Homework Set 01 — C. Greene's solutions

Problem 1.3 (a) A charge Q spread uniformly over a sphere has $\sigma = Q/4\pi R^2$ for sphere radius R

$$\Rightarrow \boxed{\rho(\vec{x}) = \frac{Q}{4\pi R^2}\,\delta(r-R)}, \quad \text{where } r = |\vec{x}|$$

Quick check: $\int \rho(\vec{x})\,d^3x = 4\pi \int \frac{Q}{4\pi R^2}\,\delta(r-R)\,r^2 dr$
$= Q$ ✓ checks!

(b) Cylindrical shell with charge λ per unit length, radius b (length not specified)

$\Rightarrow$ a length L has charge λL spread over area $2\pi b L$

$$\Rightarrow \sigma = \frac{\lambda L}{2\pi b L} = \frac{\lambda}{2\pi b} \text{ and so } \boxed{\rho(\vec{x}) = \frac{\lambda}{2\pi b}\,\delta(\rho - b)}$$

where ρ is the cylindrical radial coordinate

Check $\int_{\text{length } L} \rho(\vec{x})\,d^3x = L\,\frac{\lambda}{2\pi b}\,2\pi \int \rho\,d\rho\,\delta(\rho-b) = \lambda L$ ✓ checks

(c) For a charge Q spread uniformly over a thin, flat disc, radius R,

$$\boxed{\rho(\vec{r}) = \frac{Q}{\pi R^2}\,\delta(z)\,\Theta(R-\rho)}, \quad \text{where } \Theta(x) \equiv \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases}$$

(Heaviside step function)

(d) To convert (c) to spherical coordinates (r,θ,ϕ)

recall that $z = r\cos\theta$, so $\delta(z) = \frac{1}{r}\delta(\cos\theta) = \frac{1}{r}\delta(\theta - \frac{\pi}{2})$

$$\Rightarrow \boxed{\rho(\vec{r}) = \frac{Q}{\pi R^2 r}\,\delta(\theta - \tfrac{\pi}{2})\,\Theta(R-r)}$$

Problem 1.5 Given $\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)$, where $q = e$ (magnitude)

To find the charge, use

$\rho(\vec{r}) = -\epsilon_0 \nabla^2 \Phi$, and note that $\nabla^2(AB) = \nabla\cdot(A\nabla B + B\nabla A)$
$\nabla(AB) = A\nabla B + B\nabla A$

$$\Rightarrow \rho = -\frac{q}{4\pi}\nabla\cdot\left\{\frac{1}{r}\nabla\left[e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right] + e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\nabla\frac{1}{r}\right\}$$

$$= \frac{-q}{4\pi}\left\{\frac{1}{r}\nabla^2\left[e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right] + 2\nabla\left(\frac{1}{r}\right)\cdot\nabla\left[e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right] + e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\nabla^2\frac{1}{r}\right\}$$

Recall from the back cover of the textbook,

if there is no angular dependence, then $\nabla \to \hat{r}\frac{\partial}{\partial r}$

$\nabla^2 \to \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}$

And as shown in class, $\nabla^2 \frac{1}{r} = -4\pi\,\delta(\vec{r})$

So $\rho = \frac{q}{4\pi} 4\pi\,\delta(\vec{r}) - \frac{q}{4\pi}\frac{\alpha e^{-\alpha r}}{2r^2}\left(-2 - 2\alpha r + \alpha^2 r^2\right)$

$- \frac{2q}{4\pi}\left(\frac{-1}{r^2}\right)\left(-\frac{1}{2}\alpha e^{-\alpha r}\right)(1+\alpha r)$

which simplifies to

$$\boxed{\rho(\vec{r}) = q\,\delta(\vec{r}) - \frac{q}{2}\frac{1}{4\pi}\alpha^3 e^{-\alpha r}}$$

this represents a +e charge at the origin ($r=0$) which is the proton in the hydrogen atom

and the second term is the electron cloud spread out around the proton, i.e. $-q\,|\psi(\vec{r})|^2$ in terms of the quantum is wavefunction ψ of the hydrogenic electron.

Problem 1.9 (a) parallel plate case, plates of area A, separation x has capacitance $C = \frac{\epsilon_0 A}{x}$ (derive this yourself!)

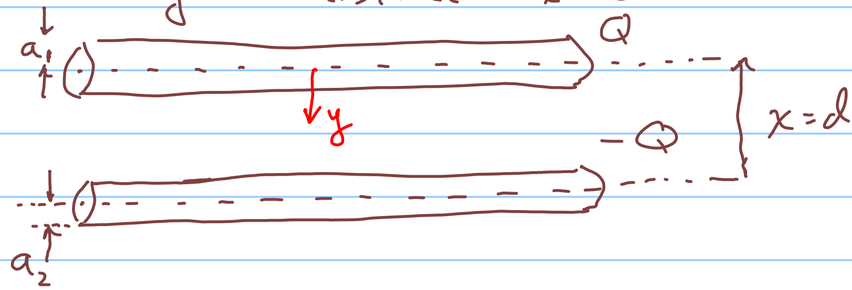
so the energy stored when the capacitor plates are charged up to $\pm Q$ is

$$U(x) = \frac{Q^2}{2C(x)} = \frac{Q^2}{2}\frac{x}{\epsilon_0 A}$$

and the force between them is

$$\boxed{F = -\frac{dU(x)}{dx} = -\frac{Q^2}{2\epsilon_0 A} = \text{attractive}}$$

Now consider 2 long, parallel cylinders of small radii a_1, a_2 separated by a distance $x = d$



From Gauss's Law applied to each cylinder separately (valid since $x \gg a_1, a_2$) the E-field at point y is

$$\vec{E}(y) = \hat{y}\left(\frac{Q}{2\pi\epsilon_0 y L} + \frac{Q}{2\pi\epsilon_0 (x-y) L}\right)$$

So the potential difference is

$$\Delta\Phi = -\int_{x-a_2}^{a_1} \frac{Q}{2\pi\epsilon_0 L}\left(\frac{1}{y} - \frac{1}{x-y}\right)dy$$

$$= \frac{Q}{2\pi\epsilon_0 L}\left[\ln\frac{x-a_2}{a_1} + \ln\frac{a_1-x}{x-a_2-x}\right]$$

or $$\Delta\Phi = \frac{Q}{2\pi\epsilon_0 L}\left\{\ln\left[x\left(1-\frac{a_2}{a_1 x}\right)\right] + \ln\left[x\left(1-\frac{a_1}{a_2 x}\right)\right]\right\}$$

(next use $\ln(1+\epsilon) \simeq \epsilon + \ldots$)

$$= \frac{Q}{2\pi\epsilon_0 L}\ln\left[\frac{(x-a_1)(x-a_2)}{a_1 a_2}\right] \simeq \frac{Q}{2\pi\epsilon_0 L}\ln\frac{x^2}{a_1 a_2} \simeq \frac{Q}{\pi\epsilon_0 L}\ln\frac{x}{a}$$

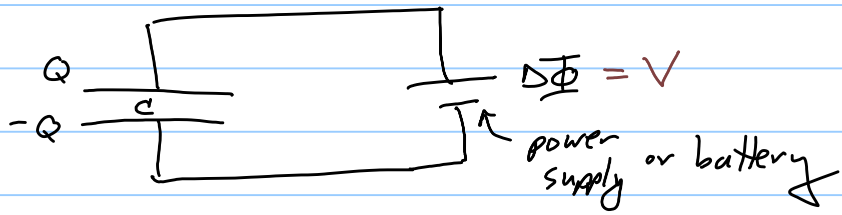
where $a = \sqrt{a_1 a_2}$

and so the capacitance in this limit is $C = \frac{Q}{\Delta\Phi} = \frac{\pi\epsilon_0 L}{\ln\frac{x}{a}}$ ← (Note to grader: students are not expected to derive this, since it was given in problem 1.7)

Now the energy as a function of x, for Q fixed, is
$$U(x) = \frac{1}{2}\frac{Q^2}{C(x)}$$ and the force is
$$F = -\frac{dU}{dx} = -\frac{1}{2} Q^2 \frac{d}{dx} \frac{\ln\frac{x}{a}}{\pi\epsilon_0 L}\Bigg|_{x=d} = \boxed{-\frac{1}{2} Q^2 \frac{1}{\pi\epsilon_0 L\, d} = F}$$
again, attractive!

(b) Now repeat the force calculation, but instead of fixed Q we assume there is a power supply holding the conductors at fixed $\Delta\Phi$ voltage difference

e.g.



For the case where V = fixed, the energy conservation argument requires us to consider work performed by the battery. i.e. the change in the internal energy in the capacitor must equal WORK done by <u>us</u> in moving the plates AND work done by the <u>battery</u>. Here is the math: $dU_{int} = F\,dx + dW_{battery}$

Now, as we change x to $x+dx$, $dU_{int} = d\left(\frac{1}{2}CV^2\right) = \frac{1}{2}V^2\frac{dC}{dx}dx$

and $dW_{battery} = V\,dQ = V\frac{d(CV)}{dx}dx = V^2\frac{dC}{dx}dx$

So for both capacitor geometries we have:
$$\frac{1}{2}V^2\frac{dC}{dx}dx = F_{us}\,dx + V^2\frac{dC}{dx}dx$$

or $$F_{us} = -\frac{1}{2}V^2\frac{dC}{dx}$$ ← this is the force <u>we</u> must supply to move the plates from x to $x+dx$ (slowly with no acceleration), and so the actual electric force on the plates is $F_{plates} = -F_{us} = \frac{V^2}{2}\frac{dC}{dx}$

<u>parallel plates</u>

or $$F_{plates} = \frac{V^2}{2}\left(-\frac{\epsilon_0 A}{x^2}\right)$$

or <u>parallel cylinders</u> $$F_{cyl} = \frac{V^2}{2}\left(\pi\epsilon_0 L\,\frac{-1}{(\ln\frac{x}{a})^2}\,\frac{1}{x}\right)$$

So in fact the answer is the <u>same</u> as in part (a), if you replace $V \to \frac{Q}{C(x)}$!