BERNOULLI NUMBERS & BERNOULLI FUNCTIONS

Derivation of the Euler-MacLaurin Formula:

9/4/84

Ref. ARFKEN, Mathematical Methods for Physicists p. 278 ff; Whittaker & Watson, Analysis, p. 125 ff

We begin by introducing the Bernoulli numbers B_n via the generating function

$$\boxed{\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}} \tag{1}$$

Taylor Series Expansion: Defines $B_n/n!$

We can obtain the Bernoulli numbers by noting from the rhs of (1) that

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = B_0 + B_1 x + B_2 \frac{1}{2!} x^2 + \dots \Rightarrow B_0 = \left[\sum B_n \frac{x^n}{n!}\right]_{x=0} \tag{2}$$

But $$\left.\frac{x}{e^x-1}\right|_{x=0} = \frac{0}{0} \rightarrow \text{l'Hopital's Rule to } \left.\frac{1}{e^x}\right|_{x=0} = 1 \tag{3}$$

$$\therefore \boxed{B_0 = 1} \tag{4}$$

Similarly:

$$B_1 = \frac{d}{dx}\left(\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\right)_{x=0} = \frac{d}{dx}\left(\frac{x}{e^x-1}\right)_{x=0} = \frac{(e^x-1)\cdot 1 - xe^x}{(e^x-1)^2} \rightarrow \frac{0}{0} \tag{5}$$

Again by l'Hopital's rule $$B_1 \rightarrow \frac{\cancel{e^x} - xe^x - \cancel{e^x}}{2(e^x-1)} \rightarrow \frac{0}{0} \tag{6}$$

Still Again by l'Hopital:

$$\boxed{B_1 \rightarrow \frac{-xe^x - e^x}{2e^x} = -\frac{1}{2}} \tag{7}$$

As can be seen, the repeated derivatives will lead to the formula [note that the $n!$ factors cancel]

$$\boxed{B_n = \frac{d^n}{dx^n}\left(\frac{x}{e^x-1}\right)_{x=0}} \tag{8}$$

To avoid the repeated derivatives we can cross multiply through in Eq. (1) to get:

$$1 = \frac{(e^x-1)}{x}\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{(\cancel{1}+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\dots\cancel{-1})}{x}\sum B_n\frac{x^n}{n!} \quad (9)$$

Then since the constant terms cancels, all other terms $\sim x$ so we can divide via x all terms

$$\boxed{\therefore\ 1 = \left(1+\frac{1}{2!}x+\frac{1}{3!}x^2+\dots+\frac{1}{k!}x^{k-1}\right)\left(B_0+B_1x+B_2\frac{x^2}{2!}+\dots\right)} \quad (10)$$

Constant term: $1 = B_0$

x^1: $0 = \frac{1}{2!}B_0 + B_1 \Rightarrow B_1 = -\frac{1}{2}$ ✓ (11)

x^2: $0 = \frac{1}{2!}B_2 + \frac{1}{2!}B_1 + \frac{1}{3!}B_0 = \frac{1}{2}B_2 + \frac{1}{2}B_1 + \frac{1}{6}B_0$

$\therefore\ B_2 + B_1 + \frac{1}{3}B_0 = 0 \Rightarrow B_2 = -B_1 - \frac{1}{3}B_0 = +\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ ✓ (12)

x^3: $0 = \frac{1}{3!}B_3 + \frac{1}{2!}\frac{1}{2!}B_2 + \frac{1}{3!}B_1 + \frac{1}{4!}B_0$

$\therefore\ B_3 = -\frac{6}{4}B_2 - B_1 - \frac{6}{24}B_0 = -\frac{3}{2}\cdot\frac{1}{6} + \frac{1}{2} - \frac{1}{4} = 0$ ✓ (13)

$5! = 120$
$4! = 24$
$3! = 6$

x^4: $0 = \frac{1}{4!}B_4 + \frac{1}{2!}\frac{1}{3!}B_3 + \frac{1}{3!}\frac{1}{2!}B_2 + \frac{1}{4!}B_1 + \frac{1}{5!}B_0$

$B_4 = -24\left(\frac{1}{12}\underset{0}{\underset{\shortparallel}{B_3}} + \frac{1}{12}B_2 + \frac{1}{24}B_1 + \frac{1}{120}B_0\right) = -\frac{24}{12}\cdot 0 - 2B_2 - B_1 - \frac{1}{5}B_0$

$= -2\frac{1}{6} + \frac{1}{2} - \frac{1}{5} = -\frac{10}{30} + \frac{15}{30} - \frac{6}{30} = -\frac{1}{30}$ ✓ (14)

Arfken notes that the odd B_n for $n \geq 3$ are all zero, but this isn't obvious at the moment:

$$\boxed{B_{2n+1} \equiv 0 \quad n \geq 1} \quad (15)$$

BERNOULLI FUNCTIONS & REVIEW OF GENERATING FUNCTIONS

Bernoulli Functions:

We can introduce the Bernoulli function by writing

Generating function for $B_n(s)$

$$\boxed{\frac{x}{e^x-1}\left(e^{xs}\right) \equiv \sum_{n=0}^{\infty} B_n(s)\,\frac{x^n}{n!}} \quad * \qquad (16)$$

Clearly the Bernoulli function and the Bernoulli numbers are related by

$$\boxed{B_n(0) = B_n \qquad ; \qquad B_0(s) \equiv 1} \leftarrow \text{follows from (10)} \qquad (17)$$

The Bernoulli function $B_n(s)$ have the following 2 important properties. First, differentiating (16) w.r.t. s we have

$$\frac{d}{ds}\left(\frac{xe^{xs}}{e^x-1}\right) = \frac{x^2 e^{xs}}{e^x-1} = \sum_{n=0}^{\infty} B_n'(s)\,\frac{x^n}{n!} \quad \Big\} \text{ Cancel a factor of } x: \qquad (18)$$

$$\therefore \frac{xe^{xs}}{e^x-1} = \sum_{n=0}^{\infty} B_n'(s)\,\frac{x^{n-1}}{n!} \quad \Big\} \text{ look at the term } \propto x^{n-1}\text{: Its coeff is } \frac{B_n'(s)}{n!} \qquad (19a)$$

* original expansion

$$\hookrightarrow = \sum_{n=0}^{\infty} B_n(s)\,\frac{x^n}{n!} \quad \Big\} \text{ look at the term } \propto x^{n-1}\text{: Its coeff is } \frac{B_{n-1}(s)}{(n-1)!} \qquad (19b)$$

From (19a,b) we then have

$$\frac{B_n'(s)}{n!} = \frac{B_{n-1}(s)}{(n-1)!} \quad \Rightarrow \quad \boxed{B_n'(s) = n\,B_{n-1}(s)} \checkmark \qquad (20)$$

Consider next [redacted] $S=1 \Rightarrow$

$$\frac{xe^x}{e^x-1} = \frac{x}{1-e^{-x}} = -\frac{(-x)}{1-e^{-x}} = \frac{(-x)}{e^{(-x)}-1} = \sum_{n=0}^{\infty} B_n\,\frac{(-x)^n}{n!} \qquad (21)$$

$\therefore S=1 \Rightarrow$

$$\sum_n B_n(1)\frac{x^n}{n!} = \cdots\cdots = \sum_n (-1)^n B_n\,\frac{x^n}{n!} \quad \Rightarrow \quad \boxed{B_n(1) = (-1)^n B_n \overset{(17)}{=} (-1)^n B_n(0)} \qquad (22) \checkmark$$

THE EULER-MACLAURIN FORMULA:

Derivation of the E-M Formula:

Start with $\int_0^1 dx\, f(x) = \int_0^1 dx\, B_0(x)\, f(x)$ (28)

Note: By inspection $B_0(s) = 1 \;\; \forall s$ [See (17)] (29)

Here the argument of B_n is x:

Then $B_1'(x) = 1 \cdot B_0(x) = 1$ [Using (20)] (30)

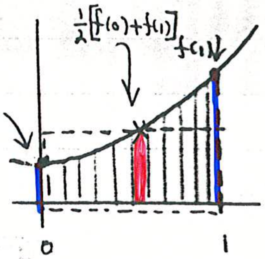
$$\therefore \int_0^1 dx\, f(x) = \int_0^1 dx\, B_1'(x)\, f(x) = B_1(x) f(x)\Big]_0^1 - \int_0^1 dx\, B_1(x)\, f'(x) \quad (31)$$

$$= f(1)\underbrace{B_1(1)}_{\underbrace{1/2}_{B_1(1)=(-1)B_1(0)=-B_1=+1/2}} - f(0)\underbrace{B_1(0)}_{-1/2} = \tfrac{1}{2}[f(1)+f(0)] \quad (32)$$

(22) ⇒

$$\boxed{\therefore \int_0^1 dx\, f(x) = \tfrac{1}{2}[f(1)+f(0)] - \int_0^1 dx\, B_1(x)\, f'(x)} \quad (33)$$

This formula makes obvious sense from the point of view of basic calculus: It represents the most naive approximation to the calculation of an integral, as shown by the following figure:



The shaded area is then approximated by the area of the rectangle which is

$$\text{Area} \simeq \text{base} \times \text{height} = 1 \cdot \tfrac{1}{2}[f(0)+f(1)] \quad (34)$$

If $f(x) = a$ then this would be the exact expression for the area, which is consistent with (33), since then $f'(x) = 0$.

Continuing as before we use (20) to write

(20) $\Rightarrow$ $$\boxed{B_2'(x) = 2B_1(x) \quad \Rightarrow B_1(x) = \frac{1}{2} B_2'(x)} \qquad (35)$$

$$\therefore \int_0^1 dx\, f(x) = \frac{1}{2}\left[f(0) + f(1)\right] - \int_0^1 dx\, \frac{1}{2} B_2'(x)\, f'(x)$$

$$= \frac{1}{2}\left[f(0) + f(1)\right] - \frac{1}{2} B_2(x) f'(x)\Big|_0^1 + \frac{1}{2}\int_0^1 dx\, B_2(x)\, f''(x) \qquad (36)$$

$$-\frac{1}{2}\left[B_2(1) f'(1) - B_2(0) f'(0)\right]$$

We can invoke the general results:

$$\boxed{\begin{array}{l} B_{2n}(1) = (-1)^{2n} B_{2n}(0) = B_{2n} \\ B_{2n+1}(1) = -B_{2n+1} = 0 \quad \text{for } n \geq 1 \\ B_{2n}(0) = B_{2n} \\ B_{2n+1}(0) = B_{2n+1} = 0 \end{array}} \qquad (37)$$

(36)

$$\therefore B_2(1) = B_2 = \frac{1}{6} = B_2(0) \quad \Rightarrow -\frac{1}{2}\left[\ldots\right] = -\frac{1}{2}\cdot\frac{1}{6}\left[f'(1) - f'(0)\right] \qquad (38)$$

$$\therefore \int_0^1 dx\, f(x) = \frac{1}{2}\left[f(0) + f(1)\right] - \frac{1}{2}\,\frac{1}{6}\left[f'(1) - f'(0)\right] + \frac{1}{2}\int_0^1 dx\, B_2(x)\, f''(x) \quad ✓ \quad (39)$$

If we find a situation in which the higher derivatives are small [because as in our case each successive derivative introduces a factor of Δ'] then we can stop with the first derivative, yielding

$$\boxed{\int_0^1 dx\, f(x) = \frac{1}{2}\left[f(0) + f(1)\right] - \frac{1}{12}\left[f'(1) - f'(0)\right] + \frac{1}{2}\int_0^1 dx\, B_2(x)\, f''(x)} \qquad (40)$$

We can proceed to write down the general expression as in (5.168 a) of ARFKEN

$$\boxed{\int_0^1 dx\, f(x) = \frac{1}{2}\left[f(1) + f(0)\right] - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(1) - f^{(2p-1)}(0)\right] + \frac{1}{(2q)!}\int_0^1 dx\, f^{(2q)}(x)\, B_{2q}(x)} \qquad (41)$$

DEFINITE INTEGRALS WITH ARBITRARY LIMITS:

Changing the limits of integration

Thus far the results in (40) and (41) hold specifically for the range of integration [0,1]. This is obvious if we recall that we have used the specific values of the Bernoulli polynomials for these values, $B_n(1)$ and $B_n(0)$. However, we now wish to extend the limits of integration first of all from $[0,\infty]$ and ultimately between any two limits. To do this let us for illustration [redacted] purposes consider (40) neglecting the remainder:

$$\therefore \quad \int_0^1 dx\, f(x) \cong \frac{1}{2}\left[f(x=1)+f(x=0)\right] - \frac{1}{12}\left[f'(x=1)-f'(x=0)\right] + \dots \qquad (42)$$

Let the indefinite integral $\int f(x)dx \equiv F(x)$. Then (42) reads as

$$F(x=1)-F(x=0) \cong \frac{1}{2}\left[f(x=1)+f(x=0)\right] - \frac{1}{12}\left[f'(x=1)-f'(x=0)\right] + \dots \qquad (43)$$

Define $y = x+1$; Then (43) $\Rightarrow$:

$$\underbrace{F(y=2)-F(y=1)}_{} \cong \frac{1}{2}\left[f(y=2)+f(y=1)\right] - \frac{1}{12}\left[f'(y=2)-f'(y=1)\right]$$

$$\equiv \int_1^2 dy\, f(y) \qquad (44)$$

Since y and x are now dummy variables of integration we see that by shifting in this way we can write

$$\boxed{\int_1^2 dx\, f(x) \cong \frac{1}{2}\left[f(2)+f(1)\right] - \frac{1}{12}\left[f'(y=2)-f'(y=1)\right] + \dots} \qquad (45)$$

We can continue this process, and doing so let us write down the first few terms in the series

$$\int_0^4 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

$$\begin{aligned}
&= \frac{1}{2}\left[f(1)+f(0)\right] - \frac{1}{12}\left[\cancel{f'(1)} - f'(0)\right] + \frac{1}{2}\int_0^1 dx\, B_2(x)\, f''(x) \\
&+ \frac{1}{2}\left[f(1)+f(2)\right] - \frac{1}{12}\left[\cancel{f'(2)} - \cancel{f'(1)}\right] + \frac{1}{2}\int_1^2 dx\, B_2(x)\, f''(x) \\
&+ \frac{1}{2}\left[f(2)+f(3)\right] - \frac{1}{12}\left[\cancel{f'(3)} - \cancel{f'(2)}\right] + \frac{1}{2}\int_2^3 dx\, B_2(x)\, f''(x) \\
&+ \frac{1}{2}\left[f(3)+f(4)\right] - \frac{1}{12}\left[f'(4) - \cancel{f'(3)}\right] + \frac{1}{2}\int_3^4 dx\, B_2(x)\, f''(x)
\end{aligned} \qquad (46)$$

$$\Rightarrow \int_0^4 f(x)\,dx = \left[\frac{1}{2}f(0) + f(1) + f(2) + f(3) + \frac{1}{2}f(4)\right] - \frac{1}{12}\left[f'(4)-f'(0)\right] + \frac{1}{2}\int_0^4 dx\, B_2(x)\, f''(x) \qquad (47)$$

For our purposes, where we intend to stop at 1st derivatives, we can generalize (47) to

$$\boxed{\int_0^n dx\, f(x) = \frac{1}{2}[f(0)+f(n)] + \sum_{j=1}^{n-1} f(j) - \frac{1}{12}[f'(n)-f'(0)] + \frac{1}{2}\int_0^n dx\, B_2(x)\, f''(x)} \quad (48)$$

If we keep expanding the remainder term then we can write

$$\boxed{\begin{aligned}\int_0^n dx\, f(x) &= \frac{1}{2}[f(0)+f(n)] + \sum_{p=1}^{n-1} f(p) - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - f^{(2p-1)}(0)\right] \\ &\quad + \underbrace{\frac{1}{(2q)!}\int_0^n f^{(2q)}(x)\, B_{2q}(x)\, dx}_{\text{remainder} \equiv R_q}\end{aligned}} \quad (49)$$

$\equiv R_q$

We are now interested in the expression for $\int_1^\infty f(x)\,dx$. So we can obtain this result by subtracting the expression in (41) from (48): [$R \equiv$ remainder]

$$\int_1^n dx\, f(x) = \left[\int_0^n - \int_0^1\right] dx\, f(x) = \frac{1}{2}[\cancel{f(0)} + f(n)] + \sum_{p=1}^{n-1} f(p) - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - \cancel{f^{(2p-1)}(0)}\right] \cancel{+R}$$

$$- \frac{1}{2}[f(1) + \cancel{f(0)}] + \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(1) - \cancel{f^{(2p-1)}(0)}\right] \cancel{-R_q} \quad (50)$$

OK! But these are not really the same

$$\therefore \int_1^n dx\, f(x) \cong \frac{1}{2}[f(n) - f(1)] + \sum_{p=1}^{n-1} f(p) - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - f^{(2p-1)}(1)\right] \quad (51)$$

See also 463.33 (135)

$$\therefore \boxed{\sum_{p=1}^{n-1} f(p) - \int_1^n dx\, f(x) \cong \frac{-1}{2}[f(n) - f(1)] + \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}\left[f^{(2p-1)}(n) - f^{(2p-1)}(1)\right]} \cancel{+\Delta R_q} \quad (52)$$

$\cancel{(R_q - R'_q)}$

If we retain only the first derivative contribution, then (52) $\Rightarrow$

$$\sum_{p=1}^{n-1} f(p) - \int_1^n dx\, f(x) \cong \frac{1}{2}[f(1) - f(n)] + \frac{1}{12}[f'(n) - f'(1)] \quad (53)$$

as $n \to \infty$:

$$\boxed{\sum_{p=1}^{\infty} f(p) - \int_1^\infty dx\, f(x) \cong \frac{1}{2}[f(1) - f(\infty)] + \frac{1}{12}[f'(\infty) - f'(1)]} \quad (54)$$

EULER-MACLAURIN FORMULA

THE EULER–MASCHERONI CONSTANT γ:

We previously encountered γ when we discussed on pp. 112,113 the following representation for $\Gamma(z)$:

$$\boxed{\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}} \qquad (1)$$

$$\gamma = \lim_{N\to\infty} \left(\sum_{1}^{N} \frac{1}{n} - \int_1^N dn \, \frac{1}{n} \right) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N \right) = 0.57721566\ldots \qquad (2)$$

Here we show how to <u>derive</u> the result in (2) <u>using the Euler–Maclaurin formula</u>. Starting from (52) we have ($p \to n$, $n \to N$)

$$\sum_{n=1}^{N-1} f(n) - \int_1^N dn \, f(n) \cong -\frac{1}{2}\left[f(N) - f(1)\right] + \sum_{n=1}^{q} \frac{1}{(2n)!} B_{2n} \left[f^{(2n-1)}(N) - f^{(2n-1)}(1) \right] + \ldots \qquad (3)$$

Add & Subtract $f(N) \Rightarrow$

$$\sum_{n=1}^{N} f(n) - \int_1^N dn \, f(n) = -\frac{1}{2}\left[f(N) - f(1)\right] + f(N) + \sum_{n=1} \ldots \qquad (4)$$

Hence

$$\boxed{\sum_{n=1}^{N} f(n) - \int_1^N dn \, f(n) \cong \frac{1}{2}\left[f(N) + f(1)\right] + \sum_{n=1}^{q} \frac{1}{(2n)!} B_{2n} \left[f^{(2n-1)}(N) - f^{(2n-1)}(1) \right] + \ldots} \qquad (5)$$

For our purposes: $f(n) = 1/n \Rightarrow f^{(1)}(n) = -1/n^2 \qquad f^{(2)}(n) = +2/n^3$

$$f^{(3)}(n) = -2\cdot 3/n^4 \qquad f^{(4)} = 2\cdot 3\cdot 4/n^5 \quad \ldots \quad f^{(m)} = (-1)^m m!/n^{m+1} \qquad (6)$$

The sum in (5) then evaluates to

$$\Sigma = \frac{1}{2!} B_2 \left[f^{(1)}(N) - f^{(1)}(1) \right] + \frac{1}{4!} B_4 \left[f^{(3)}(N) - f^{(3)}(1) \right] + \frac{1}{6!} \left[f^{(5)}(N) - f^{(5)}(1) \right] + \ldots \qquad (7)$$

$$= \frac{1}{2!} B_2 \left[-\frac{1}{N^2} + 1 \right] + \frac{1}{4!} B_4 \left[-\frac{3!}{N^4} + 3! \right] + \frac{1}{6!} B_6 \left[-\frac{5!}{N^6} + 5! \right] + \ldots \qquad (8)$$

If we now take the limit $N \to \infty$ we find (noting that $f(\infty)=0, f(1)=1$)

$$\boxed{\gamma = \left\{ \sum_{n=1}^{\infty} f(n) - \int_1^{\infty} dn\, f(n) \right\} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) - \int_1^{\infty} \frac{dn}{n} = \frac{1}{2} + \frac{1}{2}B_2 + \frac{1}{4}B_4 + \frac{1}{6}B_6 + \cdots}$$

numerically; $\gamma = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{4}\left(\frac{-1}{30}\right) + \frac{1}{6}\left(\frac{1}{42}\right)$

$= 0.5 + 0.0833 - 0.0083 + 0.0040 \cong 0.5790$

This compares to the actual value $\gamma = 0.577215\ldots$

SUMMING THE SERIES $\sum_{n=1}^{N} n^p$

1/27/84

Side Comments: Summing the Series $\sum_{n=1}^{N} n^p$:

Let us define $S_p(N) = \sum_{n=1}^{N} n^p$ $\qquad p=\text{integer}$ (1)

Such series can be summed by the following recurrence method. Begin with $p=0$

$$S_0(N) = \sum_{n=1}^{N} 1 = \underbrace{1+1+1+\dots}_{N \text{ terms}} = N \tag{2}$$

To evaluate $S_1(N)$ consider

$$\sum_{n=1}^{N}\left[(n+1)^2 - n^2\right] = [\cancel{2^2}-1^2] + [\cancel{3^2}-\cancel{2^2}] + [4^2-\cancel{3^2}] + \dots + [\cancel{(N-1+1)^2} - \cancel{(N-1)^2}] + [(N+1)^2 - \cancel{N^2}] \tag{3}$$

$$= \underline{-1 + (N+1)^2} = \cancel{-1} + (N^2+2N+\cancel{1}) = N^2+2N \,*$$

The point in (3) is that the only terms which do not get cancelled are the very lowest and the very highest.
Now the left side of (3) gives

$$\sum_{n=1}^{N}\left[\cancel{n^2}+2n+1-\cancel{n^2}\right] = \sum_{n=1}^{N}[2n+1] = \underbrace{2\sum_{n=1}^{} n}_{2S_1(N)} + \underbrace{\sum_{n=1}^{} 1}_{N} \tag{4}$$

Hence combining (3) and (4) we have: $2S_1(N) + \cancel{N} = N^2 + \cancel{2}N \,*$

$$\therefore\ \boxed{S_1(N) = \sum_{n=1}^{N} n = \tfrac{1}{2}N(N+1)} \tag{5}$$

Gradshteyn/Ryzhik p.1.

Check: $S_1(4) = 1+2+3+4 = 10 \overset{?}{=} \tfrac{1}{2}\,4\cdot 5 = 10$ ✓

$S_1(5) = 1+2+3+4+5 = 15 \overset{?}{=} \tfrac{1}{2}\,5\cdot 6 = 15$ ✓ (6)

Consider next $S_2(N)$. To evaluate this we examine lowest highest

$$\sum_{n=1}^{N}\left[(n+1)^3 - n^3\right] = [2^3-1^3] + \dots [(N+1)^3 - N^3] = \underline{-1^3 + (N+1)^3} = \cancel{-1} + (N^3+3N^2+3N+\cancel{1}) = N^3+3N^2+3N \tag{7}$$

$$= \sum_{n=1}^{N}\left[\cancel{n^3}+3n^2+3n+1-\cancel{n^3}\right] = 3\sum_{n=1}^{N} n^2 + 3\sum_{n=1}^{N} n + \sum_{n=1}^{N} 1 = 3S_2(N) + \underbrace{3\cdot\tfrac{1}{2}N(N+1)}_{\text{using (5)}} + N$$

Hence from (7) and (8)

$$3S_2(N) + \frac{3}{2}N(N+1) + N = N^3 + 3N^2 + 3N \Rightarrow 3S_2(N) = N^3 + 3N^2 + 3N - \frac{3}{2}N^2 - \frac{3}{2}N - N$$

$$= N^3 + \frac{3}{2}N^2 + \frac{1}{2}N = \frac{2N^3 + 3N^2 + N}{2} \qquad (9)$$

$$\boxed{\sum_{n=1}^{N} n^2 \equiv S_2(N) = \frac{N(2N^2+3N+1)}{6} = \frac{N(N+1)(2N+1)}{6}} \quad ✓ \; 305(228) \qquad (10)$$

Gradshteyn / Ryzhik p. 1

Proceeding in this way we can express $S_p(N)$ in terms of $S_{p-1}(N)$ etc., which thus generates the desired recurrence relation.

Lastly we evaluate $S_3(N) = \sum_{n=1}^{N} n^3$

Consider

$$\sum_{n=1}^{N}\left[(n+1)^4 - n^4\right] = \cdots = -1^4 + (N+1)^4 = \cancel{-1} + (N^4 + 4N^3 + 6N^2 + 4N \cancel{+1}) = N^4 + 4N^3 + 6N^2 + 4N \qquad (11)$$

$$= \sum_{n=1}^{N}\left[(\cancel{n^4} + 4n^3 + 6n^2 + 4n + 1) - \cancel{n^4}\right] = 4\underbrace{\sum_{n=1}^{N} n^3}_{S_3(N)} + 6S_2(N) + 4S_1(N) + N \qquad (12)$$

$$\therefore\ 4S_3(N) = N^4 + 4N^3 + 6N^2 + \cancel{4}N - 6S_2(N) - 4S_1(N) - \cancel{N}$$

$$= N^4 + 4N^3 + 6N^2 + 3N - \left[2N^3 + 3N^2 + N\right] - 2\left[N^2 + N\right] \qquad (13)$$

$$= N^4 + N^3[4-2] + N^2[6-3-2] + N[3 - \cancel{1} - 2] = N^4 + 2N^3 + N^2 = N^2(N^2 + 2N + 1)$$

$$\therefore\ \boxed{S_3(N) = \sum_{n=1}^{N} n^3 \doteq \frac{1}{4}N^2(N+1)^2 = \left[\frac{N(N+1)}{2}\right]^2} \quad ✓ \qquad (14)$$

Gradshteyn Ryzhik p. 1

Check:

$$S_3(4) = 1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100 \stackrel{?}{=} \left[\frac{4 \cdot 5}{2}\right]^2 = 100\ ✓ \qquad (14)$$

For more discussion of the evaluation of such sums see C. BENDER and S. ORSZAG's book on mathematical methods in physics entitled: ADVANCED MATHEMATICAL METHODS FOR SCIENTISTS AND ENGINEERS (McGraw-Hill, New York, 1978); see Chap. 2 and problem 2.1 p. 53.