# $2^{\underline{ND}}$ ORDER

# LINEAR DIFFERENTIAL EQUATIONS

# 2ND ORDER LINEAR DIFFERENTIAL EQUATIONS



We explore several methods for solving differential equations (originating from partial differential equations e.g. Laplace's Equation):

[1] Series Solutions (Frobenius Method)

[2] Green's Functions

[3] Integral Transforms

## SEPARATION OF VARIABLES

Partial Differential Eqns $\longrightarrow$ Ordinary Differential Equations

This introduces $n$ - ordinary differential equations and $(n-1)$ constants.

Consider the class of differential equations arising from ($k^2$=const)

$$\nabla^2 \psi(\vec{x},t) + k^2 \psi(\vec{x},t) = 0 \quad \begin{cases} k^2 = 0 \Rightarrow \text{Laplace's Eqn.} \\ k^2 > 0 \Rightarrow \text{Helmholtz Eqn.} \\ k^2 < 0 \Rightarrow \text{Diffusion Eqn.} \\ \text{- - - - - - - - - -} \\ k^2 = \text{const} \otimes \text{K.E.} \Rightarrow \text{Schrödinger Eqn.} \end{cases} \tag{1}$$

Expanding $\nabla^2$ in spherical polar coordinates gives

$$\frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right] = -k^2 \psi \tag{2}$$

Assume: $\psi = \psi(r,\theta,\Phi) = R(r)\,\Theta(\theta)\,\Phi(\phi) \Rightarrow$ (3)

$$\frac{\partial \psi}{\partial r} = \Theta \Phi \frac{\partial R}{\partial r} \;;\; \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = \Theta \Phi \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) \tag{4}$$

Similarly: $\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$ ; $\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = R\Phi\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)$ III-56, 57 (5)

$$\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2} = R\Theta\frac{1}{\sin\theta}\frac{\partial^2\Phi}{\partial\phi^2} \qquad (6)$$

Combining the previous results Eq. (2) becomes:

$$\frac{1}{r^2\sin\theta}\left[\Theta\Phi\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + R\Phi\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + R\Theta\frac{1}{\sin\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right] = -k^2R\Theta\Phi \qquad (7)$$

Divide both sides of Eq. (7) by $R(r)\Theta(\theta)\Phi(\phi)$:

$$\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}\right] = -k^2 \qquad (8)$$

At this stage we can replace all partial derivatives by total derivatives.

Cancelling various factors of $\sin\theta$ we get:

$$\frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -k^2 \qquad (9)$$

Multiplying through by $r^2\sin^2\theta$ we can isolate the term containing $\phi$.

$$\underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\text{depends only on }\phi} = r^2\sin^2\theta\left\{-k^2 - \frac{1}{r^2R}\frac{d}{dr}\left(\left(r^2\frac{dR}{dr}\right)\right) - \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right\} \qquad (10)$$

$\longleftarrow$ independent of $\phi$ $\longrightarrow$

Viewed as a function of $\phi$, the r.h.s of (10) being independent of $\phi$ leads to the result

$$\boxed{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \text{Const}(\phi) \equiv -m^2} \Rightarrow \boxed{\Phi(\phi) \sim e^{\pm im\phi}} \qquad (11)$$

And also: $r^2\sin^2\theta\left\{-k^2 - \dots\right\} = -m^2$ (12)

Eqs. (10) & (12) can then be solved for the r-dependence:

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2r^2}_{\text{depends only on } r} = \underbrace{\frac{m^2}{\sin^2\theta} - \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\text{depends only on } \theta} \qquad (13)$$

The only way in which (13) can hold is if each side = constant $\equiv Q$.

This then gives:

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2R = \frac{QR}{r^2}} \qquad (14)$$

and

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2\Theta}{\sin^2\theta} + Q\Theta = 0} \qquad (15)$$

ASSOCIATED LEGENDRE EQUATION

These equations along with $\boxed{\dfrac{d^2\Phi}{d\phi^2} + m^2\Phi = 0}$ (16)

give the solutions of the original partial differential equation.

The Associated Legendre Equation: $Q \rightarrow \ell(\ell+1)$ $\qquad \ell = \text{integer}$

Let $x = \cos\theta$ $\qquad dx = -\sin\theta\, d\theta$

Then Eq. (15) $\Rightarrow$

$$\underbrace{\frac{1}{\sin\theta}\frac{d}{d\theta}}_{-dx}\left(\underbrace{\sin^2\theta}_{(1-x^2)}\frac{d}{\underbrace{\sin\theta\, d\theta}_{-dx}}\Theta\right) - \frac{m^2\Theta}{\underbrace{\sin^2\theta}_{(1-x^2)}} + Q\Theta = 0 \qquad (17)$$

Hence:

$$\boxed{(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \Big(\underbrace{Q}_{\ell(\ell+1)} - \frac{m^2}{1-x^2}\Big)\Theta = 0} \qquad (18)$$

ASSOCIATED LEGENDRE EQUATION

$m = 0 \Rightarrow$ LEGENDRE EQUATION

The Solutions of the Associated Legendre Equation are 

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \longleftarrow \text{Legendre Polynomials} \quad (19)$$

The Spherical harmonics $Y_{\ell,m}(\theta,\phi)$ are then given by:

$$\boxed{Y_{\ell,m}(\theta,\phi) = (-1)^m \left[\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_\ell^m(\cos\theta)\, e^{im\phi} \; ; m \geq 0} \quad (20)$$

$$Y_{\ell,-m} = (-1)^m Y_{\ell,m}^* \quad (m \geq 0)$$

# Singular Points of Differential Equations

The singular points of differential equations determine some of the properties of the solutions, such as how many will arise from the series method. We note, for example, that the Associated Legendre equation has a singularity at $x = \pm 1$.

General Analysis: Write a general differential equation in the form

$$\boxed{y''(x) + P(x)y'(x) + Q(x)\,y(x) = 0} \quad (1)$$

Definitions: 1) If $P(x)$ and $Q(x)$ remain finite at $x = x_0$, then $x_0$ is called an ordinary point.

2) If $P(x)$ and/or $Q(x)$ diverges at $x_0$, $x_0$ is called a singular point.

Singular points can be further classified as follows:

2a) If $P(x)$ or $Q(x)$ diverges as $x \to x_0$ but $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ remain finite at $x_0$, then $x_0$ is called a regular = non-essential singular point

2b) If $(x-x_0)P(x)$ or $(x-x_0)^2 Q(x)$ diverges at $x_0$, then $x_0$ is said to be an irregular singular point = essential singularity

All of this applies to points in the finite part of the plane. In order to study [redacted] the behavior of the differential equations at $x \to \infty$ we proceed as in the case of complex variables (studying the complex plane at $\infty$) we let $z = 1/x$ so that $x \to \infty$ gets mapped to $z = 0$.

$$x = 1/z \;\Rightarrow\; dx = -dz/z^2 \qquad (2)$$

Then $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y(x) = 0$ (3)

Then: $\frac{d}{(-dz/z^2)}\left(\frac{d}{(-dz/z^2)}\,y\right) + P\frac{d}{(-dz/z^2)}\,y + Qy = 0$ (3)

After a little algebra:

$$\boxed{z^4\frac{d^2y}{dz^2} + \left[2z^3 - P(1/z)\,z^2\right]\frac{dy}{dz} + Q(1/z)\,y = 0} \qquad (4)$$

We can then treat this equation as we did the original equation expressed in terms of $x$: Divide (4) by $z^4$ to give

$$\frac{d^2y}{dz^2} + \underbrace{\frac{1}{z^4}\left[2z^3 - P(1/z)\,z^2\right]}_{\hat{P}(z)}\frac{dy}{dz} + \underbrace{\frac{Q(1/z)}{z^4}}_{\hat{Q}(z)}\,y = 0 \qquad (5)$$

Then if $z\hat{P}(z)$ or $z^2\hat{Q}(z)$ diverges at $z=0$, then the point $x=\infty$ is an irregular singular point. Otherwise it is a regular singular point.

EXAMPLES: a) Bessel's Equation: $x^2y'' + xy' + (x^2-n^2)\,y = 0$ (6)

Write this as: $y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$ (7)

$P(x) = \frac{1}{x}$ ; $Q(x) = \left(1 - \frac{n^2}{x^2}\right)$ $\Rightarrow$ $x=0$ is a regular singular point

and this is the only singularity in the finite part of the plane.

To study the behavior at $x \to \infty$ we note that

$\hat{P}(z) = (1/z^4)\left[2z^3 - z \cdot z^2\right] = (1/z) \Rightarrow z\hat{P}(z) = 1$ OK ✓ (8)

$\hat{Q}(z) = (1/z^4)(1 - n^2z^2) \Rightarrow z^2\hat{Q}(z) \sim 1/z^2 \Rightarrow$ Singularity at $z=0$ (9)

Hence $x = \infty$ is an irregular singular point of Bessel's Equation

(b) Legendre Equation

$$(x^2-1)y'' + 2xy' - \ell(\ell+1)y = 0 \Rightarrow y'' + \left(\frac{2x}{x^2-1}\right)y' - \frac{\ell(\ell+1)}{(x^2-1)}y = 0 \quad (10)$$

Hence $P(x) = \dfrac{2x}{(x^2-1)}$ ; $Q(x) = \dfrac{-\ell(\ell+1)}{(x^2-1)}$ (11)

Write (11) more explicitly as: $P(x) = \dfrac{2x}{(x+1)(x-1)}$ $\quad Q(x) = -\dfrac{\ell(\ell+1)}{(x+1)(x-1)}$ (12)

Hence the Legendre Equation has regular singular point at $x = \pm 1$.

To study the behavior at $x = \infty$ we have:

$$\hat{P}(z) = \frac{1}{z^4}\left[2z^3 - z^2\frac{(2/z)}{(1/z+1)(1/z-1)}\right] = \left[\frac{2}{z} - \frac{z}{z(1+z)(1-z)}\right] \quad (13)$$

$$\hat{Q}(z) = \frac{[-\ell(\ell+1)]}{(1/z+1)(1/z-1)}\frac{1}{z^4} = \frac{-\ell(\ell+1)}{z^2(1+z)(1-z)} \quad (14)$$

$\therefore$ $z = 0$ is a singular point of the Legendre Equation; but since $z\hat{P}(z)$ and $z^2\hat{Q}(z)$ are well-behaved this is a regular singular point.

Summary: The Legendre Equation has regular singular points at $x = \pm 1, \infty$.

## The Hypergeometric Equation:

Some of the more common differential equations and their singularities are tabulated by ARFKEN. One can show that equations which have 3 regular singular points can be obtained as solutions of the hypergeometric equation:

[redacted]

$$\boxed{x(x-1)y'' + [(1+a+b)x-c]y' + aby = 0} \quad (1)$$

This equation has regular singular points at $x=0,1,\infty$. By an appropriate choice of the constants $a,b,c$ this can be transformed into other specific equations such as the Legendre equation.

## The Confluent Hypergeometric Equation:

Similarly, 2ND order differential equations with one regular singular point and one irregular singular point can be obtained from the confluent hypergeometric equation, by an appropriate choice of the constants $a,c$:

$$\boxed{xy'' + (c-x)y' - ay = 0} \quad (2)$$

The naming of this as the "confluent" hypergeometric equation is derived from the fact that Eq. (2) can be derived from Eq. (1) by a "confluence" (= coming together) of 2 of the singularities of the hypergeometric equation. To see this start with Eq. (1) and write

$$z = bx \quad ; \quad b\frac{d}{dz} = \frac{d}{dx} \quad ; \quad b^2\frac{d^2}{dz^2} = \frac{d^2}{dx^2} \qquad (3)$$

Then $(1)+(3) \Rightarrow z(z-b)\dfrac{d^2y}{dz^2} + \left[(1+a+b)z - cb\right]\dfrac{dy}{dz} + aby = 0 \quad (4)$

Next divide each term by b:

$$z\left(\underline{\underline{\frac{z}{b}}}-1\right)\frac{d^2y}{dz^2}+\left[\underline{\underline{\left(\frac{1+a}{b}\right)z}}+z-c\right]\frac{dy}{dz}+ay=0 \qquad (5)$$

Finally let $b \to \infty$: The terms indicated by $=\!=\!=$ vanish in this limit $\Rightarrow$

$$\boxed{-z\frac{d^2y}{dz^2}+(z-c)\frac{dy}{dz}+ay=0} \qquad (6)$$

Which then gives the Confluent hypergeometric equation (up to a sign).

The effect of the manipulations in going from (3)–(6) is to merge the Singularities at $x=0,1$ into one singularity at $z=0$, and at the same time to convert the singularity at $\infty$ from regular to irregular.

III–66

Comparing it with Eq. 8.13, we have

$$P(x) = \frac{1}{x}, \qquad Q(x) = 1 - \frac{n^2}{x^2},$$

which shows that point $x = 0$ is a regular singularity. By inspection we see that there are no other singular points in the finite range. As $x \to \infty$ ($z \to 0$), from Eq. 8.16 we have the coefficients

$$\frac{2z^3 - z^2 \cdot z}{z^4} \quad \text{and} \quad \frac{1 - n^2 z^2}{z^4}.$$

Since the latter expression diverges as $z^4$, point $x = \infty$ is an irregular or essential singularity.

The ordinary differential equations of Section 8.2, plus two others, the hypergeometric and the confluent hypergeometric, have singular points, as shown in Table 8.3.

TABLE 8.3

| Equation | Regular singularity $x=$ | Irregular singularity $x=$ |
|---|---|---|
| 1. Hypergeometric <br> $x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0.$ | $0, 1, \infty$ | — |
| 2. Legendre[a] <br> $(1-x^2)y'' - 2xy' + l(l+1)y = 0.$ | $-1, 1, \infty$ | — |
| 3. Chebyshev <br> $(1-x^2)y'' - xy' + n^2 y = 0.$ | $-1, 1, \infty$ | — |
| 4. Confluent hypergeometric <br> $xy'' + (c-x)y' - ay = 0.$ | $0$ | $\infty$ |
| 5. Bessel <br> $x^2y'' + xy' + (x^2 - n^2)y = 0.$ | $0$ | $\infty$ |
| 6. Laguerre[a] <br> $xy'' + (1-x)y' + ay = 0.$ | $0$ | $\infty$ |
| 7. Simple harmonic oscillator <br> $y'' + \omega^2 y = 0.$ | — | $\infty$ |
| 8. Hermite <br> $y'' - 2xy' + 2\alpha y = 0.$ | — | $\infty$ |

[a] The associated equations have the same singular points.

note that for these equations the only irregular points are at $\infty$

It will be seen that the first three equations in the preceding tabulation, hypergeometric, Legendre, and Chebyshev, all have three regular singular points. The hypergeometric equation with regular singularities at 0, 1, and $\infty$ is taken as the standard, the canonical form. The solutions of the other two may then be expressed in terms of its solutions, the hypergeometric functions. This is done in Chapter 13.

In a similar manner the confluent hypergeometric equation is taken as the canonical form of a linear second-order differential equation with one regular and one irregular singular point.

# SERIES SOLUTIONS: FROBENIUS METHOD

For differential equations having only regular singularities, solutions can be found by expanding in a serious about this singularity (e.g. $x=0$).

We wish to solve the generic differential equation (homogeneous* & linear)

$$Ly(x) \equiv \frac{d^2y(x)}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y(x) = 0 \qquad (1)$$

* An inhomogeneous equation would be one in which the r.h.s. would be $F(x)$ rather than zero, as in the case of a forced harmonic oscillator. For such a case write $Ly(x) = F(x)$ and let $y_p(x)$ be a solution of this equation:

$$L\,y_p(x) = F(x) \qquad (2)$$

Then the most general solution of (2) is: $\boxed{y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x)}$ (3)

where $y_{1,2}(x)$ are solutions of the homogeneous equation $L\,y_{1,2}(x) = 0$:

Thus $$Ly(x) = \underbrace{L\,y_p(x)}_{F(x)} + \underbrace{L\left(c_1 y_1(x) + c_2 y_2(x)\right)}_{= 0} = F(x) \checkmark \qquad (4)$$

Note that a 2ND order differential equation has 2 linearly independent solutions $y_1(x)$ and $y_2(x)$. We will show that when the series solution yields only one function $y_1(x)$, we can find the other by a formal algorithm.

**Series Method:** Consider as an example the equation

$$y''(x) + \omega^2 y(x) = 0 \qquad (5)$$

We know in advance that the 2 linearly independent solutions are $\sin\omega x, \cos\omega x$. But here we want to derive these from first principles using the series method.

We begin by assuming that ($k \equiv$ index)

$$y(x) = x^k\left(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n\right) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad (6)$$

where we define $a_0 \neq 0$ ($a_0$ is the coefficient of the first nonzero term).

We will eventually determine $k$ and all the constants $a_\lambda$. Then (6) $\Rightarrow$

$$y'(x) = \sum_{\lambda=0}^{\infty}(k+\lambda)a_\lambda x^{k+\lambda-1} \quad ; \quad y''(x) = \sum_{\lambda=0}^{\infty}(k+\lambda)(k+\lambda-1)a_\lambda x^{k+\lambda-2} \qquad (7)$$

Combining (5)-(7): $$\sum_\lambda (k+\lambda)(k+\lambda-1)a_\lambda x^{k+\lambda-2} + \omega^2 \sum_\lambda a_\lambda x^{k+\lambda} = 0 \qquad (8)$$

In order for this equation to hold each term in the series (i.e. the coefficients of each power of $x$) must separately vanish. To see how this comes about, write out the first few terms:

$$\underbrace{\left[k(k-1)a_0 x^{k-2} + \omega^2 a_0 x^k\right]}_{\lambda=0} + \underbrace{\left[(k+1)k a_1 x^{k-1} + \omega^2 a_1 x^{k+1}\right]}_{\lambda=1}$$

$$+ \underbrace{\left[(k+2)(k+1)a_2 x^k + \omega^2 a_2 x^{k+2}\right]}_{\lambda=2} + \underbrace{\left[(k+3)(k+2)a_3 x^{k+1} + \omega^2 a_3 x^{k+3}\right]}_{\lambda=3} + \cdots \qquad (9)$$

We see from (9) that the lowest power of $x$, which is $x^{k-2}$ appears in only one term. Since this term must vanish by itself even though $a_0 \neq 0$ (by definition) it follows [redacted] that $k$ must satisfy

$$\boxed{k(k-1) = 0 \Rightarrow k = 0, 1} \qquad \text{(10)}$$

<u>INDICIAL EQUATION</u>

The indicial equation always arises in this way from the lowest term.

We next demand that the coefficients of the remaining independent powers of $x$ also vanish. We note from (9) that the term $\underline{\lambda = j}$ in [the second term of] Eq. (8) will give rise to the same power of $x$ as the term $\underline{\lambda = j+2}$ in the first term of Eq. (8), in both cases this being $x^{k+j}$.

Thus the net coefficient of $x^{k+j}$ is given by

$$\left[(k+j+2)(k+j+2-1)\,a_{j+2} + \omega^2 a_j\right] x^{k+j} \equiv 0 \Rightarrow \tag{11}$$

$$\boxed{a_{j+2} = \frac{-\omega^2 a_j}{(k+j+2)(k+j+1)}} \tag{12}$$

<u>RECURRENCE RELATION</u>

Together, the indicial equation ⊕ recurrence relation determine one solution of the original differential equation. From (12) it follows that $a_2$ is given in terms of $a_0$, $a_4$ in terms of $a_2$, ...., where <u>$a_0$ itself is not determined</u> (it will be one of 2 unknown constants that arise for a 2ND order differential equation). This solution has only even powers of $x$, and we are free at this point to set $a_1 \equiv 0$ (Why? <u>Because this works!</u> .... See below!!) Combining the indicial equation (10) & the recurrence relation (12), we <u>start with $k=0$</u>)

Then (12) $\Rightarrow$ $a_{j+2} = \dfrac{-\omega^2 a_j}{(j+2)(j+1)} \Rightarrow a_2 = -\dfrac{\omega^2 a_0}{2\cdot 1} = -\dfrac{\omega^2 a_0}{2!}$ (13)

$$a_4 = \frac{-\omega^2 a_2}{4\cdot 3} = +\frac{\omega^4 a_0}{4!}\;;\quad a_6 = \frac{-\omega^2 a_4}{6\cdot 5} = \frac{-\omega^6 a_0}{6!}, \;\ldots \text{ etc.} \tag{14}$$

$$\therefore\;\boxed{a_{2n} = (-1)^n \frac{\omega^{2n} a_0}{(2n)!}} \tag{15}$$

From (15) with $k=0$:

$$y(x) = x^0 a_0 \left\{ 1 - \frac{1}{2!}\omega^2 x^2 + \frac{1}{4!}\omega^4 x^4 - \frac{1}{6!}\omega^6 x^6 + \dots \right\}$$

$$\therefore \boxed{y(x) = a_0 \cos \omega x} \longleftrightarrow \boxed{k=0} \quad (16)$$

Next we examine the solution corresponding to $k=1$ in the indicial equation:
From (12) [the recurrence relation] $k=1 \Rightarrow$

$$a_{j+2} = \frac{-\omega^2 a_j}{(j+3)(j+2)} \Rightarrow a_2 = \frac{-\omega^2 a_0}{3 \cdot 2}\ ; \ a_4 = \frac{-\omega^2 a_2}{5 \cdot 4} = \frac{+\omega^4 a_0}{5!}$$

$$a_6 = \frac{-\omega^2 a_4}{7 \cdot 6} = -\frac{\omega^6 a_0}{7!}\ ; \ \dots \text{etc.} \quad (17)$$

For the $k=1$ solution then: $\boxed{a_{2n} = \dfrac{(-1)^n \omega^{2n} a_0}{(2n+1)!}}$ (18)

The solution for $k=1$ is then $y = x^1(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots)$ (19)

$$\therefore y(x) = a_0 x \left\{ 1 - \frac{\omega^2 x^2}{3!} + \frac{\omega^4 x^4}{5!} - \frac{\omega^6 x^6}{7!} + \dots \right\} \quad (20)$$

$$y = a_0 \left\{ x - \frac{\omega^2 x^3}{3!} + \frac{\omega^4 x^5}{5!} - \frac{\omega^6 x^7}{7!} + \dots \right\} \quad (21)$$

$$\boxed{\therefore y(x) = \frac{a_0}{\omega} \left\{ \omega x - \frac{\omega^3 x^3}{3!} + \frac{\omega^5 x^5}{5!} - \frac{\omega^7 x^7}{7!} + \dots \right\} = \frac{a_0}{\omega} \sin \omega x} \quad (22)$$

Thus in this situation k=1 gives a second solution in terms of another undetermined constant [This $a_0$ is not necessarily the same as previously.]

# COMMENTS ON SOLUTIONS:

[1] We will later formally prove that $\sin \omega x$ and $\cos \omega x$ are independent solutions. Knowing this we conclude that in this case (but not always!) the 2 solutions of the indicial equation gave us 2 linearly independent solutions. Since there can be at most 2 lin. indep. solutions, we have solved the equation completely. Thus we did not "lose" anything by the assumption $a_1 \equiv 0$.

[2] In general we should verify that the solutions actually solve the original equation, although here this is trivial.

[3] When a series solution leads to an infinite series, we should formally test for convergence.

## PARITY OF SOLUTIONS:

We are studying differential equations of the form:

$$L(x)\,y(x) = 0 \qquad (1)$$

In many cases of interest $L(x)$ is a linear operator having the property

$$L(-x) = \pm L(x) \qquad (2)$$

$+ \equiv$ even parity $\quad - \equiv$ odd parity [usually the parity is even].

In Eq. (1) consider:
$$\underbrace{L(x)}_{\pm L(-x)}\,y(-x) = \pm L(-x)\,y(-x) = \pm L(x')\,y(x') = 0 \qquad (3)$$

If $L(x)$ then has a well-defined parity (+ or −) then if $y(x)$ is a solution to the differential equation, so it must be that $y(-x)$ is also a solution.

It follows that <u>for any solution</u> $y(x)$ we can write

$$y(x) = \underbrace{\tfrac{1}{2}\left[y(x)+y(-x)\right]}_{\text{EVEN PARITY}} + \underbrace{\tfrac{1}{2}\left[y(x)-y(-x)\right]}_{\text{ODD PARITY}} \qquad (4)$$

Hence the solutions to such an equation can always be expressed in terms of solutions which have well defined parity. For example for the SHO equation $y''(x) + \omega^2 y(x) = 0$ we could have found the solutions $e^{+i\omega x}$, $e^{-i\omega x}$ which do not have well-defined parity. However (4) $\Rightarrow$ that linear combinations of them will:

$$\cos\omega x = \tfrac{1}{2}\left[e^{i\omega x} + e^{-i\omega x}\right] \qquad \sin\omega x = \tfrac{1}{2i}\left[e^{i\omega x} - e^{-i\omega x}\right]$$

<u>The following differential operators have even parity:</u>

SHO, Legendre, Bessel, Hermite, Chebyshev.

The Laguerre operator [radial solution for the Coulomb potential] does not.

# BESSEL'S EQUATION:

This illustrates that we do not always obtain 2 linearly independent solutions from the series expansion. Bessel's equation is:

$$x^2y'' + xy' + (x^2-n^2)y = 0 \quad ; \boxed{n \text{ is given}} \qquad (1)$$

Assume as before: $$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \quad ; a_0 \neq 0 \qquad (2)$$

From p. III-68:

$$y' = \sum_\lambda (k+\lambda) a_\lambda x^{k+\lambda-1} \quad ; \quad y'' = \sum_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} \qquad (3)$$

Eqs. (1)+(3) ⇒ $$0 = \sum_\lambda a_\lambda x^{k+\lambda} \Big\{ \underbrace{(k+\lambda)(k+\lambda-1)}_{y''} \underbrace{+(k+\lambda)}_{y'} \underbrace{-n^2}_{y} \Big\} + \sum_\lambda \underbrace{a_\lambda x^{k+\lambda+2}}_{y} \qquad (4)$$

$(k+\lambda)^2$

$$\boxed{\therefore\ 0 = \sum_\lambda a_\lambda x^{k+\lambda} \{(k+\lambda)^2 - n^2\} + \sum_\lambda a_\lambda x^{k+\lambda+2}} \qquad (5)$$

this expands about $x=0$ which is a <u>regular</u> singular point

As before the indicial equation is obtained from the lowest power of $x$ which corresponds to $\lambda=0$. The lowest power is, then $x^k$ and the requirement that its coefficient vanish yields

$$\boxed{k^2 - n^2 = 0 \Rightarrow k = \pm n} \quad \text{INDICIAL EQUATION} \qquad (6)$$

As in the SHO case <u>we must set $a_1 \equiv 0$</u> to obtain a solution, since the coefficient of $a_1$ is $$[(k+1)^2 - n^2]a_1 = [\underbrace{(k^2-n^2)}_{0 \leftarrow \text{INDICIAL EQN.}} + (2k+1)]a_1 \qquad (7)$$

Hence when the indicial equation holds the coefficient of $a_1 \neq 0$ so $a_1$ itself must be set to zero.

# Recurrence Relations

Returning to (5) above we see that the same power of $x$ arises from the $\lambda = j+2$ term in $\{\dots\}$ and the $\lambda = j$ contribution in the 2ND term.

The condition that the coefficient of $x^{k+j+2}$ vanish then gives

$$\left\{(k+j+2)^2 - n^2\right\} a_{j+2} + a_j = 0 \Rightarrow \boxed{a_{j+2} = \frac{-a_j}{(k+j+2)^2 - n^2}} \quad (8)$$

The indicial equation requires $k = \pm n$. For $\underline{k = +n}$ we have

$$(n+j+2)^2 - n^2 = (j+2)(j+2+2n) \Rightarrow \boxed{a_{j+2} = \frac{-a_j}{(j+2)(j+2n+2)}} \quad (9)$$

$k = +n$

Hence the coefficients in the series expansion are:

$$a_2 = \frac{-a_0}{2(2n+2)} = -\frac{a_0}{4(n+1)} = \frac{-a_0}{2^2 1!\ \frac{(n+1)!}{n!}} \quad (10)$$

$$a_4 = -\frac{a_2}{4\cdot(2+2n+2)} = +\frac{a_0}{8(n+2)4(n+1)} = \frac{a_0}{2^4\, 2!\ \frac{(n+2)!}{n!}} \quad (11)$$

$$a_6 = -\frac{a_4}{6(4+2n+2)} = -\frac{a_4}{12(n+3)} = \frac{-a_0}{12\cdot 32\,(n+3)(n+2)(n+1)} = \frac{-a_0}{2^6 3!\ \frac{(n+3)!}{n!}} \quad (12)$$

$\vdots$

By induction the general term is: $\boxed{a_{2p} = \frac{(-1)^p n!\ a_0}{2^{2p} p!\ (n+p)!}}$ (13) Recurrence Relation for $k = +n$

Inserting these results into the Series expansion we have:

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \xrightarrow{k=n} a_0 x^n + a_2 x^{n+2} + a_4 x^{n+4} + \dots = \sum_p a_{2p} x^{n+2p} \quad (14)$$

Thus $y(x)$ has the form (for $k=+n$)

$$y(x) = a_0 \sum_{p=0}^{\infty} \frac{(-1)^p n! \, x^{n+2p}}{2^{2p} p! (n+p)!} \qquad (15)$$

$$\therefore \quad \boxed{y(x) = a_0 2^n n! \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{x}{2}\right)^{n+2j}}{j! (n+j)!} \equiv a_0 2^n n! \, \underbrace{J_n(x)}_{\text{Bessel function}^*}} \qquad (16)$$

* Note: Some texts may differ slightly on which factors of $n!$ are included in the definition of $J_n(x)$.

The Case $k=-n$ : We can formally take over the algebra in the $k=+n$ case and simply replace $+n \to -n$. This gives

$$a_{j+2} = \frac{-a_j}{(j+2)(j+2-2n)} \qquad (17)$$

When $n \neq$ integer this leads to a second solution which is linearly independent. However, when $n = +$ integer the recurrence relation in (17) obviously blows up, whenever $j+2 = 2n$ or $j = 2(n-1)$.

Returning to the definition of $J_n(x)$ in (16) we see that when $-n$ is a negative integer then

$$J_{-n}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j-n)!} \left(\frac{x}{2}\right)^{-n+2j} \qquad (18)$$

Next define a new summation index $k = j-n$; $j=n \Rightarrow k=0$

$$\therefore J_{-n} = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)! \, k!} \left(\frac{x}{2}\right)^{-n+2k+2n} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+n)!} \left(\frac{x}{2}\right)^{n+2k} \qquad (19)$$

$$\boxed{\therefore \; J_{-n}(x) = (-1)^n J_n(x)} \qquad (20)$$

We thus see that when $k = -n$ = negative integer, the 2ND root of the indicial equation does not lead to a 2ND independent solution, in contrast to what we found earlier for the SHO. This result is part of what is contained in <u>FUCHS' THEOREM</u>:

a) The series (Frobenius) method yields at least one solution if the expansion is about a singularity which is at worst a <u>regular</u> <u>singularity</u>. Since the usual differential equations of physics have only regular singularities in the finite part of the plane $\Rightarrow$ <u>series method will</u> [redacted] <u>produce at least one solution</u>. It may also produce 2 solutions depending on the indicial equation:

b) Both roots of indicial equation being equal $\Rightarrow$ <u>one solution</u>

c) If the 2 roots differ by a nonintegral number $\Rightarrow$ <u>two solutions</u>

d) If the 2 roots differ by an integer $\Rightarrow$ <u>one or two solutions</u>

Bessel SHO

# Series Method for the Legendre Equation



The series method starts from a formalism in which $y(x)$ is expressed as an infinite series, which is appropriate for $\sin\omega x, \cos\omega x, J_n(x), \ldots$ but not for solutions of other equations where we know that the solution is a finite polynomial. So how does the series method arrange to come up with finite polynomials.

Consider the Legendre equation: $\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$ (1)

As before: $y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$ ; $y'(x) = \sum_\lambda (k+\lambda)a_\lambda x^{k+\lambda-1}$ ; $y''(x) = \sum_\lambda a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}$ (2)

$$\text{Eq}(1) \Rightarrow 0 = \sum_\lambda \underbrace{(k+\lambda)(k+\lambda-1)}a_\lambda x^{k+\lambda-2} - \sum_\lambda \Big\{\underbrace{(k+\lambda)(k+\lambda-1)+2(k+\lambda)}_{(k+\lambda)(k+\lambda+1)} - n(n+1)\Big\} a_\lambda x^{k+\lambda} \quad (3)$$

$\downarrow$

INDICIAL EQUATION ($\lambda = 0$)

$$(k+0)(k+0-1) = 0 \Rightarrow \boxed{k = 0, 1} \quad (4)$$

Recurrence Relation ($k=0$) : From (3) let $\lambda \to j+2$ in first $\sum_\lambda$ :

$$(j+2)(j+2-1)a_{j+2} - \{\underbrace{j(j-1)+2j}_{j(j+1)} - n(n+1)\}a_j = 0 \quad (5)$$

$$\Rightarrow \boxed{a_{j+2} = \frac{[j(j+1) - n(n+1)]a_j}{(j+2)(j+1)}} \quad k=0 \text{ Recurrence Relation} \quad (6)$$

As before set $a_0 \neq 0$ $a_1 \equiv 0$ which gives

$$a_2 = \frac{-n(n+1)}{2\cdot 1}a_0 \ ; \ a_4 = \frac{[2(2+1) - n(n+1)]a_2}{4\cdot 3} = -\frac{n(n+1)[6-n(n+1)]}{4!}a_0 \quad (7)$$

Continuing, ...

$$a_6 = \frac{[4\cdot 5 - n(n+1)]\, a_4}{6\cdot 5} = \frac{-n(n+1)\,[6-n(n+1)]\,[20-n(n+1)]\, a_0}{6!} \qquad (8)$$

The general term is evidently given by

$$a_{2p} = \frac{-(n)(n+1)\,[6-n(n+1)]\,[20-n(n+1)]\cdots[(2p-2)(2p-1)-n(n+1)]\, a_0}{(2p)!} \qquad (9)$$

Note that at this stage (9) leads to an <u>infinite</u> <u>series</u> (in general). To see whether this series converges we use the <u>ratio</u> <u>test</u>

$$\lim_{j\to\infty} \frac{a_{j+2}\, x^{j+2}}{a_j\cdot x^j} = \lim_{j\to\infty} \frac{[j(j+1)-n(n+1)]}{(j+2)(j+1)}\, x^2 \sim \frac{j}{j+2}\, x^2 \to 1\cdot x^2 \qquad (10)$$

It follows that <u>as $x \to 1$ this series diverges</u> [See Mathews/Walker p.15: the series diverges as $\ln(1-x^2)$. We return to this below.] We will eventually show that this infinite series solution gives the 2ND solution to the Legendre equation, the one that is not the familiar Legendre polynomials. Before attacking this solution consider first the <u>$k=1$ root</u> of the indicial equation: This gives

$$[1+(j+2)][1+(j+2)-1]\, a_{j+2} - \{(1+j)(1+j+1) - n(n+1)\}\, a_j = 0 \qquad (11)$$

$$\Rightarrow \boxed{a_{j+2} = \frac{[(j+1)(j+2) - n(n+1)]\, a_j}{(j+3)(j+2)}} \qquad (12)$$

Recurrence Relation for <u>$k=1$</u>

As before $\lim_{j\to\infty} \frac{a_{j+2}\, x^{j+2}}{a_j\cdot x^j} \longrightarrow 1\cdot x^2 \longrightarrow$ diverges as $x \to 1$ (13)

Hence the $k=1$ solution does not solve the divergence problem.

## Obtaining Finite Polynomials:

Finite polynomials (rather than an $\infty$ series) are obtained when $n =$ integer in which case the series terminates:

Example: $n = 2$ $\Rightarrow$ Using Eq. (7) p. 79 we have

$$a_2 = -\frac{2\cdot 3}{2} a_0 = -3a_0 \; ; \; a_4 = \frac{[2\cdot 3 - 2\cdot 3]}{4\cdot 3} a_2 = 0 \; ; \; a_6 = \cdots \overset{\to = 0}{a_4} = 0 \ldots \qquad (14)$$

Hence for $n = 2$ the only nonzero terms are $a_0$ and $a_2 = -3a_0$. Hence for $n = 2$ the solution $y_2(x)$ to the Legendre equation is given by

$$y_2(x) = a_0 x^{0+0} + a_2 x^{0+2} = a_0(1 - 3x^2) \; ; \; a_0 \equiv -\tfrac{1}{2} \Rightarrow \boxed{y_2 = \tfrac{1}{2}(3x^2 - 1) = P_2(x)} \qquad (15)$$

Hence this series terminates and produces $P_2(x)$. The other even Legendre polynomials can be obtained similarly by setting $n = 4, 6, \ldots$

## The Legendre Polynomials $P_n(x)$ for $n =$ Odd:

These come from the $k = 1$ recurrence relation. From Eq. (12) we examine the case $n = 3$:

$$a_2 = \frac{[1\cdot 2 - 3\cdot 4]}{3\cdot 2} a_0 = -\frac{10}{6} a_0 = -\frac{5}{3} a_0 \qquad (16)$$

$$a_4 = \frac{[3\cdot 4 - 3\cdot 4]}{5\cdot 4} a_2 = 0. \qquad a_6 = a_8 = \cdots = 0$$

The solution for $n = 3$ (having used $k = 1$) is then $y_3(x) = a_0 x^{1+0} + a_2 x^{1+2}$ (17)

$$\therefore y_3(x) = a_0\left(x - \frac{5}{3}x^3\right) = \frac{a_0}{3}\left(3x - 5x^3\right) \; ; \; \frac{a_0}{3} \equiv -\frac{1}{2} \Rightarrow \boxed{y_3(x) = \tfrac{1}{2}(5x^3 - 3x) = P_3(x)} \qquad (18)$$

# SUMMARY:

[1] If $n$ = integer $\Rightarrow$ series solution terminates in a finite polynomial

[2] If $k=0 \Rightarrow$ polynomial having only <u>even</u> powers of $x$ (even parity)

If $k=1$ " " " <u>odd</u> powers of $x$ (odd parity)

[3] If $n \neq$ integer $\Rightarrow$ series diverges as $x \to \pm 1$

[4] Since the series solution for $n$ = integer gives uniquely only one of the needed 2 solutions, <u>we must still find a 2ND solution</u>

What happens when we use $k=1$ with $n$ = even or $k=0$ with $n$ = odd?

# In Search of a Second Solution

We need a 2ND linearly independent solution. How do we know when solutions are in fact linearly independent?

From our discussion last semester a set of functions $y_\lambda(x)$ [viewed as vectors in Hilbert space] are linearly independent if

$$\boxed{\sum_\lambda a_\lambda y_\lambda(x)=0 \Rightarrow a_\lambda = 0 \quad \forall \lambda} \qquad (1)$$

By contrast, if we can find a set of constants $\{a_\lambda\}$ such that (1) holds with some $a_\lambda \neq 0$, then the solutions are linearly dependent. We want to eliminate the "guesswork" that is suggested by "if we can find".

We thus focus on a method of determining linear independence which is based on the properties of the functions $y_\lambda(x)$ themselves rather than on the $a_\lambda$. To do this we repeatedly differentiate $y_\lambda(x)$ to give:

$$\begin{aligned}
&\sum_\lambda a_\lambda y_\lambda(x)=0 \\
&\sum_\lambda a_\lambda y'_\lambda(x)=0 \\
&\sum_\lambda a_\lambda y''_\lambda(x)=0 \\
&\vdots \\
&\sum_\lambda a_\lambda y^{(n-1)}_\lambda(x)=0
\end{aligned} \qquad (2)$$

Writing (2) out explicitly gives

$$\begin{aligned}
&y_1 a_1 + y_2 a_2 + \cdots + y_n a_n = 0 \equiv m_{11} a_1 + m_{12} a_2 + \cdots + m_{1n} a_n \\
&y'_1 a_1 + y'_2 a_2 + \cdots + y'_n a_n = 0 \equiv m_{21} a_1 + m_{22} a_2 + \cdots + m_{2n} a_n \\
&\vdots \\
&y^{(n-1)}_1 a_1 + y^{(n-2)}_2 a_2 + \cdots + y^{(n-1)}_n a_n \equiv m_{n1} a_1 + m_{n2} a_2 + \cdots + m_{nn} a_n = 0
\end{aligned} \qquad (3)$$

These equations can be compressed into matrix notation:

$$M \equiv \begin{pmatrix} m_{11} \cdots & m_{1n} \\ m_{n1} \cdots & m_{nn} \end{pmatrix} \qquad a \equiv \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad (4)$$

Then (3) $\Rightarrow$ $\boxed{Ma = 0}$ (5)

Recall the discussion from last semester:

$$Ma = 0 \begin{cases} \longrightarrow a = 0 \Rightarrow M^{-1} \text{ exists} \Rightarrow \det M \neq 0 \\ \longrightarrow a \neq 0 \Rightarrow M^{-1} \text{ does not exist} \Rightarrow \det M = 0 \end{cases} \qquad (6)$$

Here we want for linear independence that $a \equiv 0 \Rightarrow \det M \neq 0$. Define

$$\boxed{\det M(x) \equiv W(x) = \text{WRONSKIAN}\ ;\ W \neq 0 \Rightarrow \varphi_\lambda(x) \text{ are lin. indep.}} \quad (7)$$

<u>Note</u>: Since $W = W(x)$ it may be that $W(x_i) = 0$ for some $x_i$. This does not mean that the $\varphi_\lambda(x)$ are linearly - dependent.
However, <u>see side comment on p.87</u>

<u>Example 1</u> : Consider 2 functions $y_1(x)$ and $y_2(x)$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \qquad (8)$$

If $W(x) = 0$ then $y_1 y_2' = y_2 y_1' \Rightarrow \frac{1}{y_1} y_1' = \frac{1}{y_2} y_2'$ (9)

$$\frac{1}{y_1}\frac{dy_1}{dx} = \frac{1}{y_2}\frac{dy_2}{dx} \Rightarrow \frac{dy_1}{y_1} = \frac{dy_2}{y_2} \Rightarrow \ln y_1 = \ln y_2 + C \Rightarrow \qquad (10)$$

$e^{\ln y_1} = e^{\ln y_2 + C} \Rightarrow y_1 = \text{Const}\ y_2 \Rightarrow$ <u>$y_1(x)$ & $y_2(x)$ are linearly-dependent</u> ✓ (11)

**Example 2:** Consider next the 2 solutions we found for the SHO:

$$y_1(x) = \sin\omega x \qquad y_2(x) = \cos\omega x$$

$$W(x) = \begin{vmatrix} \sin\omega x & \cos\omega x \\ \omega\cos\omega x & -\omega\sin\omega x \end{vmatrix} = -\omega\left(\sin^2\omega x + \cos^2\omega x\right) = -\omega \neq 0 \qquad (12)$$

Hence this confirms that these solutions are linearly independent. ✓

**Example 3:** Next consider the functions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = \cosh x$

We know that these functions are linearly-dependent since $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

This means that in this case

$$1\cdot y_1 + 1\cdot y_2 - 2\cdot y_3 = 0 \equiv \sum_\lambda a_\lambda y_\lambda \quad \text{with } a_\lambda \neq 0 \qquad (13)$$

To verify that $W = 0$ in this case we have

$$W(x) = \begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = 0 \qquad (14)$$

Note: this determinant $= 0$ because these 2 rows are identical.

# CONSTRUCTING A 2ND SOLUTION:



We now show that once we are given one solution of a differential equation, we can formally obtain a 2ND linearly independent solution.

Consider the differential equation: $y''(x) + P(x)y'(x) + Q(x)y(x) = 0$ (1)

Then the Wronskian $W(x)$ is given by

$$W = y_1 y_2' - y_2 y_1' \qquad (2)$$

Differentiating $W(x)$: $W' = y_1 y_2'' + \cancel{y_1' y_2'} - y_2 y_1'' - \cancel{y_2' y_1'} = y_1 y_2'' - y_2 y_1''$ (3)

$$W' = y_1 \underbrace{\left[-P(x) y_2' - Q(x) y_2\right]}_{y_2''} - y_2 \underbrace{\left[-P(x) y_1' - Q(x) y_1\right]}_{y_1''} \qquad (4)$$

Eq(1) ⇒

$$W' = -P(x)\underbrace{\left[y_1 y_2' - y_2 y_1'\right]}_{W(x)} - Q(x)\left[\cancel{y_1 y_2} - \cancel{y_2 y_1}\right] = -P(x)\,W(x) \qquad (5)$$

Hence $W(x)$ obeys the differential equation $\boxed{W'(x) = -P(x)\,W(x)}$ (6)

It follows from (6) that if $P(x) = 0$ so that Eq.(1) looks like $y'' + Qy = 0$ then $W'(x) = 0 \Rightarrow W = \text{constant}$. We can choose the overall scale of $y_{1,2}(x)$ so that this constant is $\pm 1$. Thus $\boxed{P(x) = 0 \Rightarrow W = \pm 1 = \text{constant}}$ (7)

This agrees with what we found previously for the SHO: $W = -\omega$.

When $P(x) \neq 0$, which is generally the case, we can use (6) to find a second solution $y_2(x)$ given $y_1(x)$.

To find the 2ND solution:

$$(6) \Rightarrow \quad \frac{dW(x)}{dx} = -P\,W(x) \Rightarrow \frac{dW}{W} = -P\,dx \;;$$ Integrating from $x' = a$ to $x' = x$:

$$\ln W(x)\Big|_a^x = -\int_a^x dx' P(x') = \ln W(x) - \ln W(a) = \ln\left(\frac{W(x)}{W(a)}\right) \quad (8)$$

$$\therefore \frac{W(x)}{W(a)} = e^{-\int_a^x dx' P(x')} \Rightarrow \boxed{W(x) = W(a)\, e^{-\int_a^x dx' P(x')}} \quad (9)$$

Note that (9) $\Rightarrow$ $W(x)$ can be found [up to an overall constant $W(a)$] just by knowing $P(x)$ without having to solve the original differential eqn.

Side Comment: If $P(x')$ is finite in the interval $a \le x' \le x$ then the only way $W(x)$ [redacted] $=0$ can arise is if $W(a) = 0$. Thus either $W \equiv 0$ or else $W(x) \neq 0$ for any $x'$ in the interval $a \le x' \le x$.

To find the second solution $y_2$ given $y_1$ use (9) to write:

$$W = y_1 y_2' - y_2 y_1' = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right) \longrightarrow y_1^2 \left\{ \frac{y_1 y_2' - y_2 y_1'}{y_1^2} \right\} = W \checkmark \quad (10)$$

$$\therefore W = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a)\, e^{-\int dx' P(x')} \Rightarrow \frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{W(a)\, e^{-\int_a^x dx' P(x')}}{y_1^2} \quad (11)$$

Next we integrate this expression with respect to $x$: BE CAREFUL HERE!! The variable $x$ in (11) becomes now a dummy variable of integration, as we let $x \to x''$, which falls in the interval $b \le x'' \le x$. When we do this we find from (11)

$$\left.\frac{y_2(x'')}{y_1(x'')}\right|_b^x = W(a)\int_{x''=b}^{x''=x} dx'' \, \frac{e^{\int_a^{x''} dx' P(x')}}{y_1^2(x'')} \qquad (12)$$

The l.h.s. of (12) gives $\frac{y_2(x)}{y_1(x)} - \frac{y_2(b)}{y_1(b)} \Rightarrow \frac{y_2(x)}{y_1(x)} = \frac{y_2(b)}{y_1(b)} + W(a)\int dx'' \ldots$ (13)

Hence finally:

$$\boxed{\underset{\sim}{y_2}(x) = \frac{y_2(b)}{y_1(b)} \cdot y_1(x) + \underset{\sim}{y_1}(x)\, W(a) \int_{x''=b}^{x''=x} dx'' \, \frac{e^{-\int_a^{x''} dx' P(x')}}{y_1^2(x'')}} \qquad (14)$$

The first term in (14) simply reproduces the already known solution $y_1(x)$ and hence can be dropped. We can also drop the constant $b$ which also does not lead to a new solution.

When $\underline{P(x)=0}$ then $e^{-\int \cdots} = 1$ and Eq. (14) simplifies to

$$\boxed{y_2(x) = y_1(x)\, W(a) \int^x dx'' \, \frac{1}{y_1^2(x'')}} \qquad (15)$$

<u>Application:</u> For $y''(x) + \omega^2 y(x) = 0 \Rightarrow P(x) = 0$. Suppose that via the series method we have found $y_1(x) = \sin \omega x$. Then since $P(x) = 0$,

$$y_2(x) = (\sin \omega x) \underset{1}{\underbrace{W(a)}} \int^x dx'' \, \frac{1}{\sin^2 \omega x''} \quad ; \text{ recall: } \frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-1}{\sin^2 x} \qquad (16)$$

Hence $\int dx'' \frac{1}{\sin^2 x''} = -\cot x'' \Rightarrow y_2(x) \overset{\text{ignoring constants}}{\cong} \sin \omega x\,(-\cot \omega x) = -\cos \omega x$ ✓ (17)

Hence given $y_1(x) = \sin \omega x$, we find $y_2(x) = \cos \omega x$ upto constants ✓

# DETAILED PROPERTIES OF SPECIAL FUNCTIONS

We now examine some of the detailed properties of special functions beginning with the solutions of the Legendre equation

## Legendre Functions:

Differential Equation: $(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$ (1)

S-L Form: $\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] + n(n+1)P_n(x) = 0$ (2)

Generating Function: $g(t,x) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)\ ; \ 0<t<1$ (3)

Rodrigues' Formula: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ (4)

Orthonormality: $\int_{-1}^{1} dx\, P_n(x) P_m(x) = \frac{2}{2n+1}\delta_{nm}$ (5)

Recurrence Relations: Using the generating function $g(t,x)$ we can derive a variety of recurrence relations among the $P_n(x)$ and their derivatives by differentiating with respect to $t$ or $x$. Consider, for example:

$$\frac{\partial g(t,x)}{\partial t} = (x-t)\left(1-2tx+t^2\right)^{-3/2} = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) \quad (6)$$

$$\Rightarrow \frac{x-t}{(1-2tx+t^2)^{1/2}} = (1-2tx+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x) \quad (7)$$

On the l.h.s. of the previous equation we can expand $(1-2tx+t^2)^{-1/2}$ using (3). So altogether we find

$$(x-t)\sum_{n=0}^{\infty} t^n P_n(x) - (1-2tx+t^2)\sum_{n=0}^{\infty} n\, t^{n-1} P_n(x) = 0 \qquad (8)$$

This equation is a power series in $t$, and hence the coefficient of each power must separately vanish. Expanding (8) into 5 separate terms gives:

$$\sum_n t^n x P_n(x) - \sum_n t^{n+1} P_n(x) - \sum_n n\, t^{n-1} P_n(x) + \sum_n 2n t^n x P_n(x) - \sum_n n\, t^{n+1} P_n(x) = 0 \qquad (9)$$

Same powers of t

Hence:

$$\sum_n t^n \left[x P_n(x) + 2nx P_n(x)\right] + \sum_n t^{n+1}\left[-P_n(x) - nP_n(x)\right] - \sum_n n\, t^{n-1} P_n(x) = 0 \qquad (10)$$

$(n \equiv s)$ $\quad (n+1 \equiv s)$ $\quad (n-1 \equiv s)$

Since $n$ is a dummy summation variable, each of these terms will generate the same powers of $n$ eventually: For example, setting $n=17$ in the 1ST term, $n=16$ in the 2ND term, and $n=18$ in the 3RD term will all generate contributions $\sim t^{17}$, whose net coefficient must vanish. Thus all 3 terms will contribute to the resulting recurrence relation. This can be done as shown above, by the formal replacements, $n \to s$, $n+1 \to s$, and $n-1 \to s$ respectively in these 3 terms. This gives:

$$\sum_{s=0}^{\infty} t^s x P_s(x)(1+2s) + \sum_{s-1=0}^{\infty} t^s P_{s-1}(x)\left[\cancel{-1} - (s-\cancel{1})\right] - \sum_{s+1=0}^{\infty} (s+1) t^s P_{s+1}(x) = 0 \qquad (11)$$

Leaving aside the lower limits on the summation, which only determine which power of $t$ receives contributions from all 3 terms in (11), we have

$$0 = (1+2s)x P_s(x) - s P_{s-1}(x) - (s+1) P_{s+1}(x) \qquad (12)$$

Finally, restoring the usual terminology by letting $s \to n$:

$$2n+1 x P_n(x) = n P_{n-1}(x) + (n+1) P_{n+1}(x) \qquad (13)$$

or
$$\boxed{(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)} \qquad (14)$$

This allows the Legendre polynomial $P_{n+1}(x)$ to be calculated in terms of the lower order polynomials $P_n(x)$ and $P_{n-1}(x)$.

Check:
$$P_0(x) = 1 \; ; \; P_1(x) = x \; ; \; P_2(x) = \frac{1}{2}(3x^2-1) \qquad (15)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) \; ; \; P_4(x) = \frac{1}{8}(35x^4-30x^2+3) \; ; \; P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

Choosing $n=1$ in (14) we get:

$$(1+1) P_2(x) \stackrel{?}{=} (2\cdot 1+1) x P_1(x) - 1 \cdot P_0(x) \qquad (16)$$

$$2 \cdot \frac{1}{2}(3x^2-1) \stackrel{?}{=} 3x \cdot x - 1 = 3x^2 - 1 \;\checkmark$$

Note on Normalization: Recall from last semester that the normalization of the $P_n(x)$ given in the standard way by 91(5) is appropriate to Legendre polynomials defined by $g(t,x)$ which implies:

$$g(t, x=1) = \frac{1}{1-t} = \sum_n t^n \equiv \sum_n t^n P_n(x=1) \Rightarrow \boxed{P_n(x=1) = 1} \qquad (17)$$

Hence the various $n$-dependent constants in (14) are appropriate to $P_n(x)$ normalized as in (17).

## Other Recurrence Relations:

We can derive another class of recurrence relations by differentiating the generating function $g(t,x)$ with respect to $x$, rather than $t$ as before:

$$\frac{\partial}{\partial x} g(t,x) = \frac{t}{(1-2tx+t^2)^{3/2}} = \frac{\partial}{\partial x}\sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=0}^{\infty} t^n P_n'(x) \Rightarrow \qquad (1)$$

$$\frac{t}{(1-2tx+t^2)^{1/2}} = (1-2tx+t^2)\sum_{n=0}^{\infty} t^n P_n'(x) \Rightarrow t\sum_{n=0}^{\infty} t^n P_n(x) = (1-2tx+t^2)\sum_{n=0}^{\infty} t^n P_n'(x) \qquad (2)$$

$$\text{Eq.(2)} \Rightarrow \underbrace{\sum_n t^{n+1} P_n(x)}_{s=n+1} - \underbrace{\sum_n t^n P_n'(x)}_{s=n} + 2\underbrace{\sum_n t^{n+1} x P_n'(x)}_{s=n+1} - \underbrace{\sum_n t^{n+2} P_n'(x)}_{s=n+2} = 0 \qquad (3)$$

As in the previous case we can collect common powers of $t$ by renaming the summation variables as shown. This gives

$$\sum_s t^s \left\{ P_{s-1} + 2xP_{s-1}' - P_s' - P_{s-2}' \right\} = 0 \Rightarrow \{\ldots\} = 0 \qquad (4)$$

By renaming the summation variable as $n$: $s-1=n,\ s=n+1,\ s-2=n-1 \Rightarrow$

$$\boxed{P_n(x) + 2xP_n'(x) - P_{n+1}'(x) - P_{n-1}'(x) = 0} \qquad (5)$$

This recurrence relation allows one to directly obtain $P_{n+1}'(x)$ in terms of $P_n(x)$ and the lower order derivatives $P_n'(x)$ and $P_{n-1}'(x)$.

*

Still other useful relations relating the $P_n$ and their derivatives can be obtained by combining (5) with the previous result in 93(14)

$$\boxed{(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = 0} \qquad (6)$$

Differentiating (6) = 93(14) with respect to $x$, and multiplying ⊗2:

$$2(2n+1)P_n + 2(2n+1)xP_n' - 2nP_{n-1}' - 2(n+1)P_{n+1}' = 0 \qquad (7)$$

Next compute $(2n+1)$ ⊗ Eq.(5):

$$(2n+1)P_n + (2n+1)2xP_n' - (2n+1)P_{n+1}' - (2n+1)P_{n-1}' = 0 \qquad (8)$$

Next subtract (8) from (7): The terms indicated by ⋀⋀⋀ cancel leaving:

$$(2n+1)P_n + \left[(\cancel{2n}+1) - \cancel{2n}\right]P_{n-1}' + \left[(\cancel{2n}+1) - 2(\cancel{n}+1)\right]P_{n+1}' = 0 \qquad (9)$$

$$\therefore \boxed{(2n+1)P_n(x) + P_{n-1}'(x) - P_{n+1}'(x) = 0} \qquad (10)$$

This is yet another relation allowing $P_{n+1}'(x)$ to be directly obtained in terms of the lower order derivative $P_{n-1}'(x)$ and $P_n(x)$ itself. Clearly this allows an iterative procedure in which successively higher-order derivatives are obtained directly in terms of lower-order derivatives.

## OTHER USES OF THE GENERATING FUNCTION $g(t,x)$:

We have seen on p. 93 that the $P_n(x)$ defined by $g(t,x)$ satisfy the normalization condition: $P_n(x=1)=1$. We now use $g(t,x)$ to evaluate $P_n(-1)$:

$$\frac{1}{(1-2tx+t^2)^{1/2}} \xrightarrow{x=-1} \frac{1}{(\underbrace{1+2t+t^2}_{(1+t)^2})^{1/2}} = \frac{1}{(1+t)} = 1 - t + t^2 - t^3 + \cdots \qquad (11)$$

$$\xrightarrow{x=-1} \sum_{n=0}^{\infty} t^n P_n(x=-1)$$

this implies $\boxed{P_n(x=-1) = (-1)^n}$ (12)

Combined with the previous result we have $\boxed{P_n(x=\pm 1) = (\pm 1)^n}$ (13)

We can also use $g(t,x)$ to evaluate $P_n(x=0)$:

$$g(t,x=0) = \frac{1}{(1-2tx+t^2)^{1/2}}\Bigg|_{x=0} = \frac{1}{(1+t^2)^{1/2}} \equiv \sum_n t^n P_n(0) \qquad (14)$$

$$(1+t^2)^{-1/2} = 1-\frac{1}{2}t^2+\frac{3}{8}t^4+\dots+(-1)^n\frac{(1\cdot3\cdot5\cdots[2n-1])}{2^n n!}t^{2n} \qquad (15)$$

We note that since only even powers of $t$ appear in (15), the coefficients of all odd powers of $t$ vanish and hence

$$\boxed{P_{2n+1}(x=0)=0} \qquad (16)$$

This makes sense in terms of the PARITY ARGUMENT given on p.72: Recall that

$$\left.\begin{aligned} &P_{2n}(x) \text{ depends only on } x^0,\ x^2,\ x^4,\dots \\ &P_{2n+1}(x) \text{ depends only on } x,\ x^3,\ x^5,\dots \end{aligned}\right\} \text{ see also next page}$$

$\therefore$ When $x\to 0$ there is no constant term in $P_{2n+1}(x)$ which survives $\Rightarrow P_{2n+1}\equiv 0$.

By contrast, $P_{2n}$ does have a constant term which survives when $x=0$, and we now evaluate it. From (15) $P_{2n}(0)$ is the coefficient of $t^{2n}$ and hence:

$$P_{2n}(0) = \frac{1\cdot3\cdot5\cdots(2n-1)(-1)^n}{2^n n!} = \frac{(-1)^n(2n-1)!!}{2^n n!} \qquad (17)$$

This can be written in another form by noting that

$$(2n)!! = (2n)(2n-2)(2n-4)\dots 6\cdot4\cdot2 = [2(n)][2(n-1)][2(n-2)]\cdots[2(3)][2(2)][2(1)] = 2^n n! \qquad (8)$$

Hence altogether:

$$\boxed{P_{2n}(0) = \frac{(-1)^n(2n-1)!!}{2^n n!} = \frac{(-1)^n(2n-1)!!}{(2n)!!}} \qquad (18)$$

Side Comments: It is easy to show that the analog of (8) for odd $n$ is

$$\boxed{(2n+1)!! = \frac{(2n+1)!}{2n!!} = \frac{(2n+1)!}{2^n n!}} \qquad (19)$$

## Return to the Parity Argument:

Yet another way to see that $P_{2n+1}(x=0)=0$ is to use the generating function $g(t,x)$:

$$g(t,x) = \frac{1}{(1-2tx+t^2)^{1/2}} = \sum_{n=0}^{\infty} t^n P_n(x) = g(-t,-x) \tag{20}$$

$$= \sum_{n=0}^{\infty} (-t)^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n t^n P_n(-x)$$

Hence $(20) \Rightarrow \sum_n t^n P_n(x) = \sum_n t^n (-1)^n P_n(-x) \Rightarrow \boxed{P_n(x) = (-1)^n P_n(-x)}$ (21)

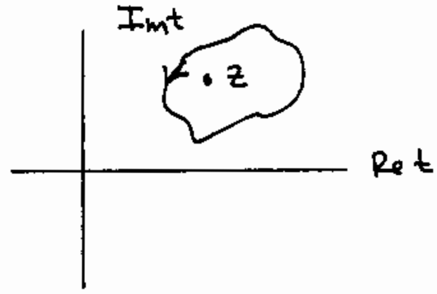
$$(21) \Rightarrow \left.\begin{array}{ll} P_n(-x) = +P_n(x) & n = \text{even} \\ P_n(-x) = -P_n(x) & n = \text{odd} \end{array}\right\}$$

$\hookrightarrow \therefore P_n(0) = 0$ for $n = \text{odd}$

# INTEGRAL REPRESENTATIONS OF LEGENDRE FUNCTIONS

For some purposes it is convenient to express $P_n(x)$ in terms of an integral of some other function. To see how this works consider:

$$f(z) = \frac{1}{2\pi i}\oint dt\,\frac{f(t)}{t-z} \qquad (1)$$




CAUCHY FORMULA

Let $f(z) = (z^2-1)^n \Rightarrow$

$$(z^2-1)^n = \frac{1}{2\pi i}\oint dt\,\frac{(t^2-1)^n}{t-z} \qquad (2)$$

We next take n-derivatives of both sides of (2) noting that

$$\frac{d}{dz}\oint \cdots = \oint \cdots \frac{1}{(t-z)^2}\;;\quad \frac{d^2}{dz^2}\oint \cdots = 2\oint \cdots \frac{1}{(t-z)^3} \qquad (3)$$

$$\therefore\; \frac{d^n}{dz^n}\oint \cdots = n!\oint \cdots \frac{1}{(t-z)^{n+1}} \qquad (4)$$

Combining (2) & (4) $\Rightarrow$

$$\underbrace{\frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n}_{P_n(z)} = \frac{1}{2\pi i}\frac{1}{2^n \cancel{n!}}\underbrace{\frac{d^n}{dz^n}\oint dt\,\frac{(t^2-1)^n}{t-z}}_{\cancel{n!}\oint dt\,\frac{(t^2-1)^n}{(t-z)^{n+1}}} \qquad (5)$$

Rodrigues' Formula $\longrightarrow P_n(z)$

Hence altogether:

$z$ = Complex

$$\boxed{P_n(z) = \frac{2^{-n}}{2\pi i}\oint dt\,\frac{(t^2-1)^n}{(t-z)^{n+1}}} \qquad (6)$$

SCHLÄPLI INTEGRAL REPRESENTATION

One can verify that $P_n(z)$ defined by (6) satisfies the original Legendre equation:

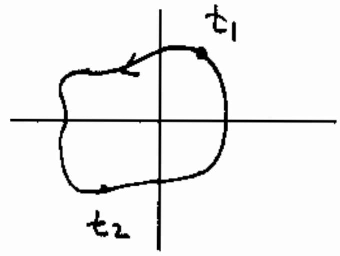
$$(1-z^2)P_n''(z) - 2zP_n'(z) + n(n+1)P_n(z) = \frac{2^{-n}}{2\pi i}\oint dt\,(t^2-1)^n\left\{\frac{(n+2)(n+1)(1-z^2)}{(t-z)^{n+3}} - \frac{2(n+1)z}{(t-z)^{n+2}} + \frac{n(n+1)}{(t-z)^{n+1}}\right\} \qquad (7)$$

After some algebra (which you should do!) it can be shown that the r.h.s. of Eq. (7) can be written in the form:

$$(1-z^2)P_n''(z) - 2z P_n'(z) + n(n+1) P_n(z) = \frac{z^{-n}}{2\pi i}(n+1)\oint dt \frac{d}{dt}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right] \quad (8)$$

If $n = \text{integer}^{*}$ the r.h.s. of (50) has the form

$$\oint \ldots = \oint dt \frac{d}{dt} f(t) = \oint d f(t) = \int_{t_1}^{t_2} d f(t) + \int_{t_2}^{t_1} d f(t)$$




$$= \left[f(t_2) - f(t_1)\right] + \left[f(t_1) - f(t_2)\right] = 0 \quad (9)$$

[redacted] Combining Eqs. (8) & (9) we conclude that the integral representation (SCHLÄFLI) in (6) does [redacted] indeed satisfy the Legendre equation. [* The restriction to $n$ = integer serves to avoid any possible problems from branch cuts, which might arise when $n \neq$ integer.]

## LAPLACE'S INTEGRAL REPRESENTATION:

Starting from the SCHLÄFLI integral representation in (6) we can derive yet another integral representation which is quite useful. Change variables so that for $z$ fixed

$$t \to z + \sqrt{z^2-1}\; e^{i\phi} \Rightarrow dt = i\sqrt{z^2-1}\; e^{i\phi} d\phi \quad (10)$$

Returning to the SCHLÄFLI integral in (6) we then have:

$$(t^2-1)^n = \left[\left(z+\sqrt{z^2-1}\; e^{i\phi}\right)^2 - 1\right]^n = \ldots \text{algebra} \ldots (z^2-1)^{n/2} e^{in\phi} 2^n \left[\sqrt{z^2-1}\cos\phi + z\right]^n \quad (11)$$

Similarly: $$(t-z)^{n+1} = \left[\sqrt{z^2-1}\; e^{i\phi}\right]^{n+1} = (z^2-1)^{(n+1)/2} e^{i(n+1)\phi} \quad (12)$$

and $$dt = i(z^2-1)^{1/2} e^{i\phi} d\phi \quad (13)$$

Inserting Eqs. (11)–(13) into (6) we find:

$$P_n(z) = \frac{\cancel{2^{-n}}}{2\pi \boxed{i}} \oint \frac{\underbrace{\left\{ \cancel{(z^2-1)}^{n/2} e^{in\phi}\, \cancel{2^n} \left[\sqrt{z^2-1}\,\cos\phi + z\right]^n \right\}}_{(t^2-1)^n} \underbrace{\left\{ \boxed{i}\, \cancel{(z^2-1)}^{1/2} e^{i\phi}\, d\phi \right\}}_{dt}}{\underbrace{\left\{ \cancel{(z^2-1)}^{(n+1)/2} e^{i(n+1)\phi} \right\}}_{(t-z)^{n+1}}} \qquad (14)$$

The indicated terms cancel against one another, so that in the end we find

$$\boxed{P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \left[\sqrt{z^2-1}\,\cos\phi + z\right]^n} \qquad (15)$$

LAPLACE'S FIRST INTEGRAL REPRESENTATION

Since we obtained this integral from the substitution in (10) $(t-z) = \sqrt{z^2-1}\, e^{i\phi}$, the contour in (15) is a circle of radius $|(z^2-1)^{1/2}|$ around the point $z$ (which is fixed). We also require that $\mathrm{Re}\, z > 0$ so that the contour encloses the point $t=1$ but not $t=-1$.

We can derive yet another integral [redacted] representation by noting that the original Legendre equation is invariant under

$$\boxed{n \to -n-1} \qquad (16)$$

Check: $(1-z^2)P_n''(z) - 2zP_n'(z) + n(n+1)P_n(z) = 0$ (17)

$n \to (-n-1)$, $(n+1) \to (-n-1+1) \Rightarrow n(n+1) \to (-n-1)(-n) = n(n+1)$ ✓

Hence the [redacted] integral in (15) will also be a solution if we write

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \left[\sqrt{z^2-1}\,\cos\phi + z\right]^{-n-1} \qquad (18)$$

We can use the integral in (18) to generate the generating function:

Begin with the substitution

$$\sqrt{z^2-1}\,\cos\phi + z = t \quad\Rightarrow\quad d\phi = \frac{-dt}{\sqrt{z^2-1}}\,\frac{1}{\sin\phi} \qquad (19)$$

Note that $\cos\phi = \dfrac{t-z}{\sqrt{z^2-1}} \Rightarrow \sin\phi = \sqrt{1-\cos^2\phi} = \left\{\dfrac{1-2tz+t^2}{1-z^2}\right\}^{1/2}$ (20)

Hence from (19) & (20): $d\phi = \dfrac{-dt}{\cancel{\sqrt{z^2-1}}}\left\{\dfrac{\cancel{1-z^2}}{1-2tz+t^2}\right\}^{1/2} = -dt\left\{\dfrac{-1}{1-2tz+t^2}\right\}^{1/2} = \dfrac{-i\,dt}{\sqrt{1-2tz+t^2}}$ (21)

Combining Eqs. (18)–(21): $P_n(z) = \dfrac{1}{2\pi}\oint dt\left(\dfrac{-i}{\sqrt{1-2tz+t^2}}\right)\left(t^{-n-1}\right)$ (22)

$$\therefore\ \boxed{P_n(z) = \frac{1}{2\pi i}\oint dt\,\frac{(1-2tz+t^2)^{-1/2}}{t^{n+1}}} \qquad (23)$$

Recall from last semester <u>Cauchy's Derivative Formula</u>:

$$\frac{f^{(n)}(t_0)}{n!} = \frac{\partial^n}{\partial t^n} f(t)\Big|_{t_0}\cdot\frac{1}{n!} = \frac{1}{2\pi i}\oint dt\,\frac{f(t)}{(t-t_0)^{n+1}} \qquad (24)$$

Making the identifications $t_0 = 0$; $f(t) = (1-2tz+t^2)^{-1/2} \Rightarrow$ (25)

$$\boxed{P_n(z) = \frac{1}{n!}\frac{\partial^n}{\partial t^n}\left(\frac{1}{\sqrt{1-2tz+t^2}}\right)_{t=0}} \qquad (26)$$

Finally (!) it is easy to show that <u>(26) is exactly equivalent to the usual generating function</u>:

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{n=0}^{\infty} t^n P_n(z) \qquad (27)$$

To see the connection between (26) & (27) write

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{n=0}^{\infty} t^n P_n(z) = P_0 + t\cdot P_1(z) + t^2 P_2(z) + \cdots + t^m P_m(z) + \cdots \qquad (28)$$

Then, for example:

$$\frac{1}{2!}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\sqrt{\cdots}}\right)_{t=0} = \frac{1}{2!}\left\{ 2\cdot 1 P_2(z) + 3\cdot 2\cancel{t P_3(z)} + 4\cdot 3 \cancel{t^2 P_4(z)} + \cdots + m(m-1)\cancel{t^{m-2}} P_m(z) + \cdots \right\}_{t=0} \qquad (29)$$

$$= P_2(z) \checkmark$$

Hence

$$\boxed{\sum_{n=0}^{\infty} t^n P_n(z) = \frac{1}{\sqrt{1-2tz+t^2}} \Longleftrightarrow P_n(z) = \frac{1}{n!}\frac{\partial^n}{\partial t^n}\left(\frac{1}{\sqrt{1-2tz+t^2}}\right)_{t=0}} \qquad (30)$$

## PARTIAL SUMMARY

1) Starting with the Legendre equation $(1-z^2)P_n''(z) - 2zP_n'(z) + n(n+1)P_n(z) = 0$ the series solution $P_n(z) = \sum_{\lambda=0}^{\infty} a_\lambda z^{k+\lambda}$ leads to finite polynomials.

2) As we showed last semester, these polynomials can be reproduced by the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n \qquad (31)$$

(3) From 97(5) we saw that Rodrigues' formula $\Rightarrow$ SCHLÄFLI INTEGRAL REP. for $P_n(z)$.

(4) By appropriate substitutions SCHLÄFLI $\Rightarrow$ LAPLACE INTEGRAL REP.

(5) By yet another set of substitutions the LAPLACE INTEGRAL $\Rightarrow$ (23)

$$P_n(z) = \frac{1}{2\pi i}\oint dt \frac{(1-2tz+t^2)^{-1/2}}{t^{n+1}} \Longrightarrow P_n(z) = \frac{1}{n!}\frac{\partial^n}{\partial t^n}\left(\frac{1}{\sqrt{1-2tz+t^2}}\right)_{t=0}$$

= generating function

# Finding the 2nd Solution of the Legendre Equation

We saw on p.88 that for a differential equation of the form

$$y''(x) + P(x)\,y'(x) + Q(x)\,y(x) = 0 \qquad (1)$$

we could start from a known solution $y_1(x)$ to find a second solution $y_2(x)$:

$$\boxed{y_2(x) = (\text{Constant})\, y_1(x) \int^{x} dx'' \, \frac{e^{-\int_a^{x''} dx'\, P(x')}}{[y_1(x'')]^2}} \qquad (2)$$

Note: Do not confuse the generic function $P(x)$ with the Legendre polynomial $P_n(x)$!!

For the Legendre equation we can write: $y''(x) + \left(\frac{-2x}{1-x^2}\right) y'(x) + \frac{n(n+1)}{1-x^2}\, y(x) = 0$ (3)

Hence: $$P(x) = \frac{-2x}{1-x^2} = \frac{-2x}{(1+x)(1-x)} \Rightarrow \int_a^{x''} dx'\, P(x') = \int_a^{x''} dx' \left(\frac{-2x'}{(1+x')(1-x')}\right)$$

Note: $$\frac{1}{2}\left\{\frac{1}{1+x'} - \frac{1}{1-x'}\right\} = \frac{-x'}{(1+x')(1-x')} \qquad (4)$$

Hence $$\int dx' \ldots = 2 \cdot \frac{1}{2} \int_a^{x''} dx' \left\{\frac{1}{1+x'} - \frac{1}{1-x'}\right\} = \ln(1+x')\Big|_a^{x''} + (-1)^2 \ln(1-x')\Big|_a^{x''} \qquad (5)$$

(-1) for coeff.
(-1) from integration

Hence $$\int dx' \ldots = +\ln\left\{(1+x')(1-x')\right\}_a^{x''} = \ln(1-x''^2) - \ln(1-a^2) \qquad (6)$$

Combining (2) & (6) we find:

$$e^{-\int dx' \ldots} = e^{-\ln(1-x''^2)} \underbrace{e^{\ln(1-a^2)}}_{\text{constant}} = \underbrace{(\text{Const})}_{C_1} \frac{1}{1-x''^2} \qquad (7)$$

It follows that given $y_1(x) \equiv P_n(x)$ we can write for $y_2(x) = Q_n(x)$:

$$Q_n(x) = C_{1n} P_n(x) \int^x dx'' \frac{1}{C_{2n}(1-x''^2)\left[P_n(x'')\right]^2} \qquad (8)$$

Clearly the constants $C_{1n}$ and $C_{2n}$ can be combined into a single constant $B_n = C_{1n}/C_{2n}$. We further note from p.88 that the original solution $y_1(x)$ (in this case $P_n(x)$) is also part of the general solution for $y_2(x)$, so that the general form of the second solution $Q_n$ is (letting $x'' \to x$)

$$\boxed{Q_n(z) = A_n P_n(z) + B_n P_n(z) \int^z \frac{dx}{(1-x^2)\left[P_n(x)\right]^2} = P_n(z)\left\{A_n + B_n \int^z \frac{dx}{(1-x^2)\left[P_n(x)\right]^2}\right\}} \qquad (9)$$

Second solution

$Q_n$ as given by (9) will be a 2ND linearly independent solution. However, if we want $Q_n$ to correspond to some standard "textbook" solution we should choose $A_n$ & $B_n$ appropriately. For example, for the case $n=0$ $P_0(z) = 1$ and hence

$$Q_0 = A_0 + B_0 \int^z \frac{dx}{1-x^2} = A_0 + B_0 \underbrace{\int dx \, \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)}_{\frac{1}{2}\left\{\ln(1+x) - \ln(1-x)\right\} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)} \qquad (10)$$

indefinite integral

Hence

$$Q_0 = A_0 + B_0 \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) \qquad (11)$$

If we expand the $\ln(\ldots)$ term in an infinite series we find:

$$(11) \Rightarrow \quad Q_0 = A_0 + B_0\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots + \frac{z^{2s+1}}{2s+1} + \cdots\right) \qquad (12)$$

We can compare (12) to the infinite (non-terminating) series solution to the Legendre equation obtained for $n=0$ by choosing $k=1$. From Eq. 81(12) we would find:

$$Q_0 = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots \qquad (13)$$

We see that to get (12) & (13) to agree we should choose $A_0 = 0$ & $B_0 = 1$. Then from (13) & (11) we have:

$$\boxed{Q_0(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)} \qquad (14)$$

As expected, Q(z) is not a finite polynomial, and also diverges at $z = \pm 1$

The higher order $Q_n$ can be obtained in a similar manner. Repeated application of various recurrence relations leads to:

$$\boxed{Q_n = \frac{1}{2}P_n(z)\ln\left(\frac{1+z}{1-z}\right) - \frac{(2n-1)}{1\cdot n}P_{n-1}(z) - \frac{(2n-5)}{3(n-1)}P_{n-3}(z) - \cdots} \qquad (15)$$

Evidently, for this expression to make sense it must be that Re z falls in the range:

$$-1 < \mathrm{Re}\, z < 1 \qquad (16)$$

However, we can define $Q_n(z)$ for Re z outside this range via a process of analytic continuation: Here this involves the replacement

$$\ln\left(\frac{1+z}{1-z}\right) \longrightarrow \ln\left(\frac{z+1}{z-1}\right) \qquad (17)$$

## SUMMARY

We saw from 81(12) that for $n=0$, the $k=1$ solution leads to an infinite series, and hence $k=1$ is the "wrong" choice to obtain a finite polynomial. However, we see from (14) & (17) that this series is indeed a legitimate solution, which can be summed to give the result in (14).

The "moral" is that even if we were not smart enough to find the 2ND solution in this way, we could always find it starting from ~~p~~(88) Eq. (14) as we did.