THE GAMMA (Γ) FUNCTION

This function is somewhat different from other functions we have studied in that it is not a solution of a 2ND order differential equation. However the gamma function (along with the related beta functions) plays an important role in many areas of mathematical physics.

<u>Fundamental Definition</u>:
$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n\cdot n^z}{z(z+1)(z+2)\cdots(z+n)} \qquad (1)$$

$$= \lim_{n\to\infty} \frac{n^z}{z}\,\frac{1}{z+1}\,\frac{2}{z+2}\,\frac{3}{z+3}\cdots\frac{n}{z+n} \qquad (2)$$

In (1) we evaluate
$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n\cdot n^{z+1}}{(z+1)(z+2)(z+3)\cdots\underbrace{(z+1+n-1)}_{(z+n)}(z+1+n)} \qquad (3)$$

Multiplying & dividing by z we have:
$$\Gamma(z+1) = \lim_{n\to\infty} \underbrace{\left\{\frac{1\cdot 2\cdot 3\cdots n^z}{z(z+1)\cdots(z+n)}\right\}}_{\Gamma(z)} \underbrace{\frac{n^1\, z}{z+n+1}}_{\cong z\frac{n}{n}\to z} \qquad (4)$$

Hence from (4)

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)} \qquad (5)$$

This can be taken as the defining equation for $\Gamma(z)$, the analog of what a differential equation would be for a conventional function like a Legendre polynomial. <u>We can evaluate $\Gamma(z)$ for integral z</u> by writing from Eq. (1)

$$\Gamma(1) = \lim_{n\to\infty} \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}\cdot n^1}{\cancel{1}\cdot(\cancel{1+1})\cdot(\cancel{1+2})\cdots(\cancel{1+n-1})(1+n)} = \lim_{n\to\infty} \frac{n}{1+n} = 1 \qquad (6)$$

Then using (5): $\Gamma(2) = 1\,\Gamma(1) = 1$; $\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1$; $\Gamma(4) = 3\,\Gamma(3) = 3!$

Hence
$$\boxed{\Gamma(n) = (n-1)!} \qquad (7)$$

We can use (7) to evaluate $0! = 1$ as follows: For integral n,

$$n(n-1)! = n! \Rightarrow (n-1)! = n!/n \Rightarrow \text{for any } z, \ \boxed{(z-1)! = z!/z} \quad (8)$$

Now in (8) let $z = 1 \Rightarrow \boxed{0! = 1!/1 = 1} \quad (9)$

Continuing in this manner, let $z=0$ in (8) $\Rightarrow (-1)! = 0!/0 = 1/0 = \infty$ (10)

Similarly, $(-2)! = (-1)!/-1 = \infty$. Hence in general the preceding argument suggests that

$$\boxed{n! = \infty, \quad \text{when } n \text{ is negative}} \quad (11)$$

The preceding results in (8) - (11) can be made more rigorous by considering the integral representation of $\Gamma(z)$.

<u>Integral Representation of $\Gamma(z)$:</u>

Consider $I_0 \equiv \int_0^\infty dt\, e^{-t} = -e^{-t}\Big|_0^\infty = 1$ (12)

Then by a partial integration: $I_1 = \int_0^\infty dt\, \underset{}{t e^{-t}} = \underbrace{-te^{-t}\Big|_0^\infty}_{0} - \int_0^\infty (-) dt\, e^{-t} = 1$ (13)

$$I_2 = \int_0^\infty \underbrace{t^2}_{u} \underbrace{e^{-t} dt}_{dV} = \underbrace{t^2 (-) e^{-t}\Big|_0^\infty}_{0} - \int_0^\infty dt\, 2t(-)e^{-t} = +2\int_0^\infty dt\, te^{-t} = 2 \cdot 1 \quad (14)$$

<u>Generalizing</u>: $I_n = \int_0^\infty dt\, t^n e^{-t} = \int_0^\infty \underbrace{t^n}_{u} \cdot \underbrace{e^{-t} dt}_{dV} = (-)t^n e^{-t}\Big|_0^\infty + \underbrace{\int_0^\infty dt\, n t^{n-1} e^{-t}}_{n I_{n-1}}$ (15)

$$\therefore \boxed{I_n = n I_{n-1} \qquad n \geq 2} \quad (16)$$

If we identify $I_n \equiv \Gamma(n+1) \Rightarrow I_{n-1} = \Gamma(n)$ then (16) $\Rightarrow \Gamma(n+1) = n\Gamma(n)$ ✓

This is the correct recurrence relation for $\Gamma(n)$ as can be seen from (5). Hence we conclude that <u>for integral n:</u>

$$\boxed{\Gamma(n) = I_{n-1} = \int_0^\infty dt\, t^{n-1} e^{-t}} \quad (17)$$

The expression in (17) can be generalized to define $\Gamma(z)$ for a complex number z

$$\boxed{\Gamma(z) = \int_0^\infty dt\ t^{z-1} e^{-t} \qquad \underset{\to \,=\,(n-1)!\ z=n}{\operatorname{Re} z > 0}} \quad (18)$$

The restriction that $\operatorname{Re} z > 0$ is needed to prevent the integral from diverging at the lower limit $t=0$. For $z = n =$ integer the integral gives $\Gamma(n) = (n-1)!$ as expected. Values of z which are half-integral can be obtained from the basic recurrence relation $\Gamma(n+1) = n\Gamma(n)$ starting from the result

$$\boxed{\Gamma(1/2) = \sqrt{\pi}} \quad (19)$$

To show this: Start with (18) and let $t = x^2 \quad dt = 2x\,dx \qquad (20)$

$$\Gamma(1/2) = \int_0^\infty dt\, e^{-t} t^{-1/2} = \int_0^\infty (\cancel{2x}\,dx)\, e^{-x^2} (\cancel{x^2})^{-1/2} = 2\int_0^\infty dx\, e^{-x^2} \qquad (21)$$

$$I \equiv \left[\Gamma(1/2)\right]^2 = \left(2\int_0^\infty dx\, e^{-x^2}\right)\left(2\int_0^\infty dy\, e^{-y^2}\right) = \iint_{-\infty}^{\infty} dx\,dy\, e^{-(x^2+y^2)} = \iint e^{-r^2}\, r\,dr\,d\theta \qquad (22)$$

where $x = r\cos\theta\ ;\ y = r\sin\theta$

$$I = \int_0^\infty dr \int_0^{2\pi} d\theta\ \ r e^{-r^2} = 2\pi \int_0^\infty dr\, e^{-r^2}\cdot r \quad ;\ \text{let } s = r^2 \quad ds = 2r\,dr$$

$$\therefore I = 2\pi \int_0^\infty ds\, \frac{e^{-s}}{2} = \pi \Rightarrow \left[\Gamma(1/2)\right]^2 = \pi \Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}} \quad (23)$$

Another Widely Used Relation:

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}} \quad (24)$$

This will be derived in detail later; for now we sketch one proof of (24).

We begin with another representation of $\Gamma(z)$ due to Weierstrass:

$$\boxed{\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}} \quad (25)$$

[See ARFKEN]

$$\gamma = \underline{\text{EULER-MASCHERONI}} \text{ constant} = \lim_{N\to\infty}\left(\sum_1^N \frac{1}{n} - \int_1^N dn \frac{1}{n}\right) = \quad (26)$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^N \frac{1}{n} - \ln N\right) = 0.57721566\ldots$$

From (25):
$$\frac{1}{\Gamma(z)}\frac{1}{\Gamma(-z)} = \left\{z \cancel{e^{\gamma z}} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)\cancel{e^{-z/n}}\right\}\left\{-z \cancel{e^{-\gamma z}} \prod_{m=1}^{\infty}\left(1-\frac{z}{m}\right)\cancel{e^{z/m}}\right\}$$

$$= -z^2 \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \quad (27)$$

<u>Note on Infinite Products:</u> We can always write $(n_1 n_2 n_3 \ldots)(m_1 m_2 m_3 \ldots) =$
$(m_1 n_1)(m_2 n_2)(m_3 n_3)\ldots$ (28)

This explains why the factors $e^{-z/n}\, e^{z/m}$ cancel.

This also explains why we have written

$$(1+z/n)(1-z/m) \longrightarrow (1-z^2/n^2) \quad (29)$$

We next wish to show that <u>the r.h.s. of (27) is proportional to $\sin\pi z$.</u>

To see this we use an <u>infinite product representation for $\sin z$:</u>

$$\boxed{\sin z = z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)} \quad (30)$$

This can be made plausible by noting that for a finite polynomial $f_n(z)$ we can write:

$$f_n(z) = (z-z_1)(z-z_2)\cdots(z-z_n) = \prod_{i=1}^{n}(z-z_i) \quad (31)$$

where the z_i are the roots of $f_n(z_i)=0$. This is the <u>FUNDAMENTAL THEOREM OF ALGEBRA</u>

From (31) it is in any case not surprising that for a function such as $\sin z$ which has an infinite number of roots that it can be expressed as an infinite product. These roots are at $z = 0, \pm n\pi \Rightarrow z^2 = 0, n^2\pi^2$.

This justifies the overall factor of z (root at $z=0$) and the factor $(1 - z^2/n^2\pi^2)$.

It then follows from (30) that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{\pi^2 z^2}{n^2 \pi^2}\right) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{32}$$

Combining (27) & (32):

$$\frac{1}{\Gamma(z)} \frac{1}{\Gamma(-z)} = -z^2 \underbrace{\left[\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\right]}_{\frac{\sin \pi z}{\pi z}} = -\frac{z}{\pi} \sin \pi z \tag{33}$$

Inverting (33):

$$\Gamma(z)\Gamma(-z) = \frac{-\pi}{z \sin \pi z} \tag{34}$$

Next use the basic recurrence relation: $\Gamma(z+1) = z\,\Gamma(z) \xrightarrow{z \to -z} \Gamma(-z+1) = -z\,\Gamma(-z)$ (35)

Hence

$$\boxed{\Gamma(1-z) = -z\,\Gamma(-z)} \tag{36}$$

Combining (34) & (36) $\Rightarrow$

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{-\pi}{z \sin \pi z} \cdot (-z) = \frac{\pi}{\sin \pi z}} \tag{37}$$

Side Comment: WALLIS' FORMULA FOR $\pi/2$

It is of interest to have formulas for computing π. To date π has been calculated by a group in Japan to more than 10^9 places. Here is one (of many!) formulas for calculating π. Returning to (30) substitute $z = \pi/2$:

$$\sin \pi/2 = 1 = \frac{\pi}{2} \prod_{n=1}^{\infty}\left(1 - \frac{(\pi/2)^2}{n^2\pi^2}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty}\left(\frac{4n^2 - 1}{4n^2}\right) \tag{38}$$

Since this is an infinite product its reciprocal is just the reciprocal of each term:

Hence
$$\boxed{\pi/2 = \frac{1}{\prod_{n=1}^{\infty}\cdots} = \prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2-1}\right)} \quad (39)$$

WALLIS' FORMULA

Numerical Results: $\pi/2 = 1.570796327\ldots$; write $4n^2 = (2n)^2$ (40)
$4n^2-1 = (2n-1)(2n+1)$

$n=1 \Rightarrow \frac{\pi}{2} \cong \frac{(2\cdot1)^2}{1\cdot3} = \frac{4}{3} = 1.333\ldots$

$n=2 \Rightarrow \frac{\pi}{2} \cong \frac{4}{3}\left(\frac{4\cdot4}{3\cdot5}\right) = \frac{64}{45} = 1.422\ldots$ (41)

$n=3 \Rightarrow \frac{\pi}{2} \cong \frac{64}{45}\cdot\left(\frac{6\cdot6}{5\cdot7}\right) = \frac{64}{45}\cdot\frac{36}{35} = 1.4628\ldots$

Note that as n increases, each new approximation starts with the previous result and multiplies by a factor which gets ever closer to 1.

Singularities of $\Gamma(z)$:

We have seen previously that $\Gamma(-n) \to \infty$ when n = integer.
Here we use the formal result in (24) to prove this rigorously:

$$\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin\pi z}} \quad (42)$$

Since $\sin\pi z = 0$ when $z = \pm$ integer $\Rightarrow$ when $z = n$ = integer

Well behaved $(n-1)!$ $\longrightarrow \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin\pi n} = \infty \Rightarrow$ For $n \geq 1/2$ $\Gamma(1-n)$ must have a pole which agrees with (10) & (11)

For $n=1$, $\Gamma(1-n) = (1-n-1)! = (-1)! = \infty$, while $\Gamma(n=1) = 0! = 1$ ✓ (43)

For $n=0$, $\Gamma(1-n) = \Gamma(1) = 0!$ ok, but $\Gamma(n) = \Gamma(0) = (-1)! = \infty$ ✓

Hence the relation in (42) correctly gives all the expected poles of $\Gamma(z)$.

THE BETA FUNCTION

This is a function closely related to the Γ function, and arises often in physics applications. We begin with

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \qquad (1)$$

Consider, then, $$m!\,n! = \Gamma(m+1)\Gamma(n+1) = \lim_{a^2\to\infty} \int_0^{a^2} du\, e^{-u} u^m \cdot \int_0^\infty dv\, e^{-v} v^n \qquad (2)$$

Let $u = x^2$; $du = 2x\,dx$; $v = y^2$ $dv = 2y\,dy$; $u^2 = a^2 \Rightarrow x = a$; $v^2 = a^2 \Rightarrow y = a$ (3)

$$\therefore m!\,n! = \lim_{a\to\infty} \int_0^a (2x\,dx)\, e^{-x^2} x^{2m} \cdot \int_0^a (2y\,dy)\, e^{-y^2} y^{2n} \qquad (4)$$

$$= \lim_{a\to\infty} 4 \int_0^a dx\, e^{-x^2} x^{2m+1} \cdot \int_0^a dy\, e^{-y^2} y^{2n+1} \qquad (5)$$

Transforming to polar coordinates we have: $dx\,dy = r\,dr\,d\theta$; $x^2+y^2 = r^2$
$x = r\cos\theta$; $y = r\sin\theta$

Hence:
$$m!\,n! = \lim_{a\to\infty} 4 \underbrace{\int_0^a (r\,dr) e^{-r^2} r^{2m+1} r^{2n+1}}_{\int_0^a dr\, e^{-r^2} r^{2m+2n+3}} \int_0^{\pi/2} d\theta \cos^{2m+1}\theta \cdot \sin^{2n+1}\theta \qquad (6)$$

Next change variables again so that $r^2 = t$; $r\,dr = \frac{1}{2} dt$. Then

$$\lim_{a\to\infty} \int_0^a \cdots = \int \tfrac{1}{2} dt\, e^{-t} t^{m+n+1} = \frac{1}{2} \Gamma(m+n+2) = \tfrac{1}{2} (m+n+1)! \qquad (7)$$

Combining (6) & (7): $$m!\,n! = 4\left\{\frac{1}{2}(m+n+1)!\right\} \underbrace{\int_0^{\pi/2} d\theta \cos^{2m+1}\theta \cdot \sin^{2n+1}\theta}_{\equiv \frac{1}{2} B(m+1,\, n+1)} \qquad (8)$$

$$\boxed{\therefore B(m+1, n+1) = B(n+1, m+1) = \frac{m!\,n!}{(m+n+1)!}} \qquad (9)$$

Using the relation $\Gamma(n+1) = n!$ ⇒

$$B(m+1,n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \Rightarrow \boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}} \quad (10)$$

ALTERNATE DEFINITION OF BETA FUNCTION

The Beta function is useful in evaluating certain classes of integrals: If we substitute in Eq.(8) $t = \cos^2\theta$; $(1-t) = \sin^2\theta$ then $dt = -2\sin\theta\cos\theta\, d\theta$ and

$$B(m+1,n+1) = 2\int_0^{\pi/2} d\theta \cos^{2m+1}\theta \cdot \sin^{2n+1}\theta = \int_0^{\pi/2} \underbrace{(2d\theta\cos\theta\sin\theta)}_{-dt}\underbrace{\cos^{2m}\theta}_{t^m} \cdot \underbrace{\sin^{2n}\theta}_{(1-t)^n} \quad (11)$$

$$\boxed{\therefore\ B(m+1,n+1) = \int_0^1 dt\, t^m (1-t)^n} \quad (12)$$

The limits are obtained by noting that with $t = \cos^2\theta$, $\theta = 0 \Rightarrow t = 1$; $\theta = \pi/2 \Rightarrow t = 0$. Also $\int_1^0 (-dt)\ldots = \int_0^1 dt\ldots$

We can obtain yet another useful relation by making the substitution

$$t = \frac{u}{1+u} \Rightarrow dt = \frac{du}{(1+u)^2} \Rightarrow \boxed{B(m+1,n+1) = \int_0^\infty du\, \frac{u^m}{(1+u)^{m+n+2}}} \quad (13)$$

We can now use this result to formally derive the relation

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z \quad (14)$$

In Eq.(13) let $m = a-1$, $n = -a$ with a in the interval $0 < a < 1$.

Then

$$B(a,-a+1) = \int_0^\infty du\, \frac{u^{a-1}}{1+u}$$

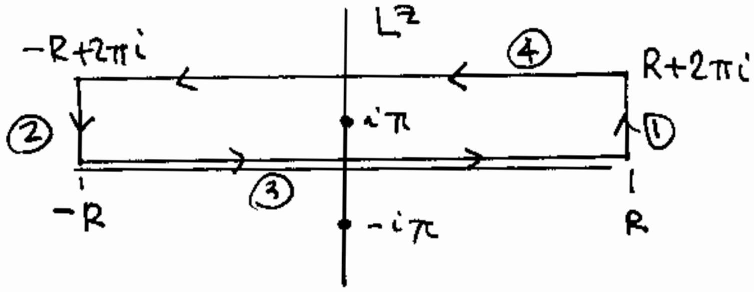
$$\hookrightarrow = \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(a+1-a)} = \Gamma(a)\Gamma(1-a)$$

$$\boxed{\Gamma(a)\Gamma(1-a) = \int_0^\infty du\, \frac{u^{a-1}}{1+u}} \quad (15)$$

There are a number of ways to evaluate this integral using contour integration. We follow the discussion of MACROBERT, <u>Functions of a Complex Variable</u>:

Begin by considering:
$$I = \int_{-\infty}^{\infty} dx\, \frac{e^{ax}}{1+e^x}\ ;\ 0<a<1 \quad (16)$$

MINI-REVIEW OF CONTOUR INTEGRATION



The integrand has poles at $z = \pm i\pi$, but for the contour shown only the pole at $z = +i\pi$ contributes. In (16) the variable is now complex:

$$I = \oint dz \, \frac{e^{az}}{1+e^{z}} \qquad 0<a<1 \tag{17}$$

From Cauchy's Formula: $\quad I = 2\pi i \left[\text{Residue at } (z=+i\pi)\right]$ (18)

Recall that the residue at z_0 is the coefficient of $1/(z-z_0)$ obtained when the integrand is expanded about z_0. Suppose we have $I = \oint dz \, g(z)$ where $g(z) = f(z)/h(z)$ with $h(z_0)=0$. Then we can expand

$$g(z) = \frac{f(z)}{h(z)} = \frac{f(z_0) + (z-z_0)f'(z_0) + \frac{1}{2!}(z-z_0)^2 f''(z_0)+\dots}{\underbrace{h(z_0)}_{\substack{/\!/ \\ 0 \text{ at pole}}} + (z-z_0)h'(z_0) + \frac{1}{2!}(z-z_0)^2 h''(z_0)+\dots} \tag{19}$$

The function $f(z_0) \neq 0$ in general: $\Rightarrow \text{Res}\{g(z)\}_{z_0} = \text{coeff of} \left(\frac{1}{z-z_0}\right)$ in (19) where this is evaluated as $z \to z_0$:

In this limit the only surviving term in the numerator is $f(z_0)$. In the denominator the leading term as $(z-z_0)\to 0$ is $(z-z_0)h'(z_0)$. Hence near z_0

$$\boxed{g(z) \cong \frac{f(z_0)}{(z-z_0)h'(z_0)} \Rightarrow \text{Residue at } z_0 = \frac{f(z_0)}{h'(z_0)}} \tag{20}$$

In the present case: $\quad g(z) = \dfrac{e^{az}}{1+e^{z}} \Rightarrow f(z) = e^{az}\ ; \ h(z) = 1+e^{z}$ (21)

Then $f(z_0) = e^{ai\pi}$;

$$h'(z_0) = \frac{d}{dz}\left(1+e^{z}\right)_{z_0} = e^{i\pi} = -1 \Rightarrow \boxed{\text{Residue} = \frac{e^{ai\pi}}{-1} = -e^{ai\pi}} \tag{22}$$

Combining (18) & (22) we then have

$$\boxed{I = \oint dz\, \frac{e^{az}}{1+e^{z}} = 2\pi i\left(-e^{ai\pi}\right) = -2\pi i\, e^{ai\pi}} \qquad (23)$$

This gives the value of $\oint$ over the entire contour shown on the previous page. We now have to extract from this the integral we want, which is just the contribution from leg ③ of the contour.

Consider first the vertical piece ① going from $z = R \to z = R + 2\pi i$

Along this piece

$$|g(z)|_{①} = \left|\frac{e^{az}}{1+e^{z}}\right| \leq \frac{|e^{az}|}{-1+|e^{z}|} = \frac{|e^{aR}|\,\overbrace{|e^{iay}|}^{=1}}{-1+|e^{R}|\,\underbrace{|e^{iy}|}_{1}} = \frac{e^{aR}}{-1+e^{R}} \qquad (24)$$

It follows that along ① $\quad |g(z)| \leq \dfrac{e^{aR}}{e^{R}-1} \xrightarrow{R\to\infty} 0 \quad$ (Recall: $a<1$) $\qquad (25)$

<u>Note</u>: the (–) sign in the denominator of (24) comes from the fact that half way up this leg $y = i\pi \Rightarrow e^{i\pi} = -1$. Hence the minimum value of the denominator (and hence the maximum of $g(z)$) occurs there.

Consider next the vertical piece ②: We can replace R by $-R$ in (24):

$$\therefore\ |g(z)|_{②} \leq \frac{e^{-aR}}{1-e^{-R}} \xrightarrow{R\to\infty} 0 \qquad (26)$$

Whereas in (25) it was the denominator which made $|g(z)|_{①} \to 0$, now in (26) it is the numerator which does the job.

We are thus left with the two horizontal pieces ③ and ④:

$$\oint \cdots = \oint_{③} + \oint_{④} \overset{R\to\infty}{=} \int_{x=-\infty}^{\infty} dx\, \frac{e^{ax}}{1+e^{x}} + \int_{x=+\infty}^{x=-\infty} dx\, \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} \qquad (27)$$

Note that in evaluating ④ the integration limits are really $\pm\infty + 2\pi i$. However, in the complex plane we have effectively

$$\pm\infty + 2\pi i \triangleq \pm\infty$$

In the denominator of (27) $e^{x+2\pi i} = e^x$. But in the numerator

$$e^{a(x+2\pi i)} = e^{ax}\, e^{a2\pi i} \longleftarrow \text{this} \neq 1 \text{ since } a \text{ can be any number in } 0<a<1$$

Hence from (27):
$$\oint \cdots = \int_{-\infty}^{\infty} dx\, \frac{e^{ax}}{1+e^x} - \int_{-\infty}^{\infty} dx\, \frac{e^{ax} e^{2\pi i a}}{1+e^x} = \left(1-e^{a2\pi i}\right)\int_{-\infty}^{\infty} dx\, \frac{e^{ax}}{1+e^x} \tag{28}$$

From (23) we know the value of $\oint$ from the residue theorem: Hence

$$\oint \cdots = \underbrace{-2\pi i e^{a i\pi}}_{(23)} = \left(1-e^{a2\pi i}\right)\int_{-\infty}^{\infty} dx\, \frac{e^{ax}}{1+e^x} \Longleftarrow (28) \tag{29}$$

Hence: $$\int_{-\infty}^{\infty} dx\, \frac{e^{ax}}{1+e^x} = \frac{+2\pi i\, e^{a\pi i}}{e^{a2\pi i}-1} = \frac{2\pi i\, e^{a\pi i}}{e^{a\pi i}\left(e^{a\pi i} - e^{-a\pi i}\right)} \tag{30}$$

Finally:
$$\boxed{I = \int_{-\infty}^{\infty} dx\, \frac{e^{ax}}{1+e^x} = \frac{\pi}{\sin a\pi}} \tag{31}$$

To get to the actual integral we want substitute $y = e^x$ in (31):

$x=\infty \Rightarrow y=\infty$; $x=-\infty \Rightarrow y=0$; $dx = dy/y$

$$\therefore I = \int_0^{\infty} \frac{dy}{y}\, \frac{y^a}{1+y} = \int_0^{\infty} dy\, \frac{y^{a-1}}{1+y} = \frac{\pi}{\sin\pi a} \tag{32}$$

Combining this with (15) above $\Rightarrow$

$$\boxed{\int_0^{\infty} dy\, \frac{y^{a-1}}{1+y} = \Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin\pi a}} \tag{33}$$

METHOD OF STEEPEST DESCENTS
(SADDLE POINT METHOD)

From p. 111(18) we saw that $\Gamma(z)$ can be represented by the integral

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \qquad \mathrm{Re}\, z > 0 \Rightarrow \boxed{n! = \Gamma(n+1) = \int_0^\infty dt\, t^n e^{-t}} \quad (1)$$

Now in many applications (e.g. Statistical mechanics) we encounter combinatoric problems when $n \gg 1$. We would like to find a simple formula for $n!$ when $n \gg 1$. Using the integral representation in (1) and the method of steepest descents we can obtain an approximate formula for $n!$:

$$\boxed{n! \cong \sqrt{2\pi n}\; n^n e^{-n}} \quad (2) \qquad \text{STIRLING'S FORMULA}$$

The Method: Consider an integral of the form $\boxed{I(s) = \int_C dz\, g(z)\, e^{s f(z)}}$ (3)

Eq. (1) is not quite in this form yet, but we will recast the integral to make it have this form. The contour C is defined by whatever the integral representation is: For $\Gamma(z)$ it is the interval $[0, \infty]$ along the real axis, but it will in general be some contour in the $\angle z$ plane.

The basic idea is that if $\mathrm{Re}\, f(z) \to 0$ as $z \to \pm\infty$ (e.g. e^{-sz^2}) then we can approximate the actual contour by a straight line which coincides with the contour in the vicinity of the maximum of $f(z) \equiv f(z_0)$. This then gives the formula we want.

Details: Write $f(z) = u(x,y) + i v(x,y) \Rightarrow I(s) = \int_0 dz\, g(z)\, e^{s u(x,y)}\, e^{i s v(x,y)}$ (4)

The integral will be dominated by the region where $u(x,y) \simeq$ maximum provided $v(x,y) = v(x_0, y_0) \equiv v_0 \cong$ constant. Then

$$\boxed{I(s) \cong g(z_0)\, e^{i s v_0} \int dz\, e^{s u(x,y)}} \quad (5)$$

To find the maximum of $u(x,y)$:

$$\left.\begin{aligned}\frac{\partial u(x,y)}{\partial x} &= \frac{\partial u(x,y)}{\partial y} = 0\\ &\quad\ \ \| \\ &\quad -\frac{\partial v}{\partial x} = 0\end{aligned}\right\} \text{Recall that if } f(z) \text{ is analytic then } \frac{df}{dz} = \frac{\partial f}{\partial x} = \underbrace{\frac{\partial u}{\partial x}}_{0} + i\underbrace{\frac{\partial v}{\partial x}}_{0} = 0 \qquad (6)$$

$$\boxed{\text{Hence to find the maximum of } u(x,y) \text{ simply evaluate } df/dz = 0 \Rightarrow z_0} \quad (7)$$

CAUTION! Recall from LIOUVILLE'S THEOREM that an analytic function cannot have an absolute maximum in the complex plane, (*) so $df/dz = 0$ does not give an absolute maximum: What it does give is a saddle point, which is a maximum of $u(x,y)$ and a minimum of $v(x,y)$ which is what we want. By orienting the integration path appropriately at z_0 we can move along a curve where z_0 is a maximum, and this gives the approximation we want.

(*) Recall from last semester:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \equiv \nabla^2 u(x,y)$$

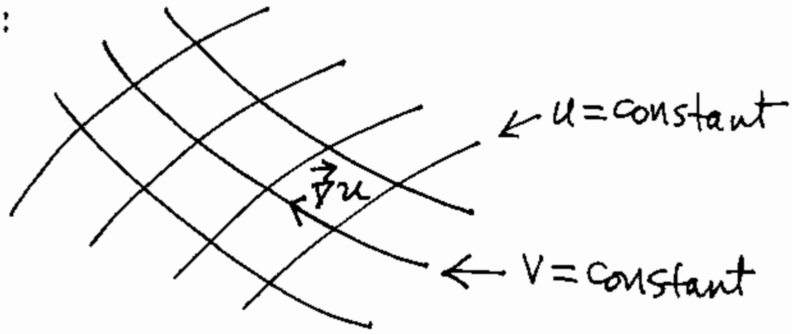
$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0 \quad \Rightarrow \boxed{\nabla^2 u = 0} \quad (8)$$

Since $\nabla^2 u = 0$ (and also $\nabla^2 v = 0$) $\Rightarrow \boxed{\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}}$ (9)

This means that if we are at an extremum where $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$, then the 2ND derivative will be positive if we move in the x-direction (indicating a minimum) but negative in the y-direction (indicating a maximum) – or vice versa. This is exactly what happens in a saddle, which is why this method is also called the saddle point method: We are approximating the integral $I(s)$ by its value in the vicinity of the saddle point.

This can be pictured graphically as follows:

x-y plane:



As indicated in the figure, the contours giving $\vec{\nabla} u$ (which gives the direction in which u is changing most rapidly) are $\perp$ to the lines $u=$ const. Since these lines are also $\perp$ to those giving $V=$ const it follows that $\vec{\nabla} u$ (i.e. maximum change in u) is in the direction $V=$ const (minimum change in V) which is exactly what we want for the integral in (5). Hence at the saddle point we will be looking for the direction of $\vec{\nabla} u$.

Computational details: $$I(s) = \int_{\mathcal{C}} dz\, g(z)\, e^{s f(z)} \qquad (10)$$

1) Compute: $\dfrac{df}{dz} = 0 \Rightarrow z_0 =$ saddle point

2) Expand: $$f(z) - f(z_0) \cong \underset{0}{\underset{\parallel}{f'(z_0)}} + \frac{1}{2}\underbrace{(z-z_0)^2}_{\text{we choose}}\underbrace{f''(z_0)}_{\text{given}} \equiv \frac{-1}{2s}t^2 \qquad (11)$$

The content of (11) is that we can choose a path in the LZ plane to ensure that the phase in (11) is negative. From the preceding discussion this will lie along $\vec{\nabla} u$.

3) To find the path: Let $(z-z_0) = \delta e^{i\alpha} \Rightarrow (z-z_0)^2 = \delta^2 e^{2i\alpha}$ (12)

Choosing α specifies the direction of the path in the LZ plane

Also, write: $f''(z_0) = |f''(z_0)|\, e^{i\phi_f}$; ϕ_f is determined for us by $f(z_0)$ (13)

Hence in Eq. (11):

$$f(z)-f(z_0) \cong \frac{1}{2}(z-z_0)^2 f''(z_0) = \frac{1}{2}\delta^2 |f''(z_0)|\, e^{2i\alpha + i\phi_f} \equiv -\frac{1}{2s}t^2 \qquad (14)$$

Note that for this method to work the r.h.s. of (14) must be negative (& real)

Hence, since α can be chosen by us, we must (and can!) choose it such that

$$\phi_f + 2\alpha = \pm\pi \Rightarrow \boxed{\alpha = \frac{1}{2}(\pm\pi - \phi_f)} \qquad (15)$$

Then $dz = d\delta\, e^{i\alpha}$; $t = \pm\delta |s f''(z_0)|^{1/2} \Rightarrow d\delta = \dfrac{\pm dt}{|s f''(z_0)|^{1/2}}$ (16)

$$\boxed{dz = \frac{\pm e^{i\alpha}\, dt}{|s f''(z_0)|^{1/2}}} \qquad (17)$$

$$\Rightarrow I(s) = \int_C dz\, g(z) e^{s f(z)} \cong g(z_0) \int_C e^{s\overbrace{[f(z_0) - \frac{1}{2s}t^2]}^{f(z)}} \overbrace{\frac{(\pm)\, dt\, e^{i\alpha}}{|s f''(z_0)|^{1/2}}}^{dz} \qquad (18)$$

Hence

$$\boxed{I(s) \cong \frac{g(z_0) e^{s f(z_0)}\, e^{i\alpha}}{|s f''(z_0)|^{1/2}} \int_{-\infty}^{\infty} dt\, e^{-t^2/2} = \frac{\sqrt{2\pi}\, g(z_0)\, e^{s f(z_0)}\, e^{i\alpha}}{|s f''(z_0)|^{1/2}}} \qquad (19)$$

MASTER FORMULA

Method of Steepest Descents: Applications:

[1] Stirling's Formula:

We begin with the integral representation of the Γ-function:

$$\Gamma(n+1) = n! = \int_0^\infty dt\, t^n e^{-t} \qquad (1)$$

To cast this in the form of 125(3): $I(s) = \int_C dz\, g(z)\, e^{sf(z)}$ (2)

We write:

$$t^n = e^{n\ln t} \Rightarrow n! = \int_0^\infty dt\, e^{-t} e^{n\ln t} = \int_0^\infty dt\, e^{n\ln t - t} \qquad (3)$$

Substitute: $t = zn \Rightarrow \ln t = \ln n + \ln z$

$$\boxed{n! = \int_0^\infty dt \cdots = \int_0^\infty (n\,dz)\, e^{n(\ln n + \ln z - z)} = n^{n+1}\int_0^\infty dz\, e^{n(\ln z - z)}} \qquad (4)$$

To connect with the notation in (2) let $n \to s$ so that

$$I(s) = s! = \int_0^\infty dz \cdot \underbrace{s^{s+1}}_{g(z)} \cdot\, e^{s\overbrace{(\ln z - z)}^{f(z)}} \qquad (5)$$

Thus:

$$f(z) = \ln z - z \Rightarrow f'(z) = \frac{1}{z} - 1 \equiv 0 \Rightarrow z = 1 \equiv z_0 \qquad (6)$$

$$\boxed{f(z_0) = -1}$$

$$f''(z_0) = -\frac{1}{z^2}\Big|_{z=z_0=1} = -1 \equiv 1 e^{\pm i\pi} \equiv 1 e^{i\phi_f} \Rightarrow \boxed{\phi_f = \pm\pi} \qquad (7)$$

We next compute the phase α in 130.1(15): $\alpha = \frac{1}{2}(\pm\pi - \phi_f)$ (8)

$$\boxed{\therefore \alpha = \frac{1}{2}(\pm\pi \pm\pi) = 0, \pm\pi \Rightarrow e^{i\alpha} = \pm 1} \qquad (9)$$

We resolve the sign ambiguity below. Inserting the various results above into the MASTER FORMULA we have:

$$S! \equiv I(s) \cong \frac{\sqrt{2\pi}\, g(z_0)\, e^{sf(z_0)}\, e^{i\alpha}}{|s f''(z_0)|^{1/2}} = \frac{\sqrt{2\pi}\cdot s^{s+1}\cdot e^{s(-1)}\cdot(\pm 1)}{|s\cdot(-1)|^{1/2}} \qquad (10)$$

$$\boxed{\therefore\ I(s) = s! \cong +\sqrt{2\pi s}\ s^s e^{-s}} \qquad (11)\ \text{STIRLING'S FORMULA}$$

Note that we have chosen the (+) phase in (10) on the common sense basis that n! must be a positive number.

Mini Review:

$$I(s) = \int_C dz\, g(z)\, e^{sf(z)} \qquad (12)$$

$$\text{Expand: } f(z) \cong f(z_0) + \tfrac{1}{2}(z-z_0)^2 f''(z_0) + \dots \quad ; \quad f''(z_0) = \rho e^{i\phi_f}, \quad (z-z_0) = \delta e^{i\alpha} \qquad (13)$$

$$f(z) = f(z_0) + \tfrac{1}{2}\delta^2 e^{2i\alpha}\cdot\rho e^{i\phi_f} = f(z_0) + \tfrac{1}{2}\rho\delta^2 e^{i(\phi_f+2\alpha)} \qquad (14)$$

$$\therefore\ u(x,y) = \mathrm{Re}\, f(z) = u_0(x_0,y_0) + \tfrac{1}{2}\rho\delta^2\cos(\phi_f+2\alpha) \qquad (15)$$

$$\therefore\ v(x,y) = \mathrm{Im}\, f(z) = v_0(x_0,y_0) + \tfrac{1}{2}\rho\delta^2\sin(\phi_f+2\alpha) \qquad (16)$$

From (16) we see that as we descend from the saddle point x_0, y_0, the condition that V_0 remain $\cong$ constant is that $\sin(\phi_f+2\alpha)=0$ [as δ varies] $\Rightarrow$

$$\phi_f + 2\alpha = 0,\ \pm\pi$$

For $(\phi_f+2\alpha)=0 \Rightarrow \cos(\phi_f+2\alpha)=+1 \Rightarrow u(x,y) \cong u_0 + \tfrac{1}{2}\rho\delta^2$;

Hence $(\phi_f+2\alpha)=0 \Rightarrow u(x,y)$ becomes more positive as δ increases away from the saddle point. This is not what we want! What we want is for $u(x,y)$ to decrease away from the saddle point, which can be achieved by choosing $(\phi_f+2\alpha)=\pm\pi$. Then $\cos(\phi_f+2\alpha)=-1 \Rightarrow u(x,y) = u_0(x_0,y_0) - \tfrac{1}{2}\rho\delta^2$

Since $v(x,y) \cong v_0(x_0,y_0) + 0$ in this case, the choice $(\phi_f+2\alpha)=\pm\pi$ works to produce the most rapid variation of u, while V remains constant.

Comments on Stirling's Formula:

Stirlings formula is a good approximation even for reasonably small n.

n	$n!$	$\sqrt{2\pi n}\, n^n e^{-n}$
5	120	118
10	3.63×10^6	3.60×10^6
4	24	23.5
3	6	5.84
2	2	1.92
1	1	0.92

Comparing $n!$ and e^n for large n:

From the Stirling formula $n! \cong \sqrt{2\pi n}\; n^n e^{-n} \Rightarrow$

$$\frac{n!}{e^n} \cong \frac{\sqrt{2\pi n}\; n^n e^{-n}}{e^n} = \sqrt{2\pi n}\; n^n e^{-2n} = \sqrt{2\pi n}\,(n/e^2)^n$$

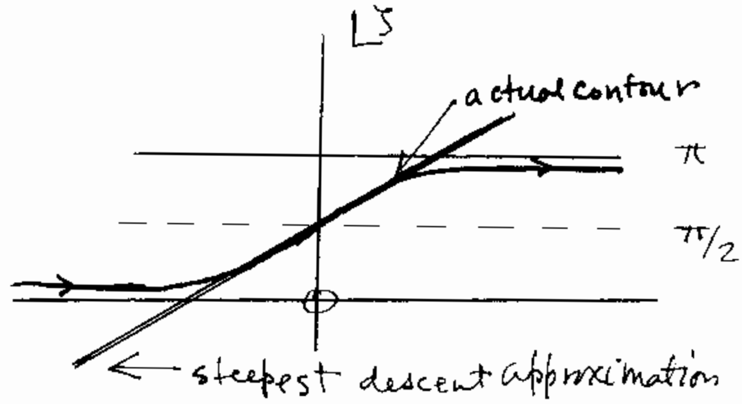
Hence for $n \geq 8$ $n/e^2 \geq 8/7.39 = 1.08 > 1 \Rightarrow (n/e^2)^n > 1 \Rightarrow \underline{n!/e^n > 1}$

Another application of the method of steepest descents:

The Hankel function $H_\lambda^{(1)}(z)$ has the following integral representation:

$$H_\lambda^{(1)}(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty+\pi i} d\zeta\; e^{(z\sinh\zeta - \lambda\zeta)} \qquad (1)$$

where the contour is as shown:



We wish to find an approximation to (1) for large z. We will assume that the region of interest is $z > \lambda$, so that we can substitute $z = \lambda\sec\beta$. (2)

Here $\beta > 0$ and in the interval $\frac{\pi}{2} > \beta > 0$. Using the method of Steepest descents we wish to show that if β is held constant so that $z \to \infty$, $\lambda \to \infty$ then

$$\boxed{H_\lambda^{(1)}(z) = H_\lambda^{(1)}(\lambda\sec\beta) \cong \frac{e^{i\lambda(\tan\beta - \beta) - i\pi/4}}{\sqrt{\frac{\pi\lambda}{2}\tan\beta}}} \qquad (3)$$

Method: In terms of β: $H_\lambda^{(1)} = \frac{1}{\pi i}\int_{-\infty}^{+\infty+\pi i} d\zeta\; e^{(\lambda\sec\beta\sinh\zeta - \lambda\zeta)}$

$$H_\lambda^{(1)} = \frac{1}{\pi i}\int_{-\infty}^{+\infty+\pi i} d\zeta\; e^{\lambda\sec\beta(\sinh\zeta - \zeta\cos\beta)} \qquad (4)$$

In terms of our MASTER FORMULA: $s \to \lambda\sec\beta$; $f(z) \to f(\zeta) = \sinh\zeta - \zeta\cos\beta$

$g(z) \to 1/\pi i$ (5)

Step[1]: The saddle point is determined by $df/d\zeta = 0 = \cosh\zeta - \cos\beta$ (6)

Recall: $\cosh\zeta = \frac{1}{2}(e^{\zeta} + e^{-\zeta})$; $\cos\beta = \frac{1}{2}(e^{i\beta} + e^{-i\beta})$ $\Rightarrow$ $\boxed{\zeta_0 = \pm i\beta}$ (7)

Both signs give same result. Use $\zeta_0 = +i\beta$.

We now know that the path of steepest descent must pass through the point $\zeta_0 = i\beta$.

Step [2]: We next compute $f''(\zeta_0)$: $f''(\zeta) = \frac{d}{d\zeta}(\cosh\zeta - \cos\beta) = \sinh\zeta$ (8)

$\therefore \sinh\zeta_0 = f''(\zeta_0) = \sinh(i\beta) = \frac{1}{2}(e^{i\beta} - e^{-i\beta}) = i\sin\beta = \sin\beta\, e^{i\pi/2}$ (9)

$$\therefore\ f''(\zeta_0) = \sin\beta\, e^{i\pi/2} \equiv \rho e^{i\phi_f} \Rightarrow \boxed{\phi_f = \pi/2} \quad (10)$$

Hence in the MASTER FORMULA: $\alpha = \frac{1}{2}(\pm\pi - \phi_f) = \frac{1}{2}(\pm\pi - \frac{\pi}{2})$

$$\boxed{\therefore \alpha = \frac{1}{2}\left(\frac{\pi}{2} \text{ or } -\frac{3\pi}{2}\right) = \pi/4 \text{ or } -3\pi/4} \quad (11)$$

These evidently give the same line: Recall from 132 (13) that α is given by:

$$(\zeta - \zeta_0) \equiv \delta e^{i\alpha} = \delta e^{i\pi/4} \quad (12)$$

so that α specifies the direction of the (straight-line) path that the path of integration takes in the method of steepest descents. Note that in the present case this is a 45° line tangent to the original contour at $\zeta_0 = i\beta$. [See figure on previous page]. Note also that $\pi/4$ & $3\pi/4$ are the same line, only differing in the sense of the contour.

Step [3]: We combine the previous results into the MASTER FORMULA:

$$I(s) \cong \frac{\sqrt{2\pi}\cdot g(z_0)\cdot e^{sf(z_0)}\cdot e^{i\alpha}}{|sf''(z_0)|^{1/2}} \rightarrow H_\lambda^{(1)}(\lambda\sec\beta) \cong \frac{\sqrt{2\pi}\cdot\left(\frac{1}{\pi i}\right)\cdot e^{\lambda\sec\beta(i\sin\beta - i\beta\cos\beta)} \otimes e^{i\pi/4}}{\sqrt{|\lambda\sec\beta|\,|i\sin\beta|}} \quad (13)$$

Noting that $\frac{1}{i}e^{i\pi/4} = e^{-i\pi/4}$ gives:

$$\boxed{H_\lambda^{(1)}(\lambda\sec\beta) \cong \frac{e^{i\lambda(\tan\beta - \beta)}\ e^{-i\pi/4}}{\sqrt{\frac{\pi\lambda}{2}\tan\beta}}} \quad (14)$$

ASYMPTOTIC EXPANSIONS

The method of steepest descents (saddle point method) leads naturally into the subject of asymptotic expansions or asymptotic series:
The approximate result contained in the MASTER FORMULA on p. 130.1 can be shown to be the first term in an asymptotic expansion.

Convergent series: $f(z) = \sum_{n=0}^{N} a_n z^n$; series approaches $f(z)$ for fixed z as $N \to \infty$ (1)

Asymptotic series: $f(z) = \sum_{n=0}^{N} b_n \frac{1}{z^n}$; Series approaches $f(z)$ for fixed N as $z \to \infty$ (2)

Example: The incomplete Gamma Function: $I(x,p) = \int_x^\infty du\, e^{-u} u^{-p}$ (3)

Evidently: $I(0,p) = \int_0^\infty du\, e^{-u} u^{-p} = \Gamma(1-p)$ (4)

Integrating (3) by parts gives: $I(x,p) = -\frac{e^{-u}}{u^p}\Big|_x^\infty - \int_x^\infty (-e^{-u})(-p\,u^{-p-1})\,du$ (5)

$$\therefore\ I(x,p) = \frac{e^{-x}}{x^p} - p\int_x^\infty du\, e^{-u} u^{-p-1} \tag{6}$$

Continuing in this manner we develop the following series:

$$I(x,p) = \frac{e^{-x}}{x^p} - p\frac{e^{-x}}{x^{p+1}} + p(p+1)\frac{e^{-x}}{x^{p+2}} - p(p+1)(p+2)\frac{e^{-x}}{x^{p+3}} + p(p+1)(p+2)(p+3)\int_x^\infty du\, e^{-u} u^{-p-4} \tag{7}$$

After many such integrations:

$$\boxed{I(x,p) = e^{-x}\left\{\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \frac{p(p+1)(p+2)}{x^{p+3}} + \cdots\right\} + \frac{(-1)^n (p+n-1)!}{(p-1)!}\int_x^\infty du\, e^{-u} u^{-p-n}} \tag{8}$$

We can verify [redacted] the general expression in (8) by noting that the expression in (7) corresponds to $n=4$, so that from (8) the coefficient of the integral should be:

$$\text{Coefficient in (7)} = (-1)^4 \frac{(p+4-1)!}{(p-1)!} = +\frac{(p+3)(p+2)(p+1)p(p-1)!}{(p-1)!} = (p+3)(p+2)(p+1)p \quad \checkmark \tag{9}$$

We next test the series in (8) for convergence using the d'Alembert ratio test:

By inspection:

$$\boxed{\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{(p+n)!}{(p+n-1)!}\frac{1}{x} = \lim_{n\to\infty}\frac{p+n}{x} = \infty} \tag{10}$$

This is the key equation to understanding what an asymptotic series is:

$$\frac{p+n}{x} \begin{cases} \xrightarrow{\ \infty\ } x\text{-fixed};\ n\to\infty \\ \xrightarrow[\ 0\ ]{} n\text{-fixed};\ x\to\infty \end{cases} \tag{11}$$

Hence there is no value of x for which the series in (8) converges formally.

Nonetheless we can show that this series is a good numerical approximation to $I(x,p)$.

Consider the partial sum $S_n(x,p)$ defined by

$$I(x,p) \equiv S_n(x,p) + (-1)^{n+1}\frac{(p+n)!}{(p-1)!}\int_x^\infty du\, e^{-u} u^{-p-n-1} \quad \Rightarrow \tag{12}$$

$$\left|I(x,p) - S_n(x,p)\right| \leq \frac{(p+n)!}{(p-1)!}\int_x^\infty du\, \underbrace{\left|e^{-u}\right|}_{\leq 1}\left|u^{-p-n-1}\right| \leq \frac{(p+n)!}{(p-1)!}\int_x^\infty du\, u^{-p-n-1}$$

$$\left|I(x,p) - S_n(x,p)\right| \leq \frac{(p+n)!}{(p-1)!}\left|\frac{1}{u^{p+n}}\left(\frac{1}{p+n}\right)\right|_\infty^x = \frac{(p+n-1)!}{(p-1)!}\frac{1}{x^{p+n}} \tag{13}$$

It follows from (13) that as $x\to\infty$ $|I(x,p)-S_n(x,p)|\to 0$, so that for fixed n $S_n(x,p)$ approaches the exact result $I(x,p)$. Hence such an asymptotic series is perfectly good for numerical computations, even though it does not formally converge to $I(x,p)$.

Numerical Results: Following ARFKEN we examine the case $I(x,p=1)$:

$$I(x,p=1)\equiv E_1(x)=\int_x^\infty du\, e^{-u}u^{-1} \Rightarrow e^x E_1(x)=\frac{1}{x}-\frac{1!}{x^2}+\frac{2!}{x^3}-\frac{3!}{x^4}+\frac{4!}{x^5}-\cdots \tag{14}$$

As we will see, another difference between a convergent series and an asymptotic series is that including more terms does not necessarily give a better numerical result. Instead there is an optimum number of terms, which in this case is $n=5$. Here are the results for $x=5$:

$$S_1=\frac{1}{x}=0.2000 \qquad S_2=\frac{1}{x}-\frac{1}{x^2}=\frac{1}{5}-\frac{1}{25}=0.1600$$

$$S_3=0.16+\frac{2}{125}=0.1760 \qquad S_4=0.1760-\frac{6}{625}=0.1664$$

$$S_5=0.1664+\frac{24}{5\times 625}=0.1741 \qquad S_6=0.1741-\frac{120}{25\times 625}=0.1664$$

$$S_7=0.1664+\frac{6!}{5^7}=0.1756 \qquad S_8=0.1756-\frac{7!}{5^8}=0.1627$$

We see that S_n for $n=$ even are all smaller than S_n for $n=$ odd. As shown in the accompanying figure, the best numerical approximation is obtained at the point of closest approach of the even and odd S_n:

$$\boxed{0.1664\le e^x E_1(x)\big|_{x=5}\le 0.1741 \quad \text{exact value} \to 0.1704} \tag{15}$$

Another Example of an Asymptotic Expansion:

Cosine and Sine Integrals

$$Ci(x) = -\int_x^\infty dt\, \frac{\cos t}{t} \quad ; \quad Si(x) = -\int_x^\infty dt\, \frac{\sin t}{t} \qquad (1)$$

We can also define the related functions: $\boxed{f(x) = Ci(x)\sin x - Si(x)\cos x}$ (2)

Hence:

$$f(x) = -\int_x^\infty \frac{dt}{t}\,(\underbrace{\sin x\cos t - \cos x\sin t}_{\sin(x-t)}) = -\int_x^\infty dt\, \frac{\sin(x-t)}{t} \qquad (3)$$

Let $y = t - x$ (for fixed x) $\Rightarrow$ $dy = dt$; $t = \infty \Rightarrow y = \infty$; $t = x \Rightarrow y = 0$

$$\boxed{\therefore f(x) = -\int_0^\infty \frac{dy}{y+x}\sin(-y) = \int_0^\infty dy\, \frac{\sin y}{y+x}} \qquad (4)$$

Similarly: $g(x) \equiv -Ci(x)\cos x - Si(x)\sin x = \int_x^\infty dt\, \frac{(\cos t\cos x + \sin t\sin x)}{t}$ (5)

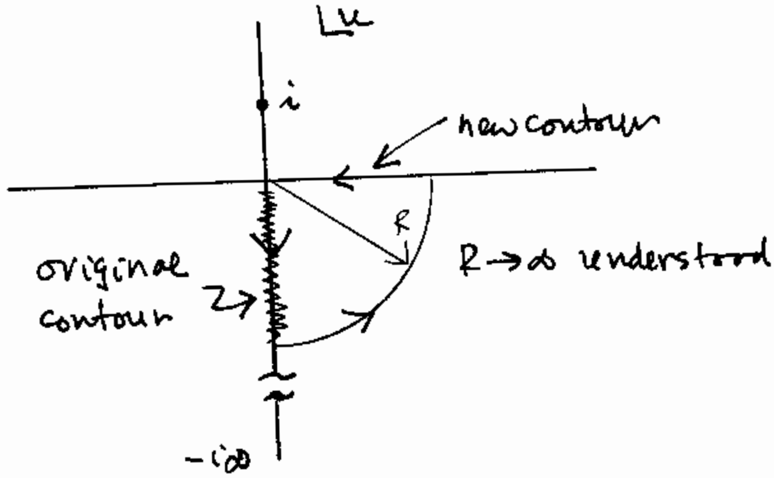
$$\boxed{\therefore g(x) = \int_x^\infty dt\, \frac{\cos(t-x)}{t} = \int_0^\infty dy\, \frac{\cos y}{y+x}} \qquad (6)$$

From (4) & (6): $g(x) + if(x) = \int_0^\infty dy\, \frac{(\cos y + i\sin y)}{y+x} = \int_0^\infty dy\, \frac{e^{iy}}{y+x}$ (7)

Let $u \equiv -iy/x$ (for fixed x) $\Rightarrow y = iux \Rightarrow dy = ix\,du$; $y = 0 \Rightarrow u = 0$
$y = \infty \Rightarrow u = -i\infty$

$$\therefore g(x) + if(x) = \int_0^{-i\infty} \frac{(ix\,du)\,e^{-ux}}{iux + x} = i\int_0^{-i\infty} \frac{du\, e^{-ux}}{1+iu} \qquad (8)$$

We wish to evaluate this integral by contour integration noting that the integrand has a singularity (simple pole) at $u = +i$.



For the contour shown $\oint_c = 0$ since the only singularity lies outside the contour.

$$\text{Thus} \quad 0 = \oint_c = \int_0^{-i\infty} + \int_{R\to\infty} + \int_\infty^0 \tag{9}$$

As usual we argue that $\int_R \to 0$ since it is damped by the exponential factor e^{-ux}.

<u>Note</u>: This argument applies because no part of $\int_R$ gets near the origin as $R\to\infty$; However the same is not true for the other two contributions in (9). Hence (9) ⇒

$$\oint = 0 = \int_0^{-i\infty} + 0 + \int_\infty^0 \Rightarrow \quad \int_0^{-i\infty} = -\int_\infty^0 = +\int_0^\infty \tag{10}$$

$$\text{Hence in (8):}\quad g(x) + i f(x) = i\int_0^\infty du\, \frac{e^{-ux}}{1+iu} = \int_0^\infty du\, \frac{(u+i)\,e^{-ux}}{1+u^2} \tag{11}$$

Equating real and imaginary parts in (11) ⇒

$$g(x) = \int_0^\infty du\, \frac{u\,e^{-ux}}{1+u^2} \quad ; \quad f(x) = \int_0^\infty du\, \frac{e^{-ux}}{1+u^2} \tag{12}$$

For these integrals to converge we must have $\mathrm{Re}\, x > 0$ so that the exponential is damped. [Also recall that from (7) & (8) $f(x)$ & $g(x)$ are real.]

We can evaluate (12) as asymptotic expansions by defining

$$uX = V \quad \Rightarrow x\,du = dV \tag{13}$$

Then $g(x) = \frac{1}{x^2}\int_0^\infty dv \, \frac{ve^{-v}}{1+v^2/x^2}$; $f(x) = \frac{1}{x}\int_0^\infty dv \, \frac{e^{-v}}{1+v^2/x^2}$ (14)

KEY POINT!!! Here is where the asymptotic expansion enters!

We wish to expand the denominators so as to evaluate the integrals by an infinite series:

$$\frac{1}{1+W} = 1 - W + W^2 - W^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n W^n \; ; \; \boxed{W = v^2/x^2} \quad (15)$$

However, <u>such an expansion only makes sense when $W = v^2/x^2 < 1$.</u> The problem is that whatever the (finite) value of x is, <u>v^2/x^2 will be >1 at some point in the integration</u>, since $0 \le v \le \infty$. Hence expanding as in (15) does not seem to make sense mathematically. However, for x sufficiently large, $v^2/x^2 > 1$ will only occur for values of v that are sufficiently large that they make a negligible contribution to the integral, due to the damping factor e^{-v}. Hence for sufficiently large x, we can use (15) in (14) knowing that the numerical error will be negligible. <u>This is specifically where the concept of an asymptotic expansion enters.</u> Combining (14) & (15) we then have:

$$f(x) \cong \frac{1}{x}\int_0^\infty dv\, e^{-v} \sum_{n=0}^{\infty} (-1)^n \frac{v^{2n}}{x^{2n}} \; ; \; g(x) \cong \frac{1}{x^2}\int_0^\infty dv\, e^{-v} \cdot v \sum_{n=0}^{\infty} (-1)^n \frac{v^{2n}}{x^{2n}} \quad (16)$$

We can evaluate these integrals term-by-term using the integral representation of $\Gamma(n+1)$:

$$\Gamma(n+1) = n! = \int_0^\infty dt\, e^{-t} t^n \;\Rightarrow \quad (17)$$

$$\underbrace{\int_0^\infty dv\, e^{-v} v^{2n} = (2n)!}_{\text{use in } f(x)} \; ; \; \underbrace{\int_0^\infty dv\, e^{-v} v^{2n+1} = (2n+1)!}_{\text{use in } g(x)} \quad (18)$$

Combining (16) & (18):

for large x

$$\boxed{f(x) \cong \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} \quad ; \quad g(x) \cong \frac{1}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}}} \quad (19)$$

We can now invert Eqs. (3) & (5) above to solve for the original functions $C_i(x), S_i(x)$:

$$\cos x\cdot f(x) + \sin x\cdot g(x) = \left\{C_i(x)\cancel{\cos x\sin x} - S_i(x)\cos^2 x\right\} + \left\{-C_i(x)\cancel{\sin x\cos x} - S_i(x)\sin^2 x\right\}$$

(20)

$$\therefore \cos x\cdot f(x) + \sin x\cdot g(x) = -S_i(x)\left[\cos^2 x + \sin^2 x\right] = -S_i(x)$$

Similarly:

$$\boxed{\begin{aligned} S_i(x) &= -\cos x\cdot f(x) - \sin x\cdot g(x) \\ C_i(x) &= \sin x\cdot f(x) - \cos x\cdot g(x)\end{aligned}} \quad (21)$$

Hence finally, combining (19) & (21):

$$\boxed{\begin{aligned} C_i(x) &\cong \frac{\sin x}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} - \frac{\cos x}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}} \\ S_i(x) &= -\frac{\cos x}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} - \frac{\sin x}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}}\end{aligned}}$$