Two-dimensional Collision of Hard Balls

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We begin with two balls colliding, one moving, one at rest.

$$\vec{v}_1 = v_1\hat{x} \tag{1}$$
$$\vec{v}_2 = 0 \tag{2}$$

Immediately move to center of mass coordinates where

$$x_{\rm cm} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} \tag{3}$$

In these coordinates, the total linear momentum is zero.

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0 \tag{4}$$
$$m_1\vec{v}_1' + m_2\vec{v}_2' = 0 \tag{5}$$

The primes denote the velocities after the collision. The effect of this momentum conservation on the kinetic energy is that the particle velocities are related by

$$\vec{v}_2 = -\frac{m_1}{m_2}\vec{v}_1, \qquad \text{and} \qquad \vec{v}_2' = -\frac{m_1}{m_2}\vec{v}_1'. \tag{6}$$

Also, the speeds before and after the collision are unchanged

$$v_1' = v_1, \qquad \text{and} \qquad v_2' = v_2. \tag{7}$$

Because $\vec{v}_1$ and $\vec{v}_2$ are so well connected, we can complete the calculation by studying $\vec{v}_1$ alone.

Our main physical consideration is the moment of impact of the balls. If we define an impact parameter, B, as the $\hat{y}$ distance between the centers of the two balls, we can define a single angle to describe the collision,

$$\sin\theta = \frac{B}{2r} \tag{8}$$

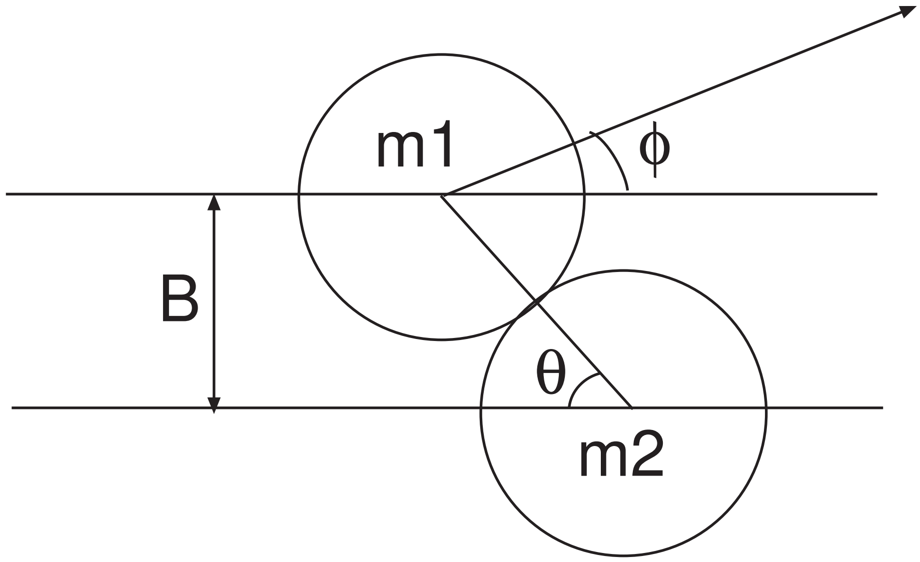
where r is the ball radius. For balls of differing radii, just use $B/(r_1 + r_2)$. During a head-on collison, $\theta = 0$. A glancing blow of m_1 above m_2 is at $\theta = \pi/2$ and below is at $\theta = -\pi/2$.

At the moment of impact, the impulse of the impact is along the point of contact of the two balls. For the first ball, we can write the impulse as

$$m_1(\vec{v}_1' - \vec{v}_1) = I(-\cos\theta\hat{x} + \sin\theta\hat{y}). \tag{9}$$

The $\hat{x}$ and $\hat{y}$ components simplify to

$$m_1 v_{1x}' - m_1 v_1 = -I\cos\theta \tag{10}$$
$$m_1 v_{1y}' = I\sin\theta. \tag{11}$$



Making these a fraction gives us the needed angle,

$$\tan\theta = \frac{m_1 v'_{1y}}{m_1 v_1 - m_1 v'_{1x}} = \frac{\sin\phi}{1-\cos\phi}. \tag{12}$$

In the last step, we replaced $v_{1x} = v_1\cos\phi$ and $v_{1y} = v_1\sin\phi$. The angle at which m_1 leaves the collision is now ϕ measured from the $\hat{x}$ axis. We now solve this equation.

$$\tan\theta - \cos\phi\tan\theta = \sin\phi \tag{13}$$

Squaring both sides gives

$$\tan^2\theta(\cos^2\phi - 2\cos\phi + 1) = \sin^2\phi = 1 - \cos^2\phi. \tag{14}$$

$$(\tan^2\theta + 1)\cos^2\phi - 2\tan^2\theta\cos\phi + \tan^2\theta - 1 = 0 \tag{15}$$

Using the quadratic formula, we find

$$\cos\phi = \frac{2\tan^2\theta \pm \sqrt{4\tan^4\theta - 4(\tan^2\theta+1)(\tan^2\theta-1)}}{2(\tan^2\theta+1)}. \tag{16}$$

Simplifying shows

$$\cos\phi = \frac{\tan^2\theta \pm 1}{\tan^2\theta + 1}. \tag{17}$$

Clearly we take the negative root to find

$$\cos\phi = 2\sin^2\theta - 1 = 1 - 2\cos^2\theta. \tag{18}$$