Exam 1, Fall 2025, Physics 630 – C. Greene's solutions

Problem 1. Short problems:

(a) **Give a brief derivation** of the integral form of Gauss's Law, starting from one of Maxwell's equations.

(b) Use Gauss's Law to **derive the electric potential AND the electric field** at a distance $\rho > a$ from the center of an infinitely long cylinder of radius a carrying constant surface charge σ in units of Coulombs/m^2.

(c) An individual is holding in place a charge of Q=7 Coulombs at the coordinate system origin (0,0,0) in an electric field directed along the z-axis, equal to $\vec{E} = E_0\hat{z}$, where $E_0 = 3$ V/m. She now moves the charge very slowly to the following location in Cartesian coordinates: $P = (x, y, z) = (1, 3, 5)$. **How much work did she perform?**

#1(a) Solution

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \qquad (1)$$

Consider a closed surface S that bounds a volume V, and integrate (1) over V:

$$\int_V d^3x\, \nabla\cdot\vec{E} = \frac{\int d^3x\, \rho_{\text{charge dens.}}}{\epsilon_0}$$

and by the divergence theorem

$$\Rightarrow \boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{\text{enclosed in }V}}{\epsilon_0}}$$ derived!

(b) Solution $\vec{E}$ must be directed normal to surface by symmetry, i.e. along $\hat{\rho}$

$\Rightarrow$ Choose a cylindrical Gaussian surface of length L, radius $\rho > a$

Then $\oint_S \vec{E}\cdot d\vec{a} = 2\pi\rho L\, E(\rho) = \frac{Q_{encl}}{\epsilon_0} = \frac{2\pi a L\sigma}{\epsilon_0}$ (since $\rho > a$)

$$\Rightarrow \boxed{\vec{E}(\rho) = \hat{\rho}\,\frac{a\sigma}{\rho\,\epsilon_0}} \qquad \Phi(\rho) = -\int_{R_{large}}^{\rho} \vec{E}\cdot d\vec{\rho} = -\frac{a\sigma}{\epsilon_0}\left(\ln\rho - \underbrace{\ln R_{large}}\right)$$

usually neglect this arbitrary constant

$$\Rightarrow \boxed{\Phi(\rho) = -\frac{a\sigma}{\epsilon_0}\ln\rho}$$

1(c) solution Her work is

$$W = \int_{(0,0,0)}^{(1,3,5)} \vec{F}_{hers}\cdot d\vec{l} = -QE_0\int_0^L dz = \boxed{-QE_0L = W}$$

$-105\,\text{J} =$ her work is NEGATIVE!

Problem 2. **Write down or derive** the Dirichlet Green's function for the interior of a cube of side L that is centered at the origin.

#2 Solution $G = \Sigma\, X(x)Y(y)Z(z)$

$X_{n_1}(x) = \sqrt{\frac{2}{L}}\sin\frac{n_1\pi}{L}\left(x+\frac{L}{2}\right), \; -\frac{L}{2} \le x \le \frac{L}{2}$

$Y_{n_2}(y) = \sqrt{\frac{2}{L}}\sin\frac{n_2\pi}{L}\left(y+\frac{L}{2}\right), \; -\frac{L}{2} \le y \le \frac{L}{2}$

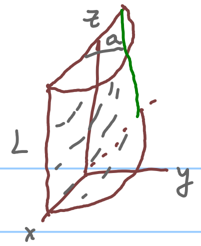
$Z_{n_3}(z) = \sqrt{\frac{2}{L}}\sin\frac{n_3\pi}{L}\left(z+\frac{L}{2}\right), \; -\frac{L}{2} \le z \le \frac{L}{2}$

eigenfunctions in all 3 coordinates and we know the completeness relation

n_1, n_2, n_3 are all positive integers and $-\nabla^2 XYZ = (n_1^2+n_2^2+n_3^2)\frac{\pi^2}{L^2}$? $-\nabla^2 G = 4\pi\delta(\vec{x}-\vec{x}')$

$$\Rightarrow \text{expect } G(\vec{x},\vec{x}') = \sum_{n_x n_y n_z} \frac{X_{n_1}(x)X_{n_1}(x')\,Y_{n_2}(y)Y_{n_2}(y')\,Z_{n_3}(z)Z_{n_3}(z')\,4\pi}{(n_1^2+n_2^2+n_3^2)\,\pi^2/L^2}$$

#3(a) solution



$\nabla^2 G = -4\pi\delta(\vec{x}-\vec{x}\,')$ $\quad 0 \le \phi \le \pi$

Problem 3. A hollow right-circular cylinder has its axis aligned with the z-axis, and it is cut in half along the x-z plane and half of it is discarded, In cylindrical coordinates, the surfaces of the remaining half-cylinder are the round portion at $0 \le z \le L$, and $0 \le \phi \le \pi$, the flat bottom and top at $0 \le \rho < a$ and $z = 0$ or $z = L$as well as the flat planar part where the cut was made at $y = 0$, $-a \le x \le a$ and $0 \le z \le L$.

(a) **Sketch** this remaining half cylinder, **write** the homogeneous partial differential equation and the separated ordinary differential equations in cylindrical coordinates, and **solve** for the Dirichlet Green's function inside it. Specifically, find an expression for the Dirichlet Green's function in the interior of this cylinder.

(b) Now consider the following problem: All surfaces of the cylinder except the flat top cap at $z = L$ are held at zero potential; that top surface has a known, specified electric potential $V_0(\rho,\phi)$. A single point charge Q is placed inside at the position $\rho = a/2$, $\phi = \pi/2$, $z = L/2$. **Find** an expression for the electric potential everywhere inside this half-cylinder.

The homogeneous equation, valid when $\vec{x} \ne \vec{x}\,'$ only, is $\nabla^2 G = 0 = \left(\frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G = 0$

Separable homogeneous solutions look like

$\psi(\rho,\phi,z) = R(\rho)\,P(\phi)\,Z(z)$, and they obey the following:

$$P''(\phi) + \nu^2 P(\phi) = 0 \quad , \quad Z''(z) - k^2 Z(z) = 0$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \left(\frac{\nu^2}{\rho^2} - k^2\right)R = 0.$$ This choice of signs gives oscillatory solutions in ρ,ϕ and exponential type in z.

Since $P(0) = 0 = P(\pi)$ is one B.C., we see at once that $P(\phi) = \sqrt{\frac{2}{\pi}}\sin\nu\phi$, $\nu = 1,2,3,\ldots$

and the radial solution regular at $\rho = 0$ is $J_\nu(k\rho)$

and we demand that $J_\nu(ka) = 0$, whereby $k = \left\{\frac{x_{\nu 1}}{a}, \frac{x_{\nu 2}}{a}, \ldots\right\}$

and $x_{\nu i} = i^{th}$ root of $J_\nu(x) = 0$.

Lastly we need 2 solutions for $Z(z)$ vanishing at the boundaries, namely $u_1(z) = \sinh kz$ and $\sinh k(z-L) = u_2(z)$, i.e. $u_1(0) = 0 = u_2(L)$

and $k = \frac{x_{\nu 1}}{a}, \frac{x_{\nu 2}}{a}, \ldots$ etc. from the Bessel function roots

Using Method 2, then the GF is immediately: (Note $N_{\nu i}^{-2} \equiv \int_0^a J_\nu^2\left(\frac{x_{\nu i}\rho}{a}\right)\rho\, d\rho$)

$$G(\vec{x},\vec{x}\,') = \frac{2}{\pi}\sum_{\nu=1}^{\infty}\sum_{i=1}^{\infty} \sin\nu\phi\,\sin\nu\phi'\, N_{\nu i}^2 J_\nu\left(\frac{x_{\nu i}\rho}{a}\right) J_\nu\left(\frac{x_{\nu i}\rho'}{a}\right) g_{\nu i}(z,z')$$

$$\text{and } g_{\nu i}(z,z') = \frac{-\sinh\left(\frac{x_{\nu i}}{a}z_<\right)\sinh\left[\frac{x_{\nu i}(z_> - L)}{a}\right] 4\pi}{W(u_1,u_2)}$$

skip calculation of this constant for now but it's straightforward: $u_1 u_2' - u_1' u_2 = W$

and to answer part (b) we will need $\left.\frac{\partial g_{\nu i}(z,z')}{\partial z'}\right|_{z'=L} = \frac{-4\pi \frac{x_{\nu i}}{a}\sinh\left(\frac{x_{\nu i}}{a}z\right)}{W}$

and then plug into:

$$\Phi(\rho,\phi,z) = \int_V q\frac{\delta(\vec{x}\,'-\vec{x}_0)}{4\pi\epsilon_0}G(\vec{x},\vec{x}\,') - \frac{1}{4\pi}\int_0^a \rho' d\rho' \int_0^\pi d\phi'\, V_0(\rho',\phi')\left.\frac{\partial G}{\partial z'}\right|_{z'=L}$$

$$= \frac{q}{4\pi\epsilon_0}G(\vec{x},\vec{x}_0) - \frac{1}{4\pi}\iint \ldots \text{ etc.}$$