Experimental Uncertainties (Errors)

Sources of Experimental Uncertainties (Experimental Errors):

All measurements are subject to some uncertainty as a wide range of errors and inaccuracies can and do happen. Measurements should be made with great care and with careful thought about what you are doing to reduce the possibility of error as much as possible.

There are three **main sources of experimental uncertainties** (experimental errors):

1. Limited accuracy of the measuring apparatus - e.g., the force sensors that we use in experiment M2 cannot determine applied force with a better accuracy than ±0.05 N.
2. Limitations and simplifications of the experimental procedure - e.g., we commonly assume that there is no air friction if objects are not moving fast. Strictly speaking, that friction is small but not equal to zero.
3. Uncontrolled changes to the environment. For example: small changes of the temperature and the humidity in the lab.

In this laboratory, we keep to a very simple form of error analysis, our purpose being more to raise your awareness of errors than to give you expertise in sophisticated methods for handling error analysis. If you ever need more information on error analysis, please check the literature[1-4].

Do not list your mistakes as experimental errors. These mistakes should have already been detected and eliminated during the preparation of the lab report. A calculator should not be listed as a source of experimental error. You may always use more significant figures (c.f. the next section) during calculations to reduce round-off error. In other words, you may always make all calculations with a better accuracy then you can do lab measurements.

Absolute and Relative Errors:

In order to learn the meaning of certain terms, consider an experiment in which we use the ultrasonic motion sensor to measure the position x of an object. The accuracy of the motion sensor has been specified by the manufacturer as ±1 mm. The **absolute error** of our

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1. Bevington, P. R., Data Reduction and Error Analysis for the Physical Sciences, New York: McGraw-Hill, 1969.
2. Taylor, J. R., An introduction to uncertainty analysis: the study of uncertainties in physical measurements, Mill Valley: University Science Books, 1982.
3. Young, H. D., Statistical Treatment of experimental data, New York: McGraw-Hill, 1962.
4. Barford N. C., Experimental Measurements: Precision, Error and Truth, Addison-Wesley, 1967.

measurements is thus ±1 mm. Note that the absolute error carries the same unit as the measured quantity.

Suppose the value measured in our first trial was equal to 50.6 cm. Then the **relative error** is equal to: (1 mm)/(50.6 cm) = (0.1 cm)/(50.6 cm) ≈ 0.002. The relative error is dimensionless. It is often expressed in percentage, as: 100%* (0.1 cm)/(50.6 cm) ≈ 0.2%.

We may express the above result using either absolute error or relative error, as follows:

$$x = 50.6 \pm 0.1 \text{ (cm)} \quad \text{(usually preferred)} \quad \text{or} \quad x = 50.6 \text{ cm} \pm 0.2\%$$

If we know the accepted value of the measured quantity (e.g., the temperature of melting point for water is equal to 0˚C or 273K), then we can calculate the percentage error as:

$$\text{Relative error} = \left| \frac{\text{Experimental value - Accepted value}}{\text{Accepted value}} \right| \times 100\%$$

When comparing experimental data with the theoretical values, then the relative error is given by a similar formula:

$$\text{Relative error} = \left| \frac{\text{Experimental value - Theoretical value}}{\text{Theoretical value}} \right| \times 100\%$$

If we do not know the accepted value of the measured quantity, but the measurements have been repeated several times for the same conditions, one can use the spread of the results themselves to estimate the experimental error.

Average Values and the Standard Deviation:

Consider the following six results of velocity measurements: 0.38, 0.38, 0.35, 0.44, 0.43, 0.42 m/s. The *average* value of these six velocity measurements is equal to: v = (0.38 + 0.38 + 0.35 + 0.44 + 0.43 + 0.42) / 6 = 0.40 m/s. Next, one needs to calculate the *deviations* from the average velocity: 0.38 - 0.40 = 0.02 m/s; 0.38 - 0.40 = 0.02 m/s; 0.40 - 0.35 = 0.05 m/s; 0.40 - 0.44 = -0.04 m/s; 0.40 - 0.43 = -0.03 m/s; 0.40 - 0.42 = -0.02 m/s.

The general formula for calculation of the **average value** x_{AV} (sometimes also called *mean value*) is as follows:

$$x_{AV} = \frac{1}{n}\left(x_1 + x_2 + x_3 + ... + x_n\right)$$

where n is the number of repeated measurements (for this example $n = 6$).

The values of the deviation from the average value are used to calculate the **experimental error**. The quantity that is used to estimate these deviations is known as the **sample standard deviation s_x** and is defined as:

$$s_x = \sqrt{\frac{1}{n-1}\left[\left(x_1 - x_{AV}\right)^2 + \left(x_2 - x_{AV}\right)^2 + ... + \left(x_n - x_{AV}\right)^2\right]}$$

The standard deviation squared - s_x^2 is the sum of squares of deviations from the average value divided by (n - 1). The number of degrees of freedom (n - 1) is the total number of measurements n minus one. The subscript usually indicates the quantity that the standard deviation is calculated for, e.g., s_v stands for the standard deviation of velocity measurements, whereas s_a is the standard deviation for acceleration data.

For the previously discussed example of velocity measurements we have $n = 6$

$$s_v = \sqrt{\frac{1}{6-1}\left[(0.02\ \text{m/s})^2 + (0.02\ \text{m/s})^2 + (0.05\ \text{m/s})^2 + (0.04\ \text{m/s})^2 + (0.01\ \text{m/s})^2 + (0.02\ \text{m/s})^2\right]} =$$

$$= \sqrt{\frac{1}{5} \cdot 0.0062(\text{m}^2/\text{s}^2)} = 0.0352\ \text{m/s} \cong 0.04\ \text{m/s}$$

We use the standard deviation as the best estimate of the experimental error. The result of measurements and error analysis should be written as:

$$\boldsymbol{v = v_{AV} \pm s_v = 0.40 \pm 0.04\ (m/s)}$$ (do not forget to write the appropriate units!)

The general format for presenting experimental results with experimental error is given by one of the following expressions:

"final result" = "average value" ± "standard deviation" (units)

$$\boldsymbol{x = x_{AV} \pm s_x}\ \textbf{units} \quad \textbf{or} \quad \boldsymbol{x = x_{AV} \pm s_x}\ \textbf{(units)} \quad \textbf{or} \quad \boldsymbol{x = (x_{AV} \pm s_x)}\ \textbf{units}$$

Obviously, the average value and the standard deviation must have the same units.

Random and Systematic Errors:

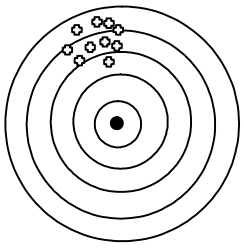
There is one important point we should make about using of the standard deviation formula as the estimate of experimental uncertainty (experimental error) in x. We have assumed implicitly that there is no "bias" in our measurements. In other words, the probability that a value of v_i will be larger than the perfect value is the same as the probability that it will be smaller than the perfect value. This is why the average (1) will provide a good estimate of the measured value. Errors of this kind are called ***random* errors** since their sign (positive or negative) is random. The random errors happen due to small factors that are beyond our

control (small amplitude building vibrations due to the car traffic outside, changes to the power line voltage, etc.), or the random nature of the experiment (e.g., a radioactive decay).

There is also a second kind of error known as ***systematic errors***. Systematic errors are instrumental or methodological errors causing consistent deviation of results in one direction from the true value. One very typical example of measurement with large systematic error is a scale that was not properly zeroed. That scale would show values that are systematically smaller (or systematically larger) value than the true weight of objects.

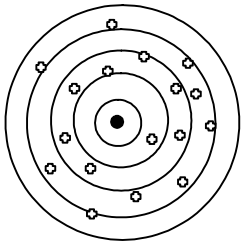
Another example is if the temperature in the room (and consequently the speed of sound) would be significantly lower than the usual 20˚C (= 68˚F). Computer needs the exact value of sound speed to properly calculate distance: $x = \Delta t * v_{sound}$. It would then provide values of x_i, which always tend to be larger than $x_{perfect}$, and the average of these x_i values could differ from $x_{perfect}$ by much more than s_x. The only way to cope with systematic errors is to carefully analyze the way your experiment works, to understand how it might be biased and by adjusting equipment. The graphs below illustrate both random and systematic errors.

Very close agreement between trials, but systematically off the target.

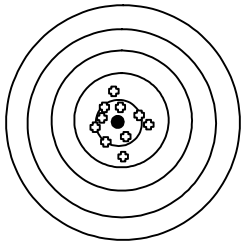


Small random error, but large systematic error.

Widespread, random variation of results.



Large random error and significant systematic error.

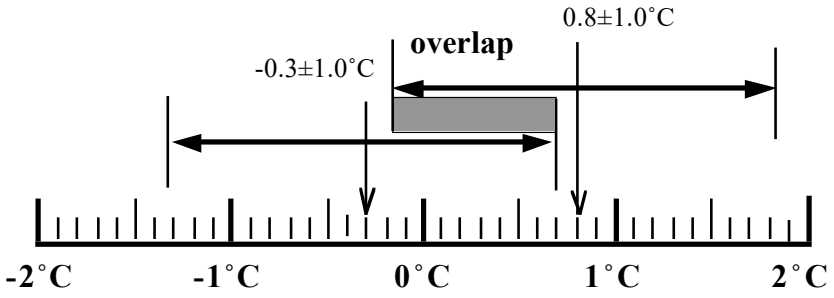


The best situation, i.e., small random error and small systematic error.

It is possible to reduce the systematic error by properly designing the experiment and adjusting experimental apparatus (for example, zeroing scales). The random error can be reduced by repeating measurements or by getting better equipment. In case of multiple measurements, we use the **average** value to reduce the random error. Standard deviation provides a good approximation of the random error. However, no matter how good is our apparatus and no matter how many times we repeat measurements **the random error cannot be completely eliminated**. A similar statement is also true for systematic error. Even the most careful calibration would not completely eliminate the systematic error. The total experimental error depends on the larger od the two types of errors - random or systematic.

Condition for Two Results to Agree:

Error estimates are necessary to be able to say whether the two independent measurements of the same thing agree *within the stated errors* or *disagree*. For example, two students measured the melting point temperature of ice and obtain results of -0.3˚C and 0.8˚C. Without an estimate of error, we cannot say whether these measurements agree. Suppose the results had been stated with errors: T_{melt} = -0.3 ± 1.0˚C and T_{melt} = 0.8 ± 1.0˚C. Since the first result admits values between -1.3˚C and 0.7˚C and the second between -0.2˚C and 1.8˚C, there is an overlap (between -0.2˚C and 0.7˚C) and **the results agree within experimental errors**.



If one cannot find an overlap between the error bands, then the results do not agree with each other.

Experimental Uncertainty (Experimental Error) for a Product of Two Measurements:

Sometimes it is necessary to combine two (or even more than two) measurements to get a needed result. A good example is a determination of work done by pulling a cart on an incline that requires measuring the force and the distance independently. Then the value of work can be calculated from a simple formula: $W = F \times s = F \times \Delta x$ (more precisely: $W = F_{AV} \times \Delta x_{AV}$).

Then, the absolute error for work W is given by the following formula:

$$s_W = F_{AV} \times s_x + \Delta x_{AV} \times s_F \quad ,$$

where, s_F and s_x are standard deviations of force and distance measurements. F_{AV} and Δx_{AV} represent average values of force and traveled distance, respectively. It is quite common that one of the two measurements is more precise, i.e., has a much smaller standard deviation. In the discussed case of work measurements, we usually know the traveled distance with a much

better accuracy, i.e., smaller experimental error, i.e., $s_x \cong 0$. Therefore, we should be able to use the approximate formula: $s_W \cong \Delta x_{AV} \times s_F$

The final result of work measurements should be written as:

$$W = F_{AV} \times \Delta x_{AV} \pm s_W \text{ (J)} = F_{AV} \times \Delta x_{AV} \pm \Delta x_{AV} \times s_F \text{ (J)}$$

"work" = "average force" × "average distance" ± "average distance" × "standard deviation of force" (in Joules)

Numerical example: Distance traveled $\Delta x = |x_1 - x_2| = \Delta x_{AV} = 1.04$ (m)

	Measured Force F (N)	Deviations from the average force: $\Delta F = F - F_{AV}$ (N)	Squared deviations from the average force: $(\Delta F)^2$ (N^2)
1	0.170	-0.007	4.9×10^{-5}
2	0.161	-0.016	25.6×10^{-5}
3	0.152	-0.025	62.5×10^{-5}
4	0.197	0.020	40.0×10^{-5}
5	0.204	0.027	72.9×10^{-5}
Average	0.177		

Average force: $F_{AV} = 0.177$ (N)

$$s_F = \sqrt{\frac{1}{4}\left(4.9\times10^{-5}N^2 + 25.6\times10^{-5}N^2 + 62.5\times10^{-5}N^2 + 40.0\times10^{-5}N^2 + 72.9\times10^{-5}N^2\right)} =$$

$$= 0.0227 \text{ N} \cong 0.023 \text{ N}$$

Standard deviation of the force measurements: $s_F = 0.023$ (N)

Standard deviation of the work done:

$$s_W \cong s_F \times \Delta x = 0.023 \text{ N} \times 1.04 \text{ m} = 0.024 \text{ (J)}$$

The final result of the work should be written as:

$$W = F_{AV} * \Delta x_{AV} \pm s_W \text{ (J)} = 0.177 \text{ (N)} * 1.04 \text{ (m)} \pm 0.024 \text{ (J)} = 0.184 \pm 0.024 \text{ (J)}$$

Significant Figures

The **numerical values of the experimental results** must be written according to specific rules. The number of significant figures that should be used in stating a result is inseparably connected with the accuracy with which the result is known.

(1) The number of significant figures in the experimental uncertainty is limited to one or (when the experimental uncertainty is small, e.g., ± 0.15) to two significant figures. You should not use more than two significant digits when stating the experimental uncertainty.

(2) Now the best estimate (usually the average value) and its uncertainty (experimental error) must always have the same number of digits after the decimal point, even if the uncertainty does not contain the same number of significant figures as the best estimate. If the uncertainty has a greater number of places after the decimal as compared to the best estimate, adding it to (or subtracting it from) the best estimate will leave the best estimate with a greater number of decimal places than your apparatus is capable of measuring. For example, 2.4 ± 0.16 implies that the result lies in the range 2.24 – 2.56. But the apparatus can measure only up to a precision of one place after the decimal. Hence the correct way to express the answer is 2.4 ± 0.2.

If the uncertainty has a smaller number of places after the decimal than the best estimate (could be because of the restriction of one or two significant figures), then your best estimate is rounded off appropriately. This is because even if your apparatus has a higher resolution (i.e., it can display more places after the decimal), it does not have that accuracy (as proved by the error having a smaller number of places after the decimal). Therefore, 2.456 ± 0.12 should be written as 2.46 ± 0.12.

(3) The value of the experimental uncertainty and rules (1) and (2) determine the number of significant digits in the best estimate (usually the average value) of the measured value.

(4) The number "zero" presents problems in significant figures. Zeros before the first non-zero number are not significant. Zeros after numbers after the decimal place are significant. Zeros before the decimal place may or may not be significant.

Each of the following numbers has four significant figures:

2.347 234.7 -0.2347 2.340 0.002347 $-2.347*10^5$

Each of the following numbers has three significant figures:

2.34 234 -0.234 0.00234 $-2.34*10^5$ $2.34*10^{-4}$

2.30 -0.230 $2.30*10^{-7}$ 0.000230 2.03 $2.30*10^4$

Each of the following numbers has two significant figures:

$2.3*10^2$ 2.3 -0.23 0.0023 $-2.3*10^5$ $2.3*10^{-4}$

(5) Writing an integer number (e.g., 350 m) presents a problem, because it does not clearly define the number of significant figures (two significant figures or three?). Use the scientific notation, such as $3.50*10^2$ m, to indicate that three significant figures are known.

6) Both measurements and experimental uncertainties must have the same units (always the case unless specified as relative error that is always dimensionless and usually written as a percentage). Please, compare the following examples:

Correct:	**Wrong:**
0.2 ± 0.3	0.20 ± 0.321
0 ± 2	0.05 ± 2.5
3.14 ± 0.01	3.14 ± 0.002
3.142 ± 0.002	3.14213 ± 0.002
3.14 ± 0.02	3.142133 ± 0.023523
$(2.34 \pm 0.15)*10^{-4}$	$(2.34 \pm 0.152)*10^{-4}$
$7.2*10^2$ (two digits)	720 (?)
$7.20*10^2$ (three digits)	720 (?)

(7) Do not mix notations - that is, do not write a measured value in scientific notation and its error as a decimal or vice versa.

(8) When multiplying or dividing measurement figures, the final answer may not have more significant figures than the *least* number of significant figures in the figures being multiplied or divided. This simply means that an answer cannot be more accurate than the least accurate measurement entering calculation, and that you cannot improve the accuracy of a measurement by doing a calculation (even if you have a 10-digit, scientific calculator). To minimize rounding errors, **you should use more significant figures during calculations and adjust the significant figures only for the final results**.

Examples: 3.5 * 22.3 = 78 *not* 78.05, 6.2 / 833 = 0.0074 *not* 0.007442977

Adding or subtracting: 42.4 - 41.62 = 0.8, 4256 - 24.7 = 4231, 33.8 + 15.63 = 49.4