Quantum Error-correction in Quantum Field Theory and Gravity

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What is the beginning of the universe? In particular, how does classical spacetime emerge from the Big Bang singularity in the early universe, and how does the spacetime end near a black hole singularity? Both the early age of the big bang and the region near the singularity of a black hole have their energy density very large in a small region[1]. In other words, the energy scale of the big bang and black hole physics is extremely high[2]. On such a small scale, one cannot ignore the effect of quantum fluctuations. Hence, understanding the mechanism of the "emergence of spacetime" requires one to know how gravity behaves at high energies where quantum effects become important. However, a direct experimental probe of physics at such extreme scales remains beyond reach because the experimental energy scale of quantum gravity is 10^{16} times higher than the highest energy scale of the currently accessible high-energy experiments.

In 1997, Juan Maldacena conjectured the holographic duality[3] as a rigorous approach to quantum gravity [3]. It claims that a quantum gravitational system is exactly equal to a quantum theory without gravity in lower spacetime dimensions living on the boundary of the quantum gravitational system. For example, the theory of quantum gravity in asymptotically $d+1$ dimensional anti-de Sitter space[4] is equivalent to a conformal field theory in d-dimension without gravity, which is called AdS/CFT duality, see figure 2a. Although holographic duality paved the new way toward quantum gravity, we still do not have a complete understanding of why, how, and when the duality holds. Recognizing the utility of quantum information theory in the studies of quantum gravity and holography accelerated the progress.

Classical information theory was established in 1948 by Claude E. Shannon. He quantified the amount of information in the unit of "bit", and mathematically formulated the information processing, such as communication, in the presence of noise [4]. Quantum mechanics was already constructed at this point in different ways, for instance, wave mechanics by Erwin Schrödinger, matrix mechanics by Werner Heisenberg, Max Born, and Pascual Jordan, and the mathematically rigorous formulation in the form of *von Neumann algebras* by John von Neumann in the 1920s. However, Shannon's information theory treated only classical information. From the 1970s, quantum formulation of information theory using the unit of "qubit" was motivated by the question; how much classical information can one obtain from a quantum system? Or, how many bits per qubit can one extract from a quantum system[5]? In the 1990s, seminal works of quantum information theory were published, for instance, the proposal of a quantum teleportation protocol by Richard Jozsa, William K. Wootters, Charles H. Bennett, Gilles Brassard, Claude Crépeau and Asher Peres in 1993 [6]. The main difference between classical and quantum information is the presence of entanglement, which is a non-local correlation that only exists in a quantum system. Its physics was theoretically and experimentally confirmed resulting in three physicists, Alain Aspect, John F. Clauser, and Anton Zeilinger, winning the Nobel Prize in 2022. Nowadays, the progress of theoretical and experimental aspects of quantum information is remarkable.

Quantum error-correction (QEC) is one of the essential frameworks in quantum information. For example, in quantum communication, signals running through a fiber could get disturbed by the interactions between the fiber and the environment. This results in the deterioration of the quality of communications. In quantum computations, avoiding or correcting errors are essential for fast and high-precision simulations. Hence, the theories and methods of quantum error-correction have been largely explored, even in the language

[1] "Small region" means that it is in order of Planck length scales, i.e., 10^{-35} meters. See figure 1 to see how small it is, compared to the other scales.

[2] "Extremely high" means energy is in order of super-Planckian energy scales, i.e., 10^9 Joules. Roughly speaking, it means that both the early period of the big bang and the region near the black hole are hot.

[3] The term "holography" was initially coined by Leonard Susskind [1] in 1995 based on the major advance work by Gerard t'Hooft [2]. It comes from the "hologram" in optics, which encodes a three-dimensional image into two-dimensional data.

[4] AdS(Anti-de Sitter) space is the maximally symmetric solution of Einstein equation with a negative cosmological constant.

[5] Nowadays, the answer to the question is formulated as Holevo's bound [5].

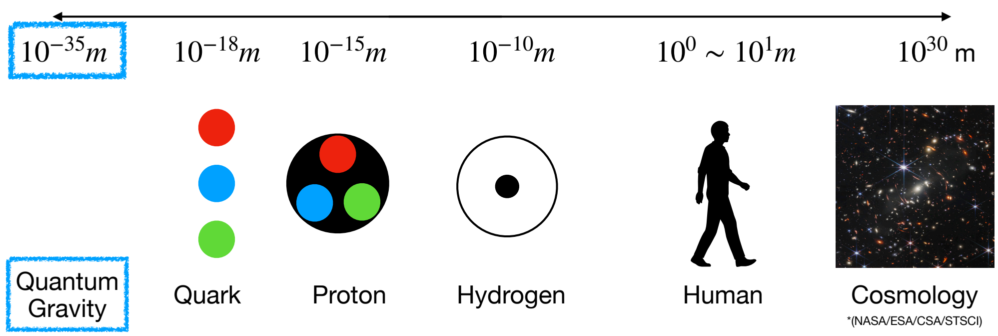


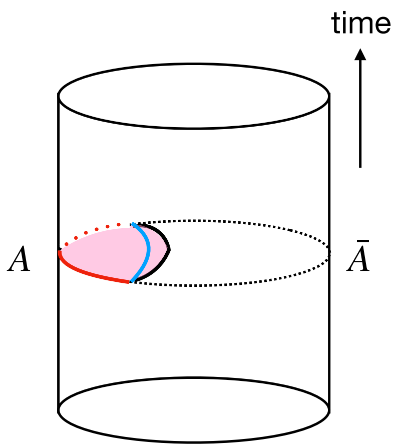
Figure 1: Length scale in the nature.

of operator algebras [7–9] including von Neumann algebras. The key is that the local errors are corrected using a non-local quantum correlation, which is entanglement. The building blocks of QEC are i) encoding, ii) error, iii) recovery, and iv) decoding. Occasionally, decoding and recovery are processed simultaneously. An encoding maps logical degrees of freedom that one wants to simulate into larger physical degrees of freedom. The key in the process is to introduce redundancies. For example, if one wants to simulate a single qubit system, and the error is a single bit-flip, i.e. $0 \to 1$, one can encode it into a three-qubit system $0 \to 000$, and $1 \to 111$. After the single bit-flip error occurs, i.e. $000 \to 001$, one can recover 0 by the majority vote among the qubits. This is the simplest classical code and did not need any use of entanglement. Various QEC codes, such as Shor code [10], and toric codes [11], use entanglement as a resource to efficiently exercise the QEC procedures.

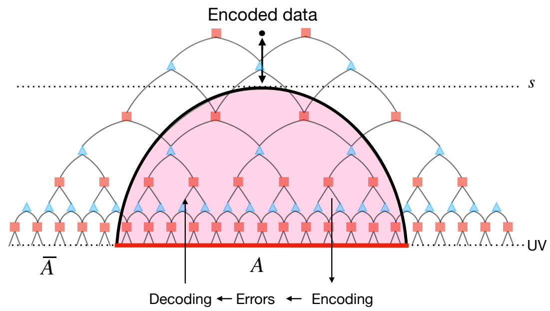
The quantum information theory and the framework of QEC play a role in the studies of quantum gravity and holography, for instance, the problem of bulk reconstruction. It asks how the information content of the bulk theory is encoded onto that of the boundary theory, figure 2a. That is, how gravity emerges from the boundary theory. In 2006, there was a proposal to reconstruct the information content of boundary region from so-called causal wedge region [12]. In the same year, Shinsei Ryu and Tadashi Takayanagi found that the information in the boundary region is reconstructed with the larger bulk region, or the entanglement wedge, than the causal wedge region. In addition, they provided the procedure to specify the bulk region recovered from a given boundary region, figure 2a. In 2015 and 2017, Ahmed Almheiri, Xi dong, and Daniel Harlow found that the bulk reconstruction can be understood in the framework of quantum error-correction [13, 14]. The application of QEC to the holographic principle is one of the main approaches.

Nima Lashakri's group and I study the structures of quantum field theory(QFT) and quantum gravity from the perspective of quantum information, especially, quantum error-correction. There are two major accomplishments in our studies; i) Providing the proper language to study QEC in QFT and gravity [15], ii) real-space renormalization group (RG) can be understood as an approximate error-correction code [16]. Although the framework of QEC in holography was already provided, two aspects should have been improved. First, the previous formulation was based on a finite-dimensional system. However, quantum field theory and quantum gravity are described by an infinite-dimensional system which, in fact, does not allow one to have some useful mathematical structures one has in a finite-dimensional system. Hence, we provided the proper tools and language to study them based on the theory of von Neumann algebras. The construction of the mathematical theory is initiated by John von Neumann to rigorously formulate quantum mechanics. Later on, the theory was developed and applied to study quantum field theory by various mathematicians. Second, the QEC code previously proposed to understand holography is exact. A QEC code is exact if one can correct an error exactly. However, there are signatures that the QEC code inherited in holography should be approximate. Then, the main problem is the formulation of approximate QEC in holography. We made important progress by claiming that the real-space renormalization group(RG) theory is an approximate error-correction. This is imperative because it suggests to us how the QEC code naturally arises in holography.

In general, the nature scales from 10^{-35} meter length scale of quantum gravity to 10^{30} meter length scale of cosmology, figure 1. Hence, we study physics scale by scale. RG theory relates different scales of physics and explores the relation between the physics of the microscopic scale and that of the macroscopic



(a)



(b)

Figure 2: (a) AdS spacetime can be drawn as a cylinder. The time of AdS spacetime flows from bottom to top as indicated by the arrow. A quantum theory without gravity, or conformal field theory (CFT), lives on the surface of the cylinder. A quantum gravitational system lives inside the cylinder. The red line is part of the boundary region. A causal wedge is a region between the red and blue lines. The entanglement wedge is the region highlighted by pink. The information on the red line is recovered from the pink region in the bulk. (b) The figure represents the discrete model of the slice at a certain time in the left figure. The model is known as multi-scale entanglement renormalization ansatz (MERA). Brian Swingle [17] proposed that this model can simulate holography. It consists of two linear maps denoted as red squares(unitary) and blue triangles(isometry). The bottom corresponds to the boundary theory, or equivalently to the UV theory. The support of any operator, for instance, region A shrinks as the RG flow goes deeper into the IR theory. After the flow passes a certain IR scale s, it reduces to a point. We explicitly simulate this using the continuous version of the above model, known as continuous MERA [16]. The same happens to a UV error acting on A. If the size of the support of the error is small enough, it shrinks to a point much faster than the operators supported on A. This led us to show that real-space RG theory is an approximate error-correction.

scale. A theory on a microscopic scale and that on a macroscopic scale are called ultraviolet(UV) theory and infrared(IR) theory, respectively. The IR theory can be encoded into the UV theory. Then, for some types of errors acting on the UV theory, we found that they gave exponentially small disturbance to the encoded information when it flows back into the IR theory, fig.2b. During the process, a recovery procedure is not required.

In conclusion, we provided the proper language to study QEC in QFT and gravity [15] using the theory of von Neumann algebras and theoretically realized real-space renormalization group (RG) as an approximate error-correction code [16]. In the next phase, we will apply our language and tools to various models in quantum field theory with and without gravity to reveal the mechanism of the emergence of spacetime.

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