac{n\pi x}{a}\right]\text{Sin}\left[\frac{m\pi y}{a}\right]}{\text{Sinh}\left[\pi\sqrt{n^2+m^2}\right]}\left(\text{Sinh}\left[\pi\sqrt{n^2+m^2}\;z/a\right]+\text{Sinh}\left[\pi\sqrt{n^2+m^2}\;(a-z)/a\right]\right)$$

In[15]:= PhiOVERV = $\sum_{j=0}^{jMAX}\sum_{k=0}^{jMAX}$ summand[2 j + 1, 2 k + 1] /. {x → a / 2, y → a / 2, z → a / 2.0, jMAX → 0}

Out[15]= 0.347546

In[16]:= PhiOVERV = $\sum_{j=0}^{jMAX}\sum_{k=0}^{jMAX}$ summand[2 j + 1, 2 k + 1] /. {x → a / 2, y → a / 2, z → a / 2.0, jMAX → 1}

Out[16]= 0.332958

In[12]:= ErrorData = Table[

{jMAX, Re[$\sum_{j=0}^{jMAX}\sum_{k=0}^{jMAX}$ summand[2 j + 1, 2 k + 1]] - 1 / 3.0 /. {x → a / 2, y → a / 2, z → a / 2.0}},

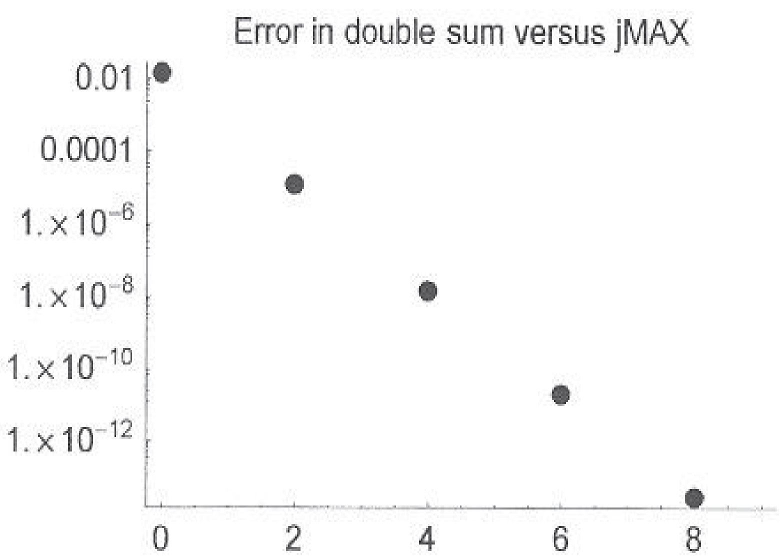
{jMAX, 0, 9}]

Out[12]= {{0, 0.0142125}, {1, -0.000375618}, {2, 0.0000117733}, {3, -3.95792×10^{-7}},
{4, 1.39906×10^{-8}}, {5, -5.11202×10^{-10}}, {6, 1.91341×10^{-11}},
{7, -7.29205×10^{-13}}, {8, 2.81866×10^{-14}}, {9, -1.09813×10^{-15}}}

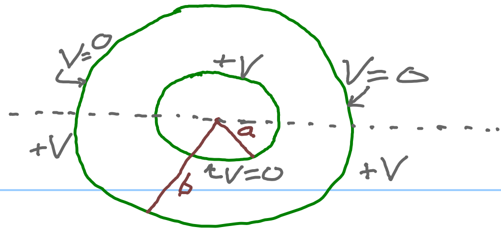
In[7]:= << Graphics`Graphics`

In[8]:= $DefaultFont = {"HelveticaBold", 16};

In[18]:= LogListPlot[ErrorData, PlotStyle → PointSize[0.03],
PlotLabel -> "Error in double sum versus jMAX"];



3.1



solution Note that we have azimuthal symmetry, so the most general form of the solution at $a<r<b$ is:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\cos\theta)$$

and we must determine the constants A_ℓ, B_ℓ using "Fourier's Trick", i.e. orthogonality of $P_\ell(\cos\theta)$

$\Rightarrow$ Our BC's are: $\Phi(a,\theta) = \sum_\ell \left(A_\ell a^\ell + B_\ell a^{-\ell-1}\right) P_\ell(\cos\theta) = \begin{cases} V, & 0 \le \theta \le \pi/2 \\ 0, & \frac{\pi}{2} < \theta \le \pi \end{cases}$

and $\Phi(b,\theta) = \sum_\ell \left(A_\ell b^\ell + B_\ell b^{-\ell-1}\right) P_\ell(\cos\theta) = \begin{cases} 0, & 0 \le \theta \le \pi/2 \\ V, & \frac{\pi}{2} < \theta \le \pi \end{cases}$

Now multiply by $P_{\ell'}(\cos\theta)$ and integrate $\int_{-1}^{1} d(\cos\theta)\,(\cdots)$

Gives $\quad \frac{2}{2\ell+1}\left(A_\ell a^\ell + B_\ell a^{-\ell-1}\right) = V \int_0^1 P_\ell(\cos\theta)\, d(\cos\theta)$

$$\frac{2}{2\ell+1}\left(A_\ell b^\ell + B_\ell b^{-\ell-1}\right) = V \int_{-1}^{0} P_\ell(\cos\theta)\, d(\cos\theta)$$

And from J's Eq. 3.28, $P_\ell = \frac{1}{2\ell+1}\left(P'_{\ell+1} - P'_{\ell-1}\right)$

Hence $\int_{-1}^{0} P_\ell(x)\,dx = \frac{1}{2\ell+1}\left[P_{\ell+1}(0) - P_{\ell+1}(-1) - P_{\ell-1}(0) + P_{\ell-1}(-1)\right]$

and recall that $P_k(0) = 0$ for all odd k

$P_k(1) = 1$, $P_k(-1) = (-1)^k$ and $P_{-1} \equiv 0$

so $\int_{-1}^{0} P_\ell(x)\,dx = \begin{cases} \frac{1}{2\ell+1}\left[-(-1)^{\ell+1} + (-1)^{\ell-1}\right] = 0 & \text{for } \ell = \text{even}, \ell \neq 0 \\ 1, \text{ for } \ell = 0 \\ \frac{P_{2k+2}(0) - P_{2k}(0)}{2(2k+1)+1}, & \text{for } \ell = \text{odd} = 2k+1,\ k = 0,1,2,\ldots \end{cases}$

and using the NIST Math Tables online (DLMF, Chap. 14),

$$P_{2k}(0) = \frac{\sqrt{\pi}}{\Gamma(k+1)\Gamma(\frac{1}{2}-k)}, \quad k = 0,1,2,\ldots$$

Mathematica simplifies this to $\int_{-1}^{0} dx\, P_{2k+1}(x) = \frac{-\sqrt{\pi}}{2\,\Gamma(2+k)\,\Gamma(\frac{1}{2}-k)} \equiv I_k$

and $\int_0^1 P_0(x)\,dx = 1$ and $\int_0^1 P_{2k+1}(x)\,dx = -I_k$

see NIST DLMF Sec 14.5(i)
https://dlmf.nist.gov/14.5

So our 2 equations and 2 unknowns for each l are:

$$\frac{2}{2l+1}\left(A_l a^l + B_l a^{-l-1}\right) = V(-I_k)$$

$$\frac{2}{2l+1}\left(A_l b^l + B_l b^{-l-1}\right) = V I_k$$

, $l = 2k+1$, (and replace $\pm I_k$ by 1 for $l=0$)

Below are the Mathematica solutions to these two equations, and example calculations

```
In[ ]:= LegFcn0[j_] := Sqrt[π] / (Gamma[j + 1] Gamma[(1 - 2 j) / 2])

In[ ]:= (LegFcn0[j + 1] - LegFcn0[j]) / (4 j + 3) // FullSimplify
```

$$\text{Out[ ]}= -\frac{\sqrt{\pi}}{2\,\text{Gamma}\left[\frac{1}{2}-j\right]\,\text{Gamma}[2+j]}$$

```
In[ ]:= Table[{j, -Sqrt[π] / (2 Gamma[1/2 - j] Gamma[2 + j])}, {j, 0, 4}]
```

$$\text{Out[ ]}= \left\{\left\{0, -\frac{1}{2}\right\}, \left\{1, \frac{1}{8}\right\}, \left\{2, -\frac{1}{16}\right\}, \left\{3, \frac{5}{128}\right\}, \left\{4, -\frac{7}{256}\right\}\right\}$$

```
In[ ]:= eqns = {2/(2 L + 1) (A a^L + B a^(-L-1)) == -V Ik, 2/(2 L + 1) (A b^L + B b^(-L-1)) == V Ik} /.
  L → 2 j + 1
```

$$\text{Out[ ]}= \left\{\frac{2\left(a^{1+2j}A + a^{-2-2j}B\right)}{1+2(1+2j)} == -\text{Ik}\,V, \frac{2\left(A\,b^{1+2j} + b^{-2-2j}B\right)}{1+2(1+2j)} == \text{Ik}\,V\right\}$$

```
In[ ]:= Solve[eqns, {A, B}]
```

$$\text{Out[ ]}= \left\{\left\{A \to -\frac{\left(a^{2+2j}+b^{2+2j}\right)\text{Ik}\,(3+4j)\,V}{2\left(a^{3+4j}-b^{3+4j}\right)}, B \to \frac{a^{2+2j}b^{2+2j}\left(a^{1+2j}+b^{1+2j}\right)\text{Ik}\,(3+4j)\,V}{2\left(a^{3+4j}-b^{3+4j}\right)}\right\}\right\}$$

```
In[ ]:= eqns0 = {2/(2 L + 1) (A a^L + B a^(-L-1)) == V, 2/(2 L + 1) (A b^L + B b^(-L-1)) == V} /.
  {L → 0}
```

$$\text{Out[ ]}= \left\{2\left(A+\frac{B}{a}\right) == 1, 2\left(A+\frac{B}{b}\right) == 1\right\}$$

```
In[ ]:= Solve[eqns0, {A, B}]
```

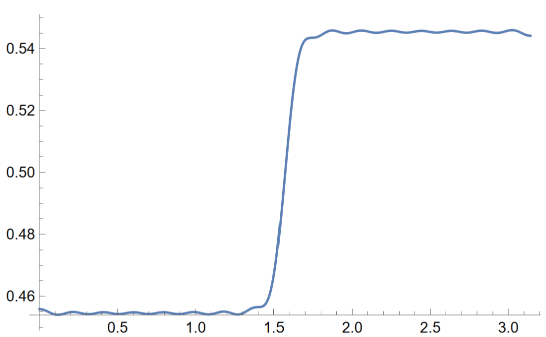
$$\text{Out[ ]}= \left\{\left\{A \to \frac{1}{2}, B \to 0\right\}\right\}$$

```
In[ ]:= I1[j_] := -Sqrt[π] / (2 Gamma[1/2 - j] Gamma[2 + j])

  V = 1;

In[ ]:= Phi[r_, θ_, jmax_] := (V = 1;
   V/2 +
    Sum[
     (-((a^(2+2j) + b^(2+2j)) I1[j] (3 + 4 j) V) / (2 (a^(3+4j) - b^(3+4j))) r^(2 j+1) +
       (a^(2+2j) b^(2+2j) (a^(1+2j) + b^(1+2j)) I1[j] (3 + 4 j) V) / (2 (a^(3+4j) - b^(3+4j))) * 1/r^(2 j+2)) *
      LegendreP[2 j + 1, Cos[θ]], {j, 0, jmax}])
```

Here are a couple of example plots of the angle dependence, for V=1, and they both look sensible

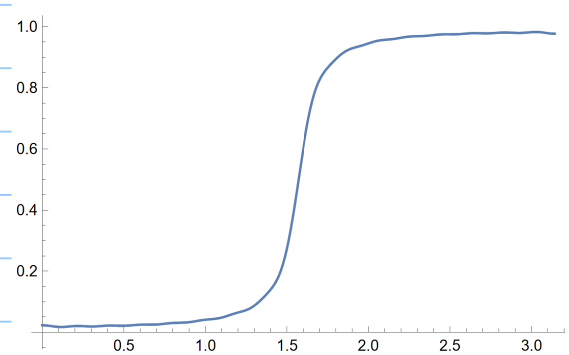
Case of similar radii spheres

```
Plot[Phi[1.1, θ, 15] /. {a → 1, b → 1.2}, {θ, 0, π}]
```

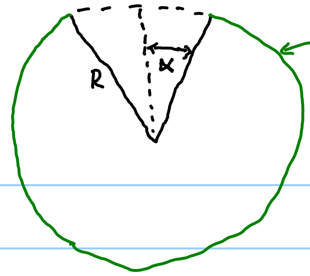


Limit of the inner sphere close to 0 radius

```
Plot[Phi[1.1, θ, 15] /. {a → 0.01, b → 1.2}, {θ, 0, π}]
```



3.2



$\sigma = \frac{Q}{4\pi R^2}$

Observe: azimuthal symmetry! and no conductors!

Solution (a) We can write expressions for the solutions inside (Φ_{in}) and outside (Φ_{out}) as:

$$\Phi_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l \left(\frac{r}{R}\right)^l P_l(\cos\theta) \qquad \Phi_{out}(r,\theta) = \sum_{l=0}^{\infty} B_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

where for convenience I added some factors of R so that A_l and B_l have dimensions of potential

Boundary conditions Where there are surface charges, the E-field must be discontinuous, i.e.

$$\frac{\sigma(\theta)}{\epsilon_0} = \vec{E}_{out}\cdot\hat{r} - \vec{E}_{in}\cdot\hat{r} = \begin{cases} \frac{\sigma}{\epsilon_0}, & \alpha<\theta<\pi \\ 0, & 0<\theta<\alpha \end{cases} \quad \text{or} \quad -\left.\frac{\partial \Phi_{out}}{\partial r}\right|_{r=R} + \left.\frac{\partial \Phi_{in}}{\partial r}\right|_{r=R} = \frac{\sigma(\theta)}{\epsilon_0}$$

and a second boundary condition is continuity of Φ at $r=R$

$$\Rightarrow \sum_{l=0}^{\infty} (A_l - B_l) P_l(\cos\theta) = 0 \quad \text{or} \quad A_l = B_l$$

$$\sum_{l=0}^{\infty}\left[\frac{l A_l}{R}\left(\frac{r}{R}\right)^{l-1} + \frac{l+1}{R}\left(\frac{R}{r}\right)^{l+2} B_l\right]_{r=R} P_l(\cos\theta) = \frac{\sigma(\theta)}{\epsilon_0}$$

$$\text{or} \quad \sum_{l=0}^{\infty} \frac{2l+1}{R} A_l P_l(\cos\theta) = \frac{\sigma(\theta)}{\epsilon_0}$$

Use Fourier's trick to obtain A_l:

$$\frac{2l+1}{R} A_l \left(\frac{2}{2l+1}\right) = \frac{Q}{\epsilon_0 4\pi R^2}\int_0^{\pi} P_l(\cos\theta)\,\sigma(\theta)\sin\theta\, d\theta$$

$$= \frac{Q}{\epsilon_0 4\pi R^2}\int_{-1}^{\cos\alpha} P_l(\cos\theta)\, d(\cos\theta)$$

and evaluate again using J. (3.28)

$$\text{or } A_l = \frac{R}{2\epsilon_0}\frac{Q}{4\pi R^2}\begin{cases} \cos\alpha + 1, & l=0 \\ \dfrac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}, & l=1,2,3,\dots \end{cases}$$

Therefore the solutions in the 2 regions are:

$$\Phi_{in}(r,\vartheta) = \frac{Q}{8\pi\epsilon_0 R}(1+\cos\alpha) + \frac{Q}{8\pi\epsilon_0 R}\sum_{\ell=1}^{\infty}\left(\frac{r}{R}\right)^{\ell}\frac{[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)]}{2\ell+1}P_\ell(\cos\vartheta)$$

$$\Phi_{out}(r,\vartheta) = \frac{Q}{8\pi\epsilon_0 r}(1+\cos\alpha) + \frac{Q}{8\pi\epsilon_0 R}\sum_{\ell=1}^{\infty}\left(\frac{R}{r}\right)^{\ell+1}\frac{[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)]}{2\ell+1}P_\ell(\cos\vartheta)$$

(b) At $r \ll R$, only the $\ell = 0, 1$ terms are significant near $r \simeq 0$

$$\Rightarrow \Phi_{in} \xrightarrow[r \ll R]{} \frac{Q}{8\pi\epsilon_0 R}(1+\cos\alpha) + \frac{Q}{8\pi\epsilon_0}\frac{r}{R^2}\left(\frac{P_2(\cos\alpha) - 1}{3}\right)P_1(\cos\vartheta)$$

Observe now that $r\cos\vartheta = z$, hence only $E_z \neq 0$ and the z-component of the electric field is

$$E_z = -\left.\frac{\partial\Phi}{\partial z}\right|_{r\to 0} \Rightarrow \vec{E} = -\hat{z}\,\frac{Q}{8\pi\epsilon_0 R^2}\left(\frac{P_2(\cos\alpha) - 1}{3}\right) \rightarrow \left(-\frac{\sin^2\alpha}{2}\right)$$

or Finally $\vec{E} = \hat{z}\,\dfrac{Q\sin^2\alpha}{16\pi\epsilon_0 R^2}$ @ $r=0$

(c) Not required