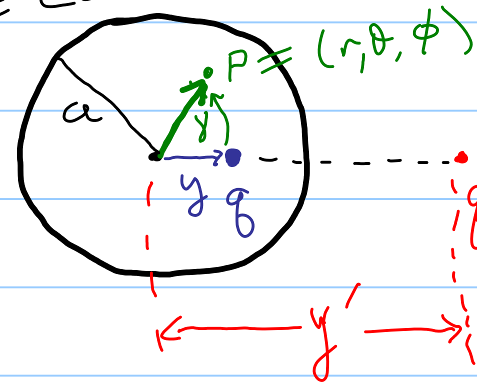
HW set 02, Physics 630, Jackson 2.2, 2.7, 2.10, 2.11

C. Greene's solutions

 # 2.2 A point charge q is INSIDE a hollow, grounded, conducting sphere of radius a.

$\Rightarrow$ we can solve this using an image charge:

outside = conductor (either finite thickness or infinitely so)



q' = image charge placed at y' along the line from the center to q.

To find q', y', demand that the real + image charge will produce $\Phi = 0$ at two points, e.g where the sphere intersects the symmetry line.

$$\Rightarrow \quad 0 = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{a-y} + \frac{q'}{y'-a}\right]$$

and

$$0 = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{a+y} + \frac{q'}{y'+a}\right]$$

Solving these 2 equations and 2 unknowns for q' and y' gives

$$q' = -\frac{a}{y}q \qquad y' = a^2/y$$

= same equations found for the case with q outside!

(a) The potential at any interior point can be written in terms of γ, the polar angle between q & P subtended from the sphere center, i.e:

$$\Phi(r,\gamma) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{[r^2+y^2-2ry\cos\gamma]^{1/2}} + \frac{(-aq/y)}{[r^2+(\frac{a^2}{y})^2-2r(\frac{a^2}{y})\cos\gamma]^{1/2}}\right.$$

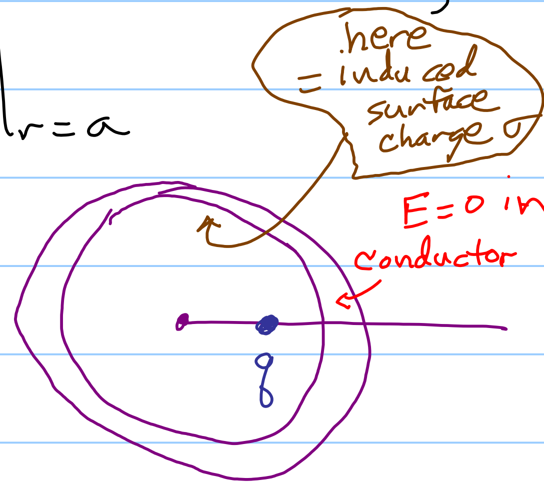
(b) Key equations we need are:

$$\vec{E} = -\nabla\Phi, \quad (\vec{E}_{out}-\vec{E}_{in})\Big|_{r=a}\cdot\hat{r}_{out} = \frac{\sigma}{\epsilon_0}$$

Since here $\vec{E}_{out}=0$ in the conductor at $r>a$,

$$\Rightarrow \quad \hat{r}\cdot\vec{E}_{in}\Big|_{r=a} = -\frac{\partial\Phi}{\partial r}\Big|_{r=a}$$

$$\Rightarrow \quad \boxed{\sigma = \epsilon_0 \frac{\partial\Phi_{in}}{\partial r}\Big|_{r=a}}$$



And this gives

$$\boxed{\sigma = \frac{-q}{4\pi a}\,\frac{a^2-y^2}{(a^2+y^2-2ay\cos\gamma)^{3/2}}}$$

and it is readily verified that $\int\sigma\,da = -q$, as expected from Gauss's Law.

(d) As Sec 2.2 showed, the force on q can be calculated simply as the force on q due to q', giving

$$\boxed{\vec{F} = \hat{z}\,\frac{1}{4\pi\epsilon_0}\,\frac{q^2 a/y}{(\frac{a^2}{y}-y)^2}}$$

, in agreement with Eq.(2.6), p. 60

(d) If the sphere is held at potential V instead of being grounded, this simply amounts to adding a constant potential V to Φ everywhere inside the sphere. If the sphere carries total charge Q, this just adds a constant to the potential everywhere inside, and that constant is in fact

$$\frac{Q+q}{4\pi\epsilon_0 a}$$

#2.7 The Dirichlet Green function for the volume V within the region $z>0$ is the sum of the point charge and image charge terms, namely

(a)

$$G_D(\vec{x},\vec{x}\,') = \frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}} - \frac{1}{\left[(x-x')^2+(y-y')^2+(z+z')^2\right]^{1/2}}$$

(b) Application of Eq. 1.44 with $\rho(\vec{x}\,')=0$ gives the potential at a point (ρ,ϕ,z) with $z\geq 0$ in cylindrical coordinates:

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}\,')\,\frac{\partial G_0(\vec{x},\vec{x}\,')}{\partial(-z')}\,da'$$

or $$\Phi(\vec{x}) = +\frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' \, V \frac{\partial G_D}{\partial z'}$$

$$= \frac{V}{4\pi}\int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' \frac{2z}{\{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2\}^{3/2}}$$

(c) On the axis, $\rho=0$, and the integral is

$$\Phi(\rho=0,z) = \frac{V}{2\pi} z \int_0^{2\pi} d\phi' \int_0^a \frac{\rho' d\rho'}{(\rho'^2+z'^2)^{3/2}}$$

$$= V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

(d) At large distances $\rho^2+z^2 \gg a^2$, we can expand the integrand, (e.g. set $R=\sqrt{\rho^2+z^2}$)

$$\frac{Vz\rho'/2\pi}{[\rho^2+z^2+\rho'^2-2\rho\rho'\cos\phi']^{3/2}} = \frac{Vz}{2\pi(\rho^2+z^2)^{3/2}} \frac{\rho'}{[1+\frac{\rho'^2}{\rho^2+z^2}-\frac{2\rho}{\rho^2+z^2}\rho'\cos\phi']^{3/2}}$$

It is simple enough to do a binomial expansion of $[\ldots]^{-3/2}$ to order R^{-4} at large R, but the next page shows how to coax the result from Mathematica.

The result, to order R^{-7} is:

$$\Phi(\rho,z) \xrightarrow[R\to\infty]{} \frac{Va^2 z}{2R^3}\left[1 - \frac{3a^2}{4R^2} + \frac{5}{8}\frac{(3a^2\rho^2+a^4)}{R^4} + \cdots\right]$$

In[6]:= Series$\left[\dfrac{x}{\left(1+\frac{x^2}{R^2}-2\,\frac{r\,x\,\mathrm{Cos}[p]}{R^2}\right)^{3/2}},\ \{R,\infty,4\}\right]$ // Normal

Out[6]= $x-\dfrac{3\,x\,(x^2-2\,r\,x\,\mathrm{Cos}[p])}{2\,R^2}+\dfrac{15\,x\,(x^2-2\,r\,x\,\mathrm{Cos}[p])^2}{8\,R^4}$

In[7]:= $\displaystyle\int_0^{2\pi}\left(x-\frac{3\,x\,(x^2-2\,r\,x\,\mathrm{Cos}[p])}{2\,R^2}+\frac{15\,x\,(x^2-2\,r\,x\,\mathrm{Cos}[p])^2}{8\,R^4}\right)dp$

Out[7]= $\dfrac{\pi\,(8\,R^4\,x+30\,r^2\,x^3-12\,R^2\,x^3+15\,x^5)}{4\,R^4}$

In[10]:= $\displaystyle\int_0^{a}\frac{\pi\,(8\,R^4\,x+30\,r^2\,x^3-12\,R^2\,x^3+15\,x^5)}{4\,R^4}\,dx$ // Expand

Out[10]= $a^2\,\pi+\dfrac{5\,a^6\,\pi}{8\,R^4}+\dfrac{15\,a^4\,\pi\,r^2}{8\,R^4}-\dfrac{3\,a^4\,\pi}{4\,R^2}$

Similarly we can make a binomial expansion of our result from (c),

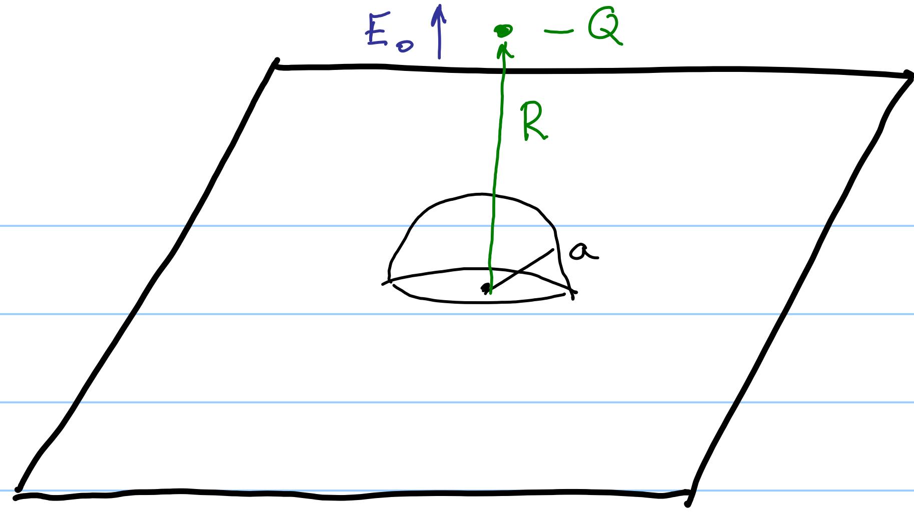
$$V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)=V\left[1-\left(1+\frac{a^2}{z^2}\right)^{-1/2}\right]$$

$$=V\left\{\cancel{1}-\left(\cancel{1}-\frac{1}{2}\frac{a^2}{z^2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\frac{a^4}{z^4}+\left(\frac{-15}{8\cdot 3!}\right)\frac{a^6}{z^6}+\dots\right)\right\}$$

$$=\frac{Va^2}{2z^2}\left\{1-\frac{3}{4}\frac{a^2}{z^2}+\frac{5}{8}\frac{a^4}{z^4}+\dots\right\}$$

and this agrees with the above result evaluated at $\rho=0$.

2.10



(a)

We can treat the uniform $\vec{E}$-field by a charge of magnitude $-Q = -4\pi\epsilon_0 R^2 E_0/2$ at distance R above the center of the boss. Then the discussion in Sec. 2.6 tells us where the image charges must go, and their magnitudes, see Eq. 2.12. Then we need the $R \to \infty$ limit of

$$\Phi = \frac{Q/4\pi\epsilon_0}{(r^2+R^2+2rR\cos\theta)^{1/2}} - \frac{Q/4\pi\epsilon_0}{(r^2+R^2-2rR\cos\theta)^{1/2}}$$

$$-\frac{aQ/4\pi\epsilon_0 R}{\left(r^2+\frac{a^4}{R^2}+\frac{2a^2r}{R}\cos\theta\right)^{1/2}} + \frac{aQ/4\pi\epsilon_0 R}{\left(r^2+\frac{a^4}{R^2}-\frac{2a^2r}{R}\cos\theta\right)^{1/2}}$$

and the $R \to \infty$ limit is (2.14), namely

$$\boxed{\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta}$$

To calculate the induced charge density, recall that in the conductor, $\Phi = 0 = E$

Using (1.22) $\vec{E}_2 \cdot \hat{n}_2 - \cancel{\vec{E}_1}^{\,0} \cdot \hat{n}_2 = \sigma/\epsilon_0$

On the boss, $\hat{n}_2 = \hat{r}$

$$\Rightarrow -\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = \frac{\sigma}{\epsilon_0} \quad \text{or} \quad \sigma = \epsilon_0 E_0 \left(1 + 2\frac{a^3}{a^3}\right)\cos\theta$$

$$\Rightarrow \boxed{\sigma_{Boss} = 3\epsilon_0 E_0 \cos\theta}$$

Outside the boss, on the plane, the charge density is instead

$$\sigma_{plane} = -\epsilon_0 \left.\frac{\partial \Phi}{\partial z}\right|_{z=0^+}$$

$$\Rightarrow \sigma_{plane} = \epsilon_0 E_0 \frac{\partial}{\partial z}\left[z - \frac{z a^3}{(\rho^2+z^2)^{3/2}}\right]\Bigg|_{z=0,\ \rho>a}$$

$$= \epsilon_0 E_0 \left(1 - \frac{a^3}{\rho^3} + \frac{3}{2}\frac{\cancel{z a^3 2z}}{\rho^5}\right) \to 0 \text{ at } z=0$$

$$\Rightarrow \boxed{\sigma_{plane} = \epsilon_0 E_0 \left(1 - \frac{a^3}{\rho^3}\right)} \text{ at } \rho > a$$

The total charge that resides on the boss is of course the following integral:

$$Q_{boss} = \int_0^1 \sigma_{Boss}\, 2\pi a^2\, d(\cos\theta)$$

$$= 3\epsilon_0 E_0 (2\pi a^2) \int_0^1 x\,dx$$

or $\boxed{Q_{boss} = 3\pi\epsilon_0 E_0 a^2}$ as we were supposed to show

2.10 (c)

The answer is already clear from the first equation written above in part (a), except we must replace $-Q \longrightarrow q$, and $R \rightarrow d$:

$$\Phi = \frac{q}{4\pi\epsilon_0}\Big\{(r^2+d^2-2rd\cos\theta)^{-1/2} - (r^2+d^2+2rd\cos\theta)^{-1/2}$$
$$+(a/d)\left(r^2+\frac{a^4}{d^2}+\frac{2a^2r}{d}\cos\theta\right)^{-1/2}$$
$$-(a/d)\left(r^2+\frac{a^4}{d^2}-\frac{2a^2r}{d}\cos\theta\right)^{-1/2}\Big\}$$

then $\sigma_{Boss} = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a}$.

Using Mathematica to differentiate, and then to integrate over the surface of the hemispherical boss, gives:

$$Q_{Boss} = -q\left[1 + \left(\frac{a^2}{d} - d\right)\frac{1}{\sqrt{a^2+d^2}}\right]$$

or simplifying,

$$\boxed{Q_{Boss} = -q\left[1 - \frac{d^2-a^2}{d\sqrt{d^2+a^2}}\right]}$$

(see next page for details)

Appendix to problem 2.10 (c): $\quad (x \equiv \cos\theta)$

$$\sigma_{Boss} = -\frac{q}{4\pi}\Big\{(-a+dx)(a^2+d^2-2adx)^{-3/2}$$
$$+\frac{d}{a}(d-ax)(a^2+d^2-2adx)^{-3/2} + (a+dx)(a^2+d^2+2adx)^{-3/2}$$
$$-\frac{d}{a}(d+ax)(a^2+d^2+2adx)^{-3/2}$$

and Mathematica gives the integral to be:

$$Q_{boss} = 2\pi a^2 \int_0^1 \sigma_{Boss}\, dx$$
$$= -\frac{q a^2}{2a^2}\Big\{2 - \frac{2d}{\sqrt{a^2+d^2}} + \frac{a^2}{d^2}\Big(\cancel{\frac{d}{a}} + \frac{d}{\sqrt{a^2+d^2}}\Big)$$
$$-\frac{a^2}{d^2}\Big(\cancel{\frac{d}{a}} - \frac{d}{\sqrt{a^2+d^2}}\Big)\Big\}$$
$$= -q\Big\{1 - \frac{1}{\sqrt{a^2+d^2}}\Big(d - \frac{a^2}{2d} - \frac{a^2}{2d}\Big)\Big\}$$
$$= -q\Big(1 - \frac{d^2-a^2}{d\sqrt{a^2+d^2}}\Big) \checkmark$$

as we were supposed to prove.

#2.11 (a) A 2D image line charge problem

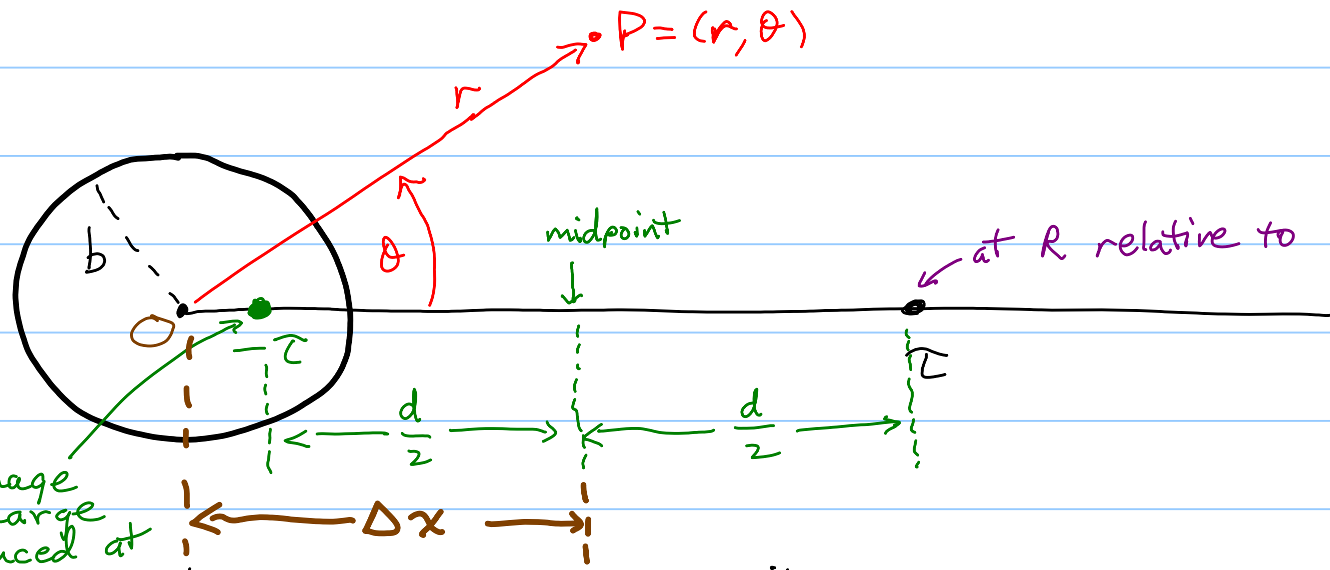


image charge placed at $x_1 = R-d$

magnitude $= -\tau$ to ensure that $\Phi \xrightarrow[r\to\infty]{} 0$ ⟵ **part of the answer to (a)**

(Aside: My solution utilizes some equations derived in the course of solving #2.8, given below as an appendix to this HW set)

From working out aspects of Jackson #2.8 (see writeup following this problem), we have

$$\Delta x = \frac{d}{2} + R - d = \frac{-d}{2}\coth\left(\frac{2\pi\epsilon_0 V}{\tau}\right),$$

which gives the position of image charge

and next, we know from #2.8,

$$b = \frac{-d}{2\sinh\left(\frac{2\pi\epsilon_0 V}{\tau}\right)}$$

Now, in this problem we are given b, τ, R, and we want to find d, V.

Simplifying notation: define $\mathcal{S} \equiv e^{2\pi\epsilon_0 V/\tau}$ + solve,

$$\Rightarrow \quad d = \frac{R^2-b^2}{R}, \quad \mathcal{S} = \frac{b}{R} = e^{2\pi\epsilon_0 V/\tau}$$

Hence

$$\boxed{V = -\frac{\tau}{2\pi\epsilon_0}\ln\frac{R}{b}}$$

and

$$\boxed{x_1 = R - d = \frac{b^2}{R}}$$

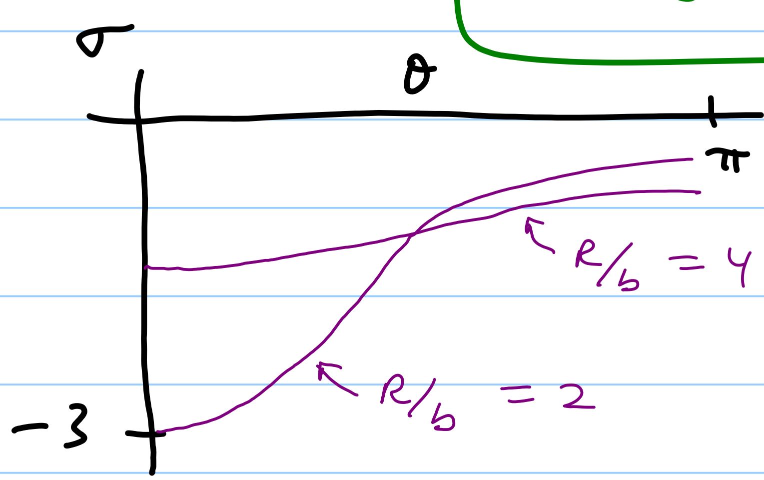
(b) Using $\ln y - \ln z = \ln\frac{y}{z}$ we have at once

$$\boxed{\Phi(r,\theta) = \frac{\tau}{4\pi\epsilon_0}\ln\left[\frac{r^2 + \left(\frac{b^2}{R}\right)^2 - \frac{2rb^2}{R}\cos\theta}{r^2 + R^2 - 2rR\cos\theta}\right]}$$

and at large distances r a series expansion gives the leading term to be

$$\boxed{\Phi(r,\theta) \xrightarrow[r\to\infty]{} \frac{(R^2 - b^2)\,\tau\cos\theta}{rR\;\;2\pi\epsilon_0}} + O(r^{-2})$$

(c) $\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial r}\Big|_{r=b^+} = \boxed{-\frac{\tau}{2\pi b}\frac{R^2 - b^2}{b^2 + R^2 - 2bR\cos\theta} = \sigma}$

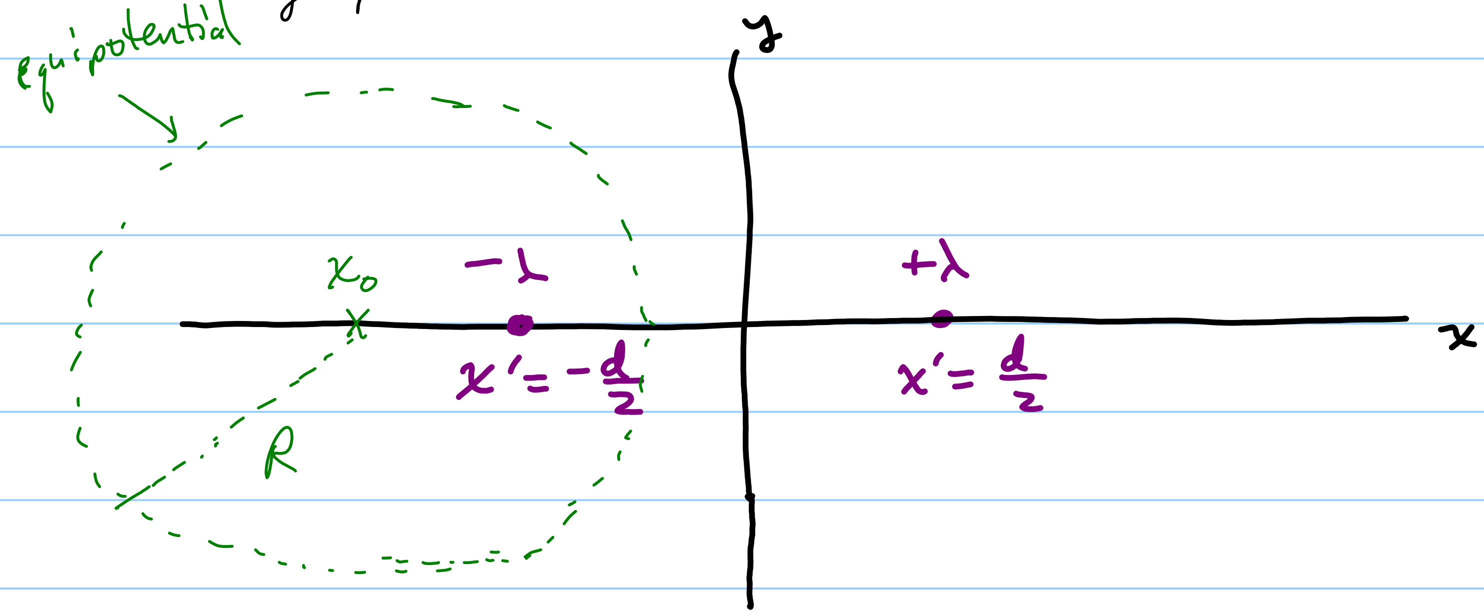


(d) F (per unit length on τ) $= \frac{-\tau^2}{2\pi\epsilon_0 d}\hat{x}$

or $\boxed{\frac{\vec{F}}{l} = \frac{-\tau^2}{2\pi\epsilon_0}\frac{R}{R^2 - b^2}\hat{x}}$

Appendix Jackson 2.8

(a) Let's study the equipotentials for two equal and opposite parallel line charges, $\pm\lambda$ aligned with the z-axis. In the xy-plane it looks like



From Gauss's Law, the potential due to a single line charge of density $+\lambda$ is

$$-\frac{\lambda}{2\pi\epsilon_0}\ln r$$

So Φ for the above diagram must be:

$$\Phi(x,y) = \frac{-\lambda}{2\pi\epsilon_0}\left(\ln\left[\left(x-\frac{d}{2}\right)^2+y^2\right]^{1/2} - \ln\left[\left(x+\frac{d}{2}\right)^2+y^2\right]^{1/2}\right)$$

$$= \frac{-\lambda}{4\pi\epsilon_0}\ln\frac{\left(x-\frac{d}{2}\right)^2+y^2}{\left(x+\frac{d}{2}\right)^2+y^2}$$

so along an equipotential, $\Phi(x,y) = V$, x and y obey $e^{-4\pi\epsilon_0 V/\lambda} = \dfrac{\left(x-\frac{d}{2}\right)^2+y^2}{\left(x+\frac{d}{2}\right)^2+y^2}$

call this $\Omega^2 =$

Now solve this for the equipotentials,

$$\Rightarrow \left(x-\frac{d}{2}\right)^2 - \delta^2\left(x+\frac{d}{2}\right)^2 + y^2 - \delta^2 y^2 = 0$$

$$\Rightarrow \quad x^2 - \frac{1+\delta^2}{1-\delta^2}\, x d + \left(\frac{d}{2}\right)^2 + y^2 = 0$$

This is the equation of a circle that is centered at $(x_0, 0)$

i.e. $(x-x_0)^2 + y^2 = R^2 = x^2 - 2x x_0 + x_0^2 + y^2$

and from this we deduce that

$$\boxed{x_0 = \frac{1+\delta^2}{1-\delta^2}\frac{d}{2}}$$

and some algebra gives $R = \dfrac{\delta d}{1-\delta^2}$ = circle radius

or plugging in the definition of δ,

$$\Rightarrow \boxed{R = \frac{d}{2\sinh\left|\frac{2\pi\epsilon_0 V}{\lambda}\right|}}$$

and

$$\boxed{x_0 = -\frac{d}{2}\coth\left(\frac{2\pi\epsilon_0 V}{\lambda}\right)}$$