THE RIEMANN ζ- FUNCTION

Some of the functions we have studied (e.g. Legendre polynomials) are solutions of differential equations. On the other hand, a function such as $\Gamma(z)$ is effectively defined by its recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z)$$

The Riemann ζ-function is defined in yet another way:

$$\boxed{\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , \quad s>1} \qquad (1)$$

To see how one may evaluate $\zeta(s)$ analytically consider carrying out a Fourier series expansion of $f(x) = x^2$: In general,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad , \quad f(x) = \text{even} \qquad (2)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad , \quad f(x) = \text{odd} \qquad (3)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, f(t)\cdot 1 \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, f(t)\cos(nt) \qquad (4)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\, f(t)\sin(nt) \qquad (5)$$

In our case, with $f(x) = x^2$, only the a_n contribute and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\cdot t^2 = \frac{2\pi^2}{3} \quad ; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dt\cdot t^2 \cos(nt) \qquad (6)$$

Let $y = nt$, then:

$$a_n = \frac{1}{\pi n^3}\int_{-n\pi}^{n\pi} dy \cdot y^2 \cos y \overset{\text{partial integration}}{=} \frac{4}{n^2}\cos(n\pi) = (-1)^n \frac{4}{n^2} \qquad (7)$$

Hence from (2) & (7):

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[(-1)^n \frac{4}{n^2}\right]\cos(nx) \qquad (8)$$

In (8) set $x = \pi \Rightarrow$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[(-1)^n \frac{4}{n^2}\right]\underbrace{\cos(n\pi)}_{(-1)^n} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2} \qquad (9)$$

Rearranging (9):

$$\boxed{\zeta(2) \equiv \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \frac{\pi^2}{6}} \qquad (10)$$

Results for other $\zeta(s)$ (e.g. $\zeta(4)$) can be obtained in a similar way.

THE RIEMANN ζ-FUNCTION & NUMBER THEORY

Consider
$$\zeta(s)\left(1-\frac{1}{2^s}\right) = \left(1+\cancel{\frac{1}{2^s}}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right) \qquad (1)$$
$$-\frac{1}{2^s}\left(\cancel{1}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)$$
$$=\left(1+\frac{1}{3^s}+\cancel{\frac{1}{4^s}}+\frac{1}{5^s}+\cancel{\frac{1}{6^s}}+\cdots\right)-\left(\cancel{\frac{1}{4^s}}+\cancel{\frac{1}{6^s}}+\frac{1}{8^s}+\cdots\right) \qquad (2)$$

We see that in this way all the terms in the original sum which are multiples of 2 are eliminated, so that

$$\boxed{\zeta(s)\left(1-\frac{1}{2^s}\right) = \left(1+\frac{1}{3^s}+\frac{1}{5^s}+\cdots\right)} \qquad (3)$$

Next consider:
$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right) = \left(1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{15^s}+\cdots\right)$$
$$\otimes\left(1-\frac{1}{3^s}\right) = \qquad (4)$$
$$\left(1+\cancel{\frac{1}{3^s}}+\frac{1}{5^s}+\frac{1}{7^s}+\cancel{\frac{1}{9^s}}+\frac{1}{11^s}+\frac{1}{13^s}+\cancel{\frac{1}{15^s}}+\cdots\right)-\left(\cancel{\frac{1}{3^s}}+\cancel{\frac{1}{9^s}}+\cancel{\frac{1}{15^s}}+\frac{1}{21^s}+\cdots\right) \qquad (5)$$
$$=\left(1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\cdots\right)$$

At this stage we have eliminated all terms in the original sum for $\zeta(s)$ which are multiples of the prime numbers 2 & 3.

Proceeding in this way it follows that the expression

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\cdots\left(1-\frac{1}{p^s}\right) \qquad (6)$$

where P is a prime number eliminates all terms in $\zeta(s)=\sum(1/n^s)$ for which n is a multiple of the primes $2, 3, 5, 7, 11, \ldots P$.

Continuing in this way we would eventually eliminate every term in $\zeta(s)$ other than 1. Hence

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\cdots\left(1-\frac{1}{p^s}\right)=\zeta(s)\prod_{P=\text{prime }2}^{\infty}\left(1-\frac{1}{p^s}\right)=1 \quad (7)$$

Solving for $\zeta(s)$,

$$\boxed{\zeta(s)=\frac{1}{\prod\limits_{\text{primes}}\left(1-\frac{1}{p^s}\right)}=\prod_{\substack{P=2\\P=\text{prime}}}^{\infty}\frac{1}{\left(1-\frac{1}{p^s}\right)}} \quad (8)$$

EULER PRIME NUMBER PRODUCT FOR $\zeta(s)$

APPLICATIONS TO COMPUTATION

Returning to Eq. (3) we have:

$$\zeta(s)\left(1-\frac{1}{2^s}\right)=\left(1+\frac{1}{3^s}+\frac{1}{5^s}\cdots\right)=\sum\left(1+\frac{1}{(\text{odd number})^s}\right) \quad (9)$$

Since $\zeta(s)$ involves a sum over all integers it follows from Eq. (9) that this expression evaluates $\zeta(s)$ to a given precision using only half as many terms as would be required had $\zeta(s)$ itself had been evaluated directly. This is also evident in Eq. (8), where the sum extends only over the primes.

USING $\zeta(s)$ TO IMPROVE CONVERGENCE OF SERIES

$\zeta(s)$ can also be used to improve the convergence of other series. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \sum_{1}^{\infty} \frac{1}{n^2(1+\frac{1}{n^2})} = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{n^2}\left(\frac{1}{1+\frac{1}{n^2}}\right) \qquad (1)$$

In the last term $\frac{1}{n^2} < 1$ for $n \geq 2$ so that we can expand

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} + \sum \frac{1}{n^2}\left(1 - \frac{1}{n^2} + \frac{1}{n^4} - \frac{1}{n^6} + \cdots\right) \qquad (2)$$

$$= \frac{1}{2} + \left(\frac{1}{n^2} - \frac{1}{n^4} + \frac{1}{n^6} + \cdots\right)$$

If we stop at the end of a finite number of terms, then as an example the exact expression is

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \sum_{1}^{\infty}\left(\underbrace{\frac{1}{n^2}}_{\zeta(2)} - \underbrace{\frac{1}{n^4}}_{\zeta(4)} + \underbrace{\frac{1}{n^6}}_{\zeta(6)} - \frac{1}{n^8+n^6}\right) + \frac{1}{2} \qquad \text{Exact} \qquad (3)$$

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Hence
$$\boxed{\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \zeta(2) - \zeta(4) + \zeta(6) - \sum_{n=1}^{\infty} \frac{1}{n^8+n^6} + \frac{1}{2}} \qquad (4)$$

Since $\zeta(2)$, $\zeta(4)$, $\zeta(6)$ are known, the remaining part of the sum converges as $1/n^8$, rather than as $1/n^2$ in the original expression. The use of $\zeta(2), \zeta(4), \zeta(6)$ to improve the convergence of the original sum is thus similar to the use of subtracted dispersion relations to improve the convergence of a dispersion