COMPLEX INTEGRATION

We wish to consider integrals of the form

$$\int_C W(z)\, dz = \int \left[u(x,y) + i v(x,y)\right]\left[dx + i dy\right] \qquad (1)$$

$z \to C$
Some contour or path

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

If x and y are literally 2-dim coordinates, they may depend parametrically on another coordinate t (e.g. time) in which case we have

$$\int_C W(z)\,dz = \int_{t_0}^{t_1} dt \left(u\frac{\partial x}{\partial t} - v\frac{\partial y}{\partial t}\right) + i\int_{t_0}^{t_1} dt\left(v\frac{\partial x}{\partial t} + u\frac{\partial y}{\partial t}\right) \qquad (2)$$

We define $\displaystyle\int_C = -\int_{-C}$ $\leftarrow$ contour traversed in opposite direction

<u>Definition</u>: Unless otherwise stated $\displaystyle\int_C$ for a closed contour is taken to be in the <u>counter clockwise</u> (ccw) direction.

Also: $\displaystyle\int_{C_1} + \int_{C_2} = \int_{C_1 + C_2}$ (obvious!)

<u>Triangle Inequality for Integrals:</u>

$$\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,|dz| \qquad (3)$$

This can be seen as follows:

$$\int f(z)\,dz \cong \sum\{f(z_1)\Delta z + f(z_2)\Delta z + \cdots\} \quad (4)$$

$$= \sum\{f(z_1)+f(z_2)+\cdots\}\Delta z \Rightarrow \left|\sum\{f(z_1)+f(z_2)+\cdots\}\Delta z\right| \leq \sum_i |f(z_i)||\Delta z| \quad (5)$$

cancellations possible — no cancellations possible

Hence $\left|\int_C f(z)\,dz\right| \leq \int |f(z)||dz|$

Recall that the usual triangle inequality is:

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (6)$$

Note that (5) can also be arrived at by writing

$$\int_C \cong \sum\{f(z_1)\Delta z + f(z_2)\Delta z + \cdots\} \Rightarrow \quad (7)$$

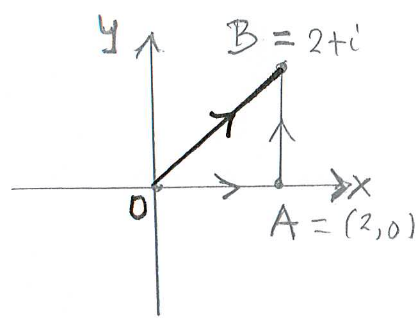
$$\left|\sum\{\cdots\}\right| = |f(z_1)\Delta z + f(z_2)\Delta z + \cdots| \leq |f(z_1)\Delta z| + |f(z_2)\Delta z| + \cdots$$

But $|f(z_1)\Delta z| = |f(z_1)||\Delta z| \Rightarrow \sum_i \leq \sum_i |f(z_i)||\Delta z|$ (8)

We will return to use this later;

<u>Example of Complex Integration:</u>

Find $I = \int_C z^2\,dz$



C is <u>either</u> 1) OB $\equiv C_1$

2) OA + AB $\equiv C_2$

<u>NOTE:</u> When a path or contour is specified in the $\angle z$ plane this defines a relationship between x and y along the path (or between r and θ in polar coordinates) so that there is <u>only 1 independent integration variable left.</u>

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1) Along C_1 : $z^2 = \underbrace{x^2-y^2}_{u} + i\,\underbrace{2xy}_{v}$; $dz = dx + i\,dy$

$$\int_{C_1} z^2\,dz = \int_{C_1} \left[(x^2-y^2)\,dx - 2xy\,dy\right] + i\int_{C_1} (x^2-y^2)\,dy + 2xy\,dx$$

Up to this point no detailed specification of the path has taken place (i.e. no relation between x & y).

Next we note that along $C_1 = OB$ $x = 2y \Rightarrow dx = 2dy$

Hence $$\int_{C_1} = \int_{y=0}^{y=1} \left(3y^2 \cdot 2dy - 4y^2 \cdot dy + i\,3y^2 \cdot dy + i\,4y^2 \cdot 2dy\right) = \frac{2}{3} + i\,\frac{11}{3}$$

2) $$\int_{C_2} = \int_{OA} z^2\,dz + \int_{AB} z^2\,dz$$

Along OA $\left\{\begin{matrix} dy = 0 \\ y = 0 \end{matrix}\right\} \Rightarrow \int_{OA} z^2\,dz \rightarrow \int_0^2 u\,dx + i\int_0^2 v\,dx$

Since $y=0 \Rightarrow v = 0$ $= \int_{OA} = \int_0^2 x^2\,dx + i\int 0 = \frac{8}{3}$

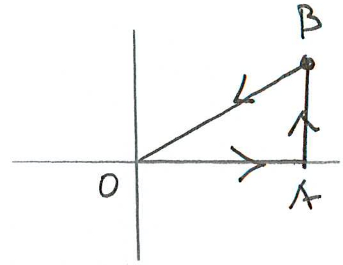
Along AB $\left\{\begin{matrix} dx = 0 \\ x = 2 \end{matrix}\right\} = \int_{AB} = \int_0^1 -v\,dy + i\int_0^1 u\,dy = -4\int_0^1 y\,dy + i\int_0^1 (4-y^2)\,dy$

$= -2 + i\,\frac{11}{3}$

Hence $$\int_{C_2} = \int_{OA} + \int_{AB} = \frac{8}{3} + \left(-2 + i\,\frac{11}{3}\right) = \frac{2}{3} + i\,\frac{11}{3}$$

This is an example of a theorem we are about to prove: $\int_{z_1}^{z_2} f(z)\,dz$ is independent of the path if $f(z)$ is analytic. Note also that for the closed path $OA+AB+(-OB)$

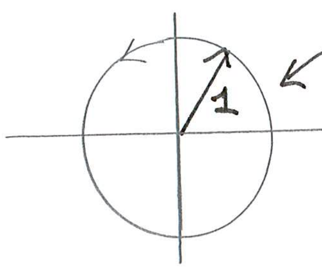
We have:
$$\oint_{OA+AB+(-OB)} = \left(\frac{2}{3}+i\frac{11}{3}\right)-\left(\frac{2}{3}+i\frac{11}{3}\right)=0$$



Similarly: for $f(z)$ analytic

$$\boxed{\oint f(z)\,dz = 0}$$

Another Example: Evaluate $\oint_C \bar{z}\,dz$ $\quad C=$ Unit circle around origin



$$\Rightarrow \oint_C \bar{z}\,dz = \int_0^{2\pi} \underbrace{1e^{-i\theta}}_{\bar{z}} \cdot \underbrace{(ie^{i\theta}d\theta)}_{dz \text{ along contour}}$$

$$= 1\int_0^{2\pi} i\,d\theta = 2\pi i \neq 0$$

Since $\bar{z}$ is not analytic $\oint \neq 0$, in contrast to previous example.

Note also: Along unit circle $\bar{z} = 1e^{-i\theta} = 1/z$. Hence (for later) we have also shown that

$$\oint_C \frac{1}{z}\,dz = 2\pi i$$

<u>Side Comment:</u> Since $f(z)=z^2$ is analytic one can also evaluate the integral directly as in real integration:

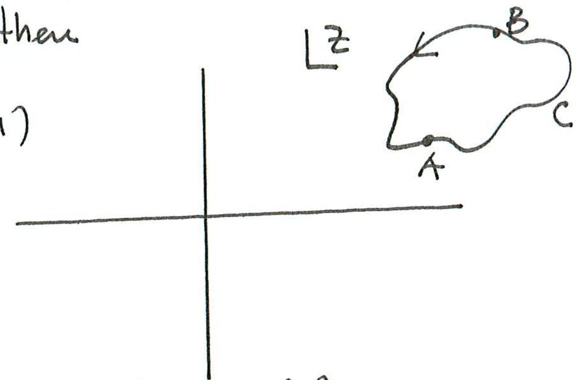
$$I = \int_0^{2+i} dz\, z^2 = \frac{1}{3} z^3 \Big|_0^{2+i} = \frac{1}{3}(2+i)^3 = \frac{1}{3}(2+11i) = \frac{2}{3} + i\frac{11}{3} \checkmark$$

This is a fundamental theorem of complex calculus which holds in any regime where $f(z)$ is analytic.

CAUCHY'S THEOREM: (IMPORTANT!!)

Thm: If $f(z)$ is analytic within and on a contour C (closed) and $f'(z)$ is continuous in this region then

$$\oint_C f(z)\,dz = 0 \qquad (1)$$



Proof:

$$\oint f(z)\,dz = \oint (u\,dx - v\,dy) + i\oint (u\,dy + v\,dx) \qquad (2)$$

We can show that $\oint = 0$ if we can show that the integrand is a perfect differential; for example:

$$\oint (u\,dx - v\,dy) = \oint d\phi = \int_A^B d\phi + \int_B^A d\phi = (\phi_B - \phi_A) + (\phi_A - \phi_B) = 0 \qquad (3)$$

To show this, consider a scalar function $\phi(x,y)$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \equiv \Phi_x\,dx + \Phi_y\,dy \qquad (4)$$

Assuming all relevant derivatives exist, then

$$\frac{\partial \Phi_x}{\partial y} = \frac{\partial^2\phi}{\partial y\,\partial x} \qquad \frac{\partial \Phi_y}{\partial x} = \frac{\partial^2\phi}{\partial x\,\partial y} \Rightarrow \boxed{\frac{\partial \Phi_x}{\partial y} = \frac{\partial \Phi_y}{\partial x}} \qquad (5)$$

Hence from (4) & (5):

$$\boxed{\Phi_x\,dx + \Phi_y\,dy = d\phi \Rightarrow \frac{\partial \Phi_x}{\partial y} = \frac{\partial \Phi_y}{\partial x}} \qquad (6)$$

We can also show that the implication goes the other way: Let $\vec{\Phi} = (\Phi_x, \Phi_y, 0)$

We have shown at the beginning of the Semester that

$$\vec{\nabla}\times\vec{\Phi} = 0 \Leftrightarrow \vec{\Phi} = \vec{\nabla}\phi \qquad (7)$$

Now if $\dfrac{\partial \Phi_x}{\partial y} = \dfrac{\partial \Phi_y}{\partial x} \Rightarrow \partial_x \Phi_y - \partial_y \Phi_x = (\vec{\nabla}\times\vec{\Phi})_z = 0 \qquad (8)$

Furthermore, in 2-dimensions where $\Phi = \Phi(x,y)$ we have

$$(\vec{\nabla}\times\vec{\Phi})_x = \underset{0}{\partial_y \Phi_z} - \underset{0}{\underbrace{\partial_z \Phi_y}} \; ; \; (\vec{\nabla}\times\Phi)_y = \underset{0}{\underbrace{\partial_z \Phi_x}} - \underset{0}{\partial_x \Phi_z} = 0 \qquad (9)$$

Hence altogether: $(\vec{\nabla}\times\vec{\Phi})_x = (\vec{\nabla}\times\vec{\Phi})_y = (\vec{\nabla}\times\vec{\Phi})_z = 0 \Rightarrow (\vec{\nabla}\times\vec{\Phi}) = 0$

$$\Downarrow$$

$$\Phi = \vec{\nabla}\phi \qquad (10)$$

Combining the previous results Cauchy's Theorem follows by the following chain of arguments:

$$\partial_y \Phi_x(x,y) \overset{1}{=} \partial_x \Phi_y(x,y) \overset{2}{\Rightarrow} \vec{\nabla}\times\vec{\Phi} = 0 \overset{3}{\Rightarrow} \vec{\Phi} = \vec{\nabla}\phi \Rightarrow \qquad (11)$$

$$d\phi \overset{4}{=} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \overset{5}{\equiv} \Phi_x\, dx + \Phi_y\, dy \qquad (12)$$

$$\therefore \; \Phi_x dx + \Phi_y dy \overset{6}{=} d\phi \Rightarrow \underline{\text{perfect differential}} \qquad (13)$$

Stated In Words:

a) If $\partial\Phi_x/\partial y = \partial\Phi_y/\partial x$ this implies $\vec{\nabla}\times\vec{\Phi} = 0$

b) If $\vec{\nabla}\times\vec{\Phi} = 0$ then $\vec{\Phi}$ can be written as $\Phi = \vec{\nabla}\phi$

(this introduces the scalar field ϕ)

c) Since ϕ is a scalar, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy \equiv \Phi_x dx + \Phi_y dy$

d) Combining a) … c) we see that an expression such as $(\Phi_x dx + \Phi_y dy)$ is a perfect differential if $(\partial_y\Phi_x = \partial_x\Phi_y)$

e) If we identify $u(x,y) = \Phi_y(x,y)$ and $v(x,y) = \Phi_x(x,y)$ then the condition for a perfect differential is just the C-R ~~condition~~ equation $\partial u/\partial x = \partial v/\partial y$. The same holds true for the other C-R relation

f) From Eq. (3) p. CV-34.1 this completes the proof.

Implications of Cauchy Theorem

a) Path Independence of $\int f(z)\,dz$

Since $\oint_{C_1+(-C_2)} f(z)\,dz = 0$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0 \Rightarrow \boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz} \quad (1)$$

This is the same statement as the path-independence of the work done moving in a "conservative-field" (e.g. gravity). The reasons are also the same (... perfect differential...)

b) Fundamental Theorem of Calculus ⊛

From a) above the function $F(z) \equiv \int_{z_0}^{z} f(z')\,dz'$ defines a unique function, since all that needs to be specified are the endpoints z_0, z

⊛ Theorem: $F(z)$ is also analytic and $F'(z) = f(z)$ (2)

Proof: $F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} \cdots - \int_{z_0}^{z} \cdots = \int_{z}^{z+\Delta z} \cdots$ along any path (e.g. a line)

Note that trivially: $f(z) = \frac{f(z)}{\Delta z} \int_{z}^{z+\Delta z} dz' = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)\,dz'$ (no prime!) (3)

Then (2) & (3) $\Rightarrow$ $\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(z') - f(z)]\,dz'$ (4)

we want to show that this $\to 0$

We note that $f(z)$ is continuous (because it is analytic) and hence for all $\epsilon>0$ $\exists$ $\delta>0$ such that

$$|f(z')-f(z)|<\epsilon \quad \text{when} \quad |z'-z|<\delta \tag{5}$$

In our case $z'-z=\Delta z$, so take $0<|\Delta z|<\delta$

Then
$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| < \frac{1}{|\Delta z|}\int_{z}^{z+\Delta z} \underbrace{|f(z')-f(z)|}_{<\epsilon}\,|dz'| \tag{6}$$

[The above uses the triangle inequality that we previously proved for integrals]

From (6):
$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| < \frac{\epsilon}{|\Delta z|}\underbrace{\int_{z}^{z+\Delta z}|dz'|}_{|\Delta z|} = \epsilon \tag{7}$$

Hence
$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| < \epsilon \quad \text{when} \quad 0<|\Delta z|<\delta \tag{8}$$

Now as $|\Delta z|\to 0$ $\quad \delta\to 0 \Rightarrow \epsilon\to 0$ so that (9)

$$\Big|\underbrace{\lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z}}_{F'(z)}-f(z)\Big|\to 0 \quad \Rightarrow \boxed{F'(z)=f(z)}$$

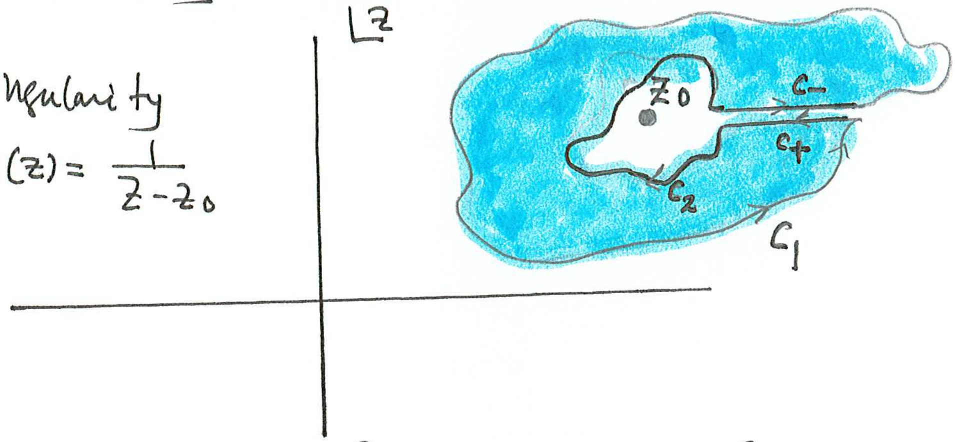
The usual version of the fundamental theorem of calculus then follows:

$$\int_\alpha^\beta f(z)\,dz = \int_{z_0}^{\beta} f(z)\,dz' - \int_{z_0}^{\alpha} f(z)\,dz = F(\beta)-F(\alpha) \quad \text{where} \tag{10}$$

$$F(z)=\int_{z_0}^{z} f(z')\,dz' \tag{11}$$

(c) Contours Containing Singularities:

$f(z)$ has a singularity at z_0, e.g. $f(z) = \frac{1}{z - z_0}$



If the task is to evaluate $\oint_{C_1} f(z)\,dz$, where C_1 is some complicated contour, we can deform the contour as shown:
The contributions along C_+ and C_- cancel (since there $f(z)$ is analytic).

Moreover

$$\int_{C_1 + C_+ + C_- + (+C_2)} = 0 \quad \Bigg\} \text{ Since there are no singularities \underline{inside} this contour.}$$

Since $\int_{C_+} + \int_{C_-} = 0 \Rightarrow \int_{C_1} + \int_{+C_2} = 0 \Rightarrow \boxed{\oint_{C_1} f(z)\,dz = \oint_{-C_2} f(z)\,dz}$

Note that this is a different result from that proved in a) on p.39, since there $f(z)$ was analytic in the region R, but here it is not: This is what is called a "multiply connected region" where the contour encloses a region within which $f(z)$ is not analytic, and where $\oint_{C1} f(z)\,dz \neq 0$.

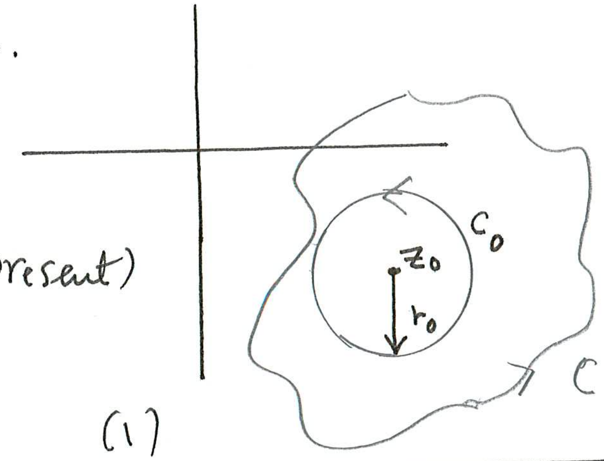
Implications: Given C_1 you can make your life much easier by replacing C_1 by a simpler contour C_2 (like a circle), provided that C_1 and C_2 enclose exactly the same singularities.

d) CAUCHY'S INTEGRAL FORMULA *

* VERY IMPORTANT !!

Let $g(z)$ denote a general function of z, which may or may not be analytic in some domain.

Then we have shown that if C and C_0 enclose the same singularities (if any are present) then



$$\oint_C g(z)\,dz = \int_{C_0} g(z)\,dz \qquad (1)$$

C_0: circle of radius r_0. $\Rightarrow$ along C_0 $\boxed{z - z_0 = r_0 e^{i\theta}}$ $\boxed{dz = i r_0 e^{i\theta}\,d\theta}$ ☆ (2)

Of special interest are functions $g(z)$ having the form

$$\boxed{g(z) = \frac{f(z)}{z - z_0} \quad ; \; f(z) \text{ is analytic within } C} \quad (3)$$

$g(z)$ is not analytic at z_0, but has the special form of non-analyticity given in (3). To evaluate $\oint_C g(z)\,dz$ we replace C by C_0 as shown. Then:

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_{C_0} dz\,\frac{f(z)}{z - z_0} = \underbrace{f(z_0)\oint_{C_0} \frac{dz}{z - z_0}}_{\mathrm{I}} + \underbrace{\oint_{C_0} dz\,\frac{[f(z) - f(z_0)]}{z - z_0}}_{\mathrm{II}} \qquad (4)$$

II: As $r_0 \to 0$ $(z - z_0) \equiv \Delta z \to 0$ and $[f(z) - f(z_0)]/\Delta z \to f'(z)$, which is analytic since $f(z)$ is analytic. Hence via Cauchy

$$\mathrm{II} = \oint dz\, f'(z) \equiv 0$$

I Hence: $\displaystyle\oint \frac{f(z)}{z - z_0}\,dz = f(z_0)\oint \frac{dz}{(z - z_0)} \overset{☆}{=} f(z_0)\int_0^{2\pi} \frac{(i\cancel{r_0}\cancel{e^{i\theta}}\,d\theta)}{\cancel{r_0}\cancel{e^{i\theta}}} = 2\pi i f(z_0)$

$$\therefore \quad \boxed{\oint dz\,\frac{f(z)}{z - z_0} = 2\pi i f(z_0)}$$

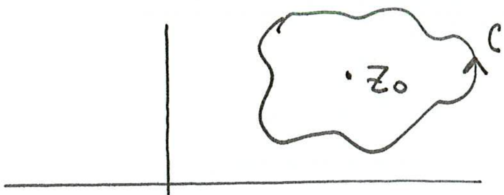
CAUCHY'S INTEGRAL FORMULA

COMMENTS:

This is one of the most important results in the theory of complex functions from both a practical & "philosophical" point of view:

practical : Evaluation of real integrals via contour integration

"philosophical":

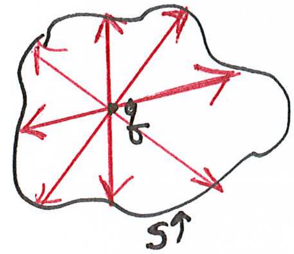


$$\frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{z-z_0} = f(z_0)$$

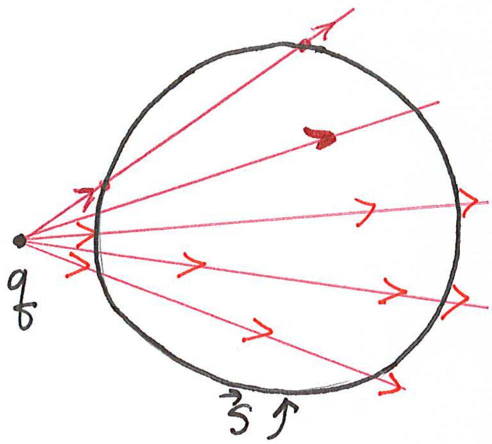
This tells us that the value of a function $f(z)$, which is analytic, can be determined at any point interior to C by knowing its value only along the boundary C of that region. [Since there are "more points" in the interior than on its boundary, analyticity buys us something.]

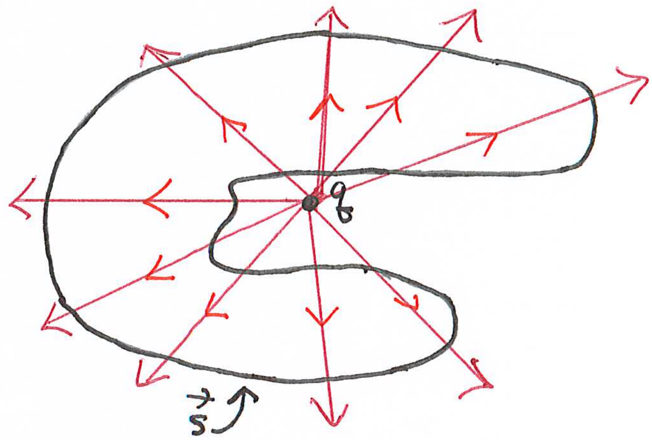
CONNECTION TO GAUSS' LAW*

* $\oint_S \vec{E}\cdot d\vec{s} = q/\epsilon_0$; q = charge inside surface



$$\oint_S \vec{E}\cdot d\vec{s} \neq 0$$





$$\oint_S \vec{E}\cdot d\vec{S} = 0$$

A singularity of $g(z)$ at z_0 in the complex plane, thus plays the same role as a charge q in 2- or 3-dimensional space. Thus $\oint_C dz\, g(z) = \oint_C dz \frac{f(z)}{z-z_0} \equiv 0$ unless z_0 is inside the contour C, just as $\oint_S \vec{E}\cdot d\vec{S} = 0$ unless q is inside $\vec{S}$

CONSEQUENCES OF CAUCHY'S INTEGRAL FORMULA

a) Derivatives of Analytic Functions

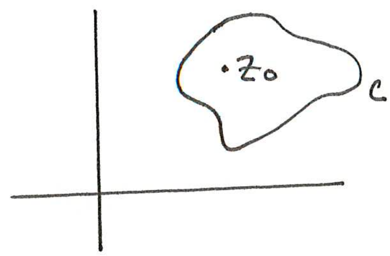
One can prove that all of the derivatives of an analytic function are analytic. This is not true for real variables: $x^{1/2}$ is differentiable everywhere, but its derivative $\sim x^{-1/2}$ has a singularity at the origin.

By contrast $z^{1/2}$ is analytic, but only because we introduced a branch cut, (e.g. along the real axis), and this eliminates the point $z=0$ where the derivative of $z^{1/2}$ would have a singularity.

To differentiate an analytic function start with CAUCHY'S INTEGRAL FORMULA:

$$\frac{d}{dz_0} f(z_0) = \frac{d}{dz_0}\left\{\frac{1}{2\pi i}\oint_C dz\, \frac{f(z)}{z-z_0}\right\} = \frac{1}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^2} \qquad (1)$$

any point inside C (a variable for these purposes)



Similarly, taking another derivative:

$$\frac{d^2}{dz_0^2} f(z_0) \equiv f''(z_0) = \frac{2!}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^3} \qquad (2)$$

For the nth derivative:

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C dz\,\frac{f(z)}{(z-z_0)^{n+1}}} \qquad (3)$$

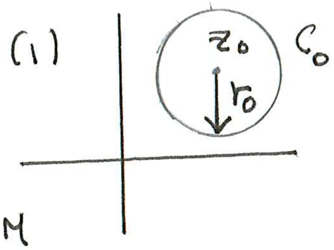
Thus $f(z)$ has all possible derivatives within C. The kth derivative is therefore continuous in C because this formula allows the $(k+1)$th derivative to be computed.

b) Liouville's Theorem:

If $f(z)$ is analytic and $|f(z)|$ is bounded for all values of z, then $f(z) = \text{constant}$.

Proof: Start with Cauchy's Integral Formula $\Rightarrow$

$$f'(z_0) = \frac{1}{2\pi i}\oint_C dz \frac{f(z)}{(z-z_0)^2} \qquad (1)$$



Choose this contour for $C = C_0$

Then: $|z - z_0| = r_0$ and

$$|f'(z_0)| \leq \left|\frac{1}{2\pi i}\right| \oint_{C_0} \underbrace{|dz|}_{2\pi r_0} \frac{\overbrace{|f(z)|}^{\leq M}}{\underbrace{|(z-z_0)|^2}_{r_0^2}} \leq \frac{1}{2\pi r_0^2} \cdot M \cdot 2\pi r_0 = \frac{M}{r_0} \qquad (2)$$

Here M denotes the maximum value that $f(z)$ assumes in the complex plane, (which we can do since by assumption $|f(z)| \leq M$).

From Eq. (2) above it follows that

$$|f'(z_0)| \leq \frac{M}{r_0} \quad \leftarrow \text{for any } r_0$$

$\therefore$ Take $r_0 \to \infty \Rightarrow |f'(z_0)| \to 0 \Rightarrow f(z_0) = \text{constant}$

Q.E.D.

Implication: If an analytic function is not a constant, then it cannot be bounded. Recall the example given previously [p. CV-20,21]

$$\sin z = \sin x \cdot \cosh y + i \sinh y \cos x$$

$$|\sin z|^2 = \sin^2 x + \underbrace{\sinh^2 y}_{\text{not bounded}}$$

(C) FUNDAMENTAL THEOREM OF ALGEBRA

If $P_m(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$ is a polynomial in z

Then $P_m(z) = 0$ has at least one root.

Proof: Assume the contrary – that $P_m(z) \neq 0$ for any z (i.e. that there is no root). Then $P_m^{-1}(z)$ is entire (i.e. analytic in the entire complex plane.) Moreover, since $P_m(z)$ is a polynomial, $|1/P_m(z)| \to 0$ as $z \to \infty \Rightarrow$ $P_m^{-1}(z)$ is bounded for all z. This follows by noting that in the finite part of the $\mathbb{C}z$ plane we can find the biggest value of $|P_m^{-1}(z)|$ and be assured that there will not be a larger value as $|z| \to \infty$.

It follows that since $|1/P_m(z)|$ is bounded, then if we assume $P_m(z) \neq 0$ anywhere, so that $1/P_m(z)$ is analytic everywhere, then $1/P_m(z)$ must be a constant $\Rightarrow$ Contradiction!

It then follows that the assumption that $P_m(z)$ does not vanish anywhere must be false $\Rightarrow$ for some z_0 $P_m(z_0) = 0$. QED ✓

This argument can be repeated by writing

$$P_m(z) = (z - z_0)\, P_{m-1}(z)$$

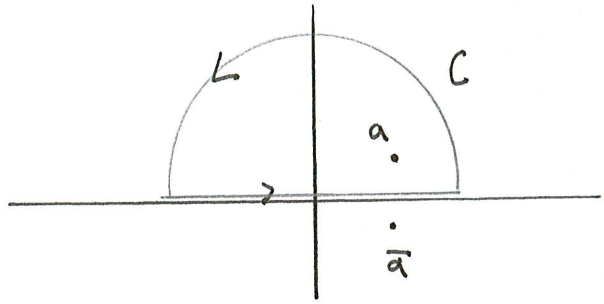
$P_{m-1}(z)$ must then have a root also, at $z_1 \Rightarrow P_m(z) = (z-z_0)(z-z_1)\, P_{m-2}(z)$.

Hence altogether:

$$\boxed{P_m(z) = \underbrace{(z-z_0)(z-z_1)\cdots(z-z_{m-1})}_{m \text{ factors}}}$$

d) HILBERT TRANSFORMS & DISPERSION RELATIONS

This formalism allows a complex function $f(z)$ to be expressed as a real integral over its real and imaginary parts. [Applications to follow!!]



$\bar{a} = a^*$

Consider:

$$I = \frac{1}{2\pi i}\oint_C dz \left\{ \frac{f(z)}{z-a} + \frac{f(z)}{z-\bar{a}} \right\} \qquad (1)$$

Cauchy's Integral Formula $\Rightarrow$ only the singularity inside C counts so that

$$I = \frac{1}{2\pi i}\oint_C dz\, \frac{f(z)}{z-a} = f(a) \qquad (2)$$

Suppose now that $f(z)$ is a function like e^{iz} which vanishes (as $z \to \infty$) along the semicircle: At any point along the semi-circle

$$e^{iz} = e^{i(x+iy)} = e^{ix}e^{-y} \xrightarrow{y\to\infty} 0 \text{ in upper half-plane (uhp)}$$

Then

$$I = \oint_C dz \ldots = \int_{-\infty}^{\infty} dx + \underbrace{\int_{\text{semi-circle}} dz}_{\to 0} \longrightarrow \int_{-\infty}^{\infty} dx \ldots \qquad (3)$$

Hence

$$f(a) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx \left\{ \frac{f(x)}{x-a} + \frac{f(x)}{x-\bar{a}} \right\} \qquad (4)$$

<u>Side Comment!</u> Note the technique that we used:

1) First we evaluate I by using Cauchy to get $\oint$ for the whole contour. This gives $I = f(a)$

2) Next we break up $\oint$ into 2 pieces, one going $\to 0$. This allows us to evaluate real integrals via complex integration.

Returning to Eq. (4) write $a = \alpha + i\beta$ $\quad \bar{a} = \alpha - i\beta$

$$\frac{f(x)}{x-a} + \frac{f(x)}{x-\bar{a}} = f(x)\left\{\frac{(x-\bar{a})+(x-a)}{(x-a)(x-\bar{a})}\right\} = f(x)\left\{\frac{2x-(\bar{a}+a)}{(x-a)(x-\bar{a})}\right\} \tag{5}$$

$$a + \bar{a} = 2\alpha$$

$$(x-a)(x-\bar{a}) = [(x-\alpha)+i\beta][(x-\alpha)-i\beta] = (x-\alpha)^2 + \beta^2 \tag{6}$$

$$\therefore\ f(x)\{\ldots\} = f(x)\cdot\frac{2(x-\alpha)}{(x-\alpha)^2+\beta^2} \tag{7}$$

$\longrightarrow u + iv$

Hence altogether:
$$I = f(a) = \frac{1}{\pi i}\int_{-\infty}^{\infty} dx\, f(x)\, \frac{(x-\alpha)}{(x-\alpha)^2+\beta^2} \tag{8}$$

$u(\alpha,\beta) + iV(\alpha,\beta)$

$$u(\alpha,\beta) + iV(\alpha,\beta) = \frac{1}{\pi i}\int_{-\infty}^{\infty} dx\, [u(x, y=0) + i\,V(x, y=0)] \cdot \frac{(x-\alpha)}{(x-\alpha)^2+\beta^2} \tag{9}$$

Equating real and imaginary parts,

$$\boxed{\begin{aligned} u(\alpha,\beta) &= \frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{(x-\alpha)V(x)}{(x-\alpha)^2+\beta^2} \\ V(\alpha,\beta) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{(x-\alpha)u(x)}{(x-\alpha)^2+\beta^2} \end{aligned}} \tag{10}$$

HILBERT TRANSFORM PAIR

$u(\alpha,\beta)$ & $V(\alpha,\beta)$

When $\beta = 0$:
$$u(\alpha) = u(\alpha,0) = P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{V(x)}{x-\alpha}$$
$$V(\alpha) = V(\alpha,0) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{u(x)}{x-\alpha} \tag{11}$$

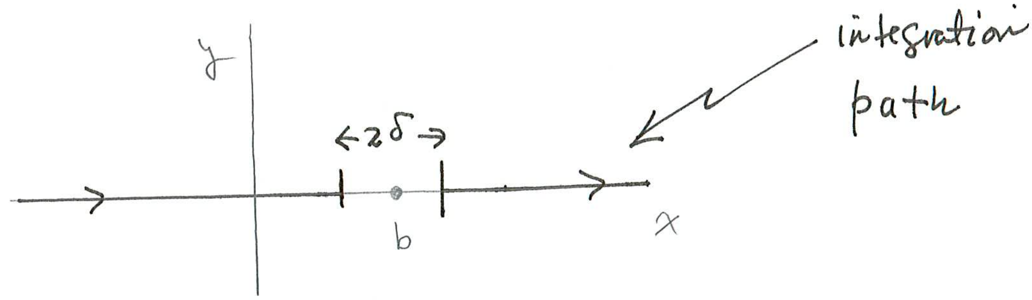
$P \equiv$ Principal Value Integration

PRINCIPAL VALUE INTEGRATION

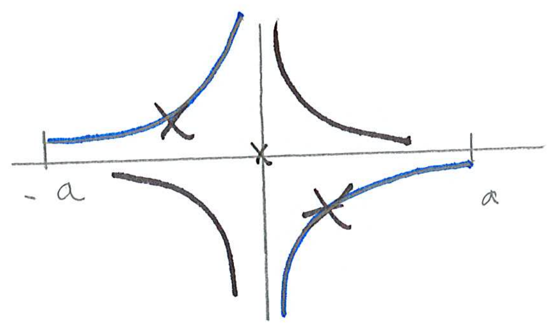
This is a formal technique for making sense of integrals such as those on p. CV-48.2 where there is a singularity (pole) along the path of integration.

$$P\int_a^c dx\, f(x) \equiv \lim_{\delta\to 0}\left[\int_a^{b-\delta} dx\, f(x) + \int_{b+\delta}^{c} dx\, f(x)\right] \qquad (1)$$

Pictorially:



Examples:



(a)

Consider $I = \int_{-a}^{a} dx \frac{1}{x}$ } not well defined since ∃ a singularity at $x = 0$.

However, by symmetry we expect to find $I = 0$. Doing a Principal Value integration we find:

$$I = P\int_{-a}^{a}\frac{dx}{x} = \lim_{\delta\to 0}\left[\int_{-a}^{0-\delta}\frac{dx}{x} + \int_{0+\delta}^{a}\frac{dx}{x}\right] \qquad (2)$$

$$= \lim_{\delta\to 0}\left\{\ln(-\delta) - \ln(-a) + \ln(a) - \ln(\delta)\right\} = \lim_{\delta\to 0} \ln\left(\frac{-\delta a}{-a\delta}\right) = \ln(1) = 0 \checkmark \qquad (3)$$

(b) For $0 < b < a$ Consider

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$$I = P\int_{-a}^{a} dx \, \frac{f(x)}{x-b}$$ ← this is often encountered in QM

To evaluate: $$I = P\left[\int_{-a}^{a} dx \, \frac{f(b)}{x-b} + \int_{-a}^{a} dx \, \frac{f(x)-f(b)}{x-b}\right] \quad (4)$$

$$= f(b)\, P\int_{-a}^{a} dx \, \frac{1}{x-b} + \int_{-a}^{a} dx \, \frac{f(x)-f(b)}{x-b} \quad (5)$$

P not needed here since $\int$ is well-behaved at $x=b$

$$\boxed{I = f(b) \ln\left(\frac{b-a}{b+a}\right) + \int_{-a}^{a} dx \, \frac{f(x)-f(b)}{x-b}} \quad 6)$$

known

well behaved

(C) DIRAC's FORMULA

Symbolically :
$$\boxed{\lim_{\epsilon\to 0}\left(\frac{1}{\omega \pm i\epsilon}\right) = P\left(\frac{1}{\omega}\right) \mp i\pi\,\delta(\omega)} \qquad (1)$$

- As with every formula involving $\delta(\omega)$, this is understood as holding under an integral sign.
- This formula is very widely used in QM!

Proof!
$$\frac{1}{\omega \pm i\epsilon} = \frac{1}{\omega \pm i\epsilon}\,\frac{\omega \mp i\epsilon}{\omega \mp i\epsilon} = \frac{\omega}{\omega^2+\epsilon^2} \mp \frac{i\epsilon}{\omega^2+\epsilon^2} \qquad (2)$$

At the beginning of the semester we established the following representation for $\delta(x)$:
$$\lim_{\epsilon\to 0}\left(\frac{1}{\pi}\,\frac{\epsilon}{x^2+\epsilon^2}\right) = \delta(x) \qquad (3)$$

Consider then Eq.(2) appearing (as it should!) in an integral:
$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{\omega \pm i\epsilon} = \underbrace{\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega\,\frac{\omega f(\omega)}{\omega^2+\epsilon^2}}_{I} \mp \lim_{\epsilon\to 0} i\int_{-\infty}^{\infty} d\omega\, f(\omega)\,\frac{\epsilon}{\omega^2+\epsilon^2} \qquad (4)$$

$$= I \mp i\pi\int_{-\infty}^{\infty} d\omega\, f(\omega)\,\delta(\omega) = I \mp i\pi f(0) \qquad (5) \leftarrow$$

Next evaluate
$$I = \lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\delta} d\omega\,\frac{f(\omega)\,\omega}{\omega^2+\epsilon^2} + \int_{\delta}^{\infty} d\omega \ldots + \int_{-\delta}^{\delta} d\omega \ldots\right\} \qquad (6) \leftarrow$$

The reason for introducing δ is to make the integrals well-behaved when the limit $\epsilon\to 0$ is taken. NOTE: (6) is an identity.

In Eq. (6) we take the limit as $\delta \to 0$ after first taking the limit $\epsilon \to 0$ in the first two terms:

$$I = \underbrace{\lim_{\delta \to 0} \left[\int_{-\infty}^{-\delta} d\omega \, \frac{f(\omega)}{\omega} + \int_{\delta}^{\infty} d\omega \, \frac{f(\omega)}{\omega} \right]}_{\equiv P \int_{-\infty}^{\infty} d\omega \, \frac{f(\omega)}{\omega}} + \underbrace{\lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{-\delta}^{\delta} d\omega \, \frac{\omega f(\omega)}{\omega^2 + \epsilon^2}}_{f(0) \underbrace{\int_{-\delta}^{\delta} d\omega \, \frac{\omega}{\omega^2+\epsilon^2}}_{\equiv 0 \text{ (odd function over a symmetric interval)}}} \tag{7}$$

(8)

Hence $$I = P \int_{-\infty}^{\infty} d\omega \, \frac{f(\omega)}{\omega} \tag{9}$$

Combining Eqs. (5), (6), and (9) then gives:

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\omega \, f(\omega) \left\{ \frac{1}{\omega \pm i\epsilon} \right\} = P \int_{-\infty}^{\infty} d\omega \, f(\omega) \left\{ \frac{1}{\omega} \right\} \mp i\pi \int_{-\infty}^{\infty} d\omega \, f(\omega) \left\{ \delta(\omega) \right\} \tag{10}$$

or symbolically: $$\boxed{\lim_{\epsilon \to 0} \left\{ \frac{1}{\omega \pm i\epsilon} \right\} = P \left\{ \frac{1}{\omega} \right\} \mp i\pi \, \delta(\omega)} \tag{11}$$

Related Identities:

$$\frac{d}{dx} \log x = \frac{1}{x} - i\pi \, \delta(x) \tag{12}$$

$$= \lim_{\epsilon \to 0} \frac{1}{x + i\epsilon} \tag{13}$$

To justify these formulas consider

$$\int_{-\epsilon}^{\epsilon} dx \left\{ \frac{d}{dx} \log x \right\} \stackrel{?}{=} \underbrace{\int_{-\epsilon}^{\epsilon} dx \left(\frac{1}{x} \right)}_{\equiv 0 \text{ by symmetry (as before!)}} \tag{14}$$

However, the l.h.s. of (14) gives

$$\int_{-\epsilon}^{\epsilon} dx \left\{ \frac{d}{dx} \log x \right\} = \log x \Big|_{-\epsilon}^{\epsilon} = \log\left(\frac{\epsilon}{-\epsilon}\right) = \log(-1) = -i\pi \quad ✓ \tag{15}$$

Hence to obtain a correct identity we should write – as in (12) –

$$\frac{d}{dx} \log x = \frac{1}{x} - i\pi\,\delta(x) \tag{16}$$

Since

$$\int_{-\epsilon}^{\epsilon} dx\,(-i\pi\,\delta(x)) = -i\pi \quad ✓ \tag{17}$$

<u>Another Identity:</u> Start with $\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$ (18)

$$\therefore \boxed{\delta(x) = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\left(\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon}\right)} \tag{19}$$

From the previous results we have yet another identity

$$\lim_{\epsilon\to 0}\left(\frac{1}{\omega+i\epsilon}\right) + \lim_{\epsilon\to 0}\left(\frac{1}{\omega-i\epsilon}\right) = \left[P\left(\frac{1}{\omega}\right) - \cancel{i\pi\delta(\omega)}\right] + \left[P\left(\frac{1}{\omega}\right) + \cancel{i\pi\delta(\omega)}\right] \tag{20}$$

$$\therefore \boxed{P\left(\frac{1}{\omega}\right) = \frac{1}{2}\lim_{\epsilon\to 0}\left[\frac{1}{\omega+i\epsilon} + \frac{1}{\omega-i\epsilon}\right]} \tag{21}$$

Return to Hilbert Transform Pairs:

From p. CV-48.2 Eq. (1):

$$u(\alpha) = P\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{v(x)}{x-\alpha} \quad ; \quad v(\alpha) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{u(x)}{x-\alpha} \tag{1}$$

$$= \underbrace{P\,\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{v(\alpha)}{x-\alpha}}_{I_1} + \underbrace{P\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{v(x)-v(\alpha)}{x-\alpha}}_{I_2} \tag{2}$$

$$I_1 = \frac{v(\alpha)}{\pi}\; P\int_{-\infty}^{\infty} \frac{dx}{x-\alpha} = \frac{v(\alpha)}{\pi}\lim_{\delta\to 0}\left\{\int_{-\infty}^{\alpha-\delta} dx\cdots + \int_{\alpha+\delta}^{\infty} dx\cdots\right\} \tag{3}$$

$$\{\ldots\} = \log(x-\alpha)\Big|_{-\infty}^{\alpha-\delta} + \log(x-\alpha)\Big|_{\alpha+\delta}^{\infty} = \lim_{L\to\infty}\left\{\begin{array}{l}\log(\cancel{\alpha}-\delta-\cancel{\alpha}) - \log(-L-\alpha) \\ +\log(L-\alpha) - \log(\cancel{\alpha}+\delta-\cancel{\alpha})\end{array}\right\} \tag{4}$$

$$= \lim_{L\to\infty}\left\{\log(-\delta) - \log(-L) + \log(L) - \log(\delta)\right\} \tag{5}$$

$$\{\ldots\} = \log(-\delta L) - \log(-L\delta) = \log\left(\frac{-\delta L}{-L\delta}\right) = \log\left(\delta/\delta\right) = \log(1) = 0 \tag{6}$$

Hence $I_1 \equiv 0$.

In I_2 the integral is well behaved so that P can be dropped. This allows (1) to be rewritten as:

$$\boxed{u(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{v(x)-v(\alpha)}{x-\alpha} \quad ; \quad v(\alpha) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{u(x)-u(\alpha)}{x-\alpha}} \tag{7}$$

Application of Hilbert Transform Pairs:

Consider the function $f(z) = i e^{iz} \equiv u(x,y) + i v(x,y)$

$$= e^{-y}(-\sin x + i\cos x) \quad (1)$$

$\therefore u(x,0) \equiv u(x) = -\sin x \quad ; \quad v(x,0) \equiv v(x) = \cos x$

Since these are the real and imaginary parts of $f(z)$ they form a Hilbert transform pair. Formally, $f(z) \to 0$ as $y \to \infty$ so that the integral we previously considered in the derivation of the Hilbert transform pair vanishes along the semi-circle in the u.h.p. Then we can write:

$$v(\alpha) = \cos\alpha = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{u(x) - u(\alpha)}{x-\alpha} = \frac{-1}{\pi}\int_{-\infty}^{\infty} dx \left[\frac{-\sin x + \sin\alpha}{x-\alpha}\right] \quad (2)$$

Taking $\alpha = 0$ then gives:

$$\cos(0) = 1 = +\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x} \Rightarrow \boxed{\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x} = \pi} \quad (3)$$

This demonstrates how Hilbert transform pairs can be used to evaluate real integrals. We will return to derive (3) by the more conventional techniques of contour integration. These techniques show the power of <u>using complex variables to evaluate real integrals</u> in an elegant way!

DISPERSION RELATIONS

A dispersion relation (as we use it) is an integral relation between two observable quantities where the integration is restricted to values of the argument that are physically meaningful.

Consider the Hilbert transform pair in Eq. (11) p. CV-48.2

$$u(\alpha) = P\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{v(x)}{x-\alpha} \; ; \quad v(\alpha) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{u(x)}{x-\alpha} \qquad (1)$$

Let us rename variables to make a connection with real problems:

$$u(\omega) = P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \; \frac{v(\omega')}{\omega'-\omega} \; ; \quad v(\omega) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \; \frac{u(\omega')}{\omega'-\omega} \qquad (2)$$

Now if ω and ω' are actually frequencies then negative frequencies are not meaningful, so that (2) is not really a dispersion relation,

However, u, v may be the real and imaginary parts of a function $f(z)$ which is the FOURIER TRANSFORM of a real ~~transform~~ function $G(t)$:

$$f(z) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dt \; G(t)\, e^{izt} \qquad (3)$$

Then $$f^*(z) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dt \; \underbrace{G^*(t)}_{G(t)}\, e^{-iz^*t} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dt \; G(t)\, e^{-iz^*t} = f(-z^*) \qquad (4)$$

If z is real $(z=\omega)$ then

$$\boxed{f^*(\omega) = f(-\omega)} \qquad (5)$$

$$[u(\omega)+iv(\omega)]^* = u(-\omega)+iv(-\omega)$$

$$\hookrightarrow u(\omega) - iv(\omega) = u(-\omega)+iv(-\omega)$$

$\Big\} \Rightarrow$ REALITY CONDITIONS

$$\boxed{\begin{array}{l} u(\omega) = u(-\omega) \Rightarrow \text{EVEN} \\ v(-\omega) = -v(\omega) \Rightarrow \text{ODD} \end{array}} \qquad (6)$$

To use the REALITY CONDITIONS, return to Eq.(2) on the previous page:

$$u(\omega) = P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \frac{v(\omega')}{\omega'-\omega} = P\frac{1}{\pi}\int_{-\infty}^{0} d\omega' \frac{v(\omega')}{\omega'-\omega} + P\frac{1}{\pi}\int_{0}^{\infty} d\omega' \frac{v(\omega')}{\omega'-\omega} \quad (7)$$

In the first $\int$ let $\omega' \to -\omega'$:

$$P\frac{1}{\pi}\int_{-\infty}^{0} \cdots \longrightarrow P\frac{1}{\pi}\int_{+\infty}^{0} (-d\omega') \frac{\overbrace{v(-\omega')}^{-v(\omega')}}{-\omega'-\omega} = P\frac{1}{\pi}\int_{0}^{\infty} d\omega' \frac{v(\omega')}{\omega'+\omega} \quad (8)$$

Combining (7) & (8):

$$u(\omega) = P\frac{1}{\pi}\int_{0}^{\infty} d\omega' v(\omega') \underbrace{\left\{\frac{1}{\omega'+\omega} + \frac{1}{\omega'-\omega}\right\}}_{\frac{2\omega'}{\omega'^2-\omega^2}} \quad (9)$$

Hence finally:

$$\boxed{u(\omega) = \frac{2}{\pi} P\int_{0}^{\infty} d\omega' \frac{\omega' v(\omega')}{\omega'^2-\omega^2}} \quad (10)$$

Note that this integral is a true dispersion relation: It expresses the real function $u(\omega)$ as an integral over the imaginary part $v(\omega')$, but restricted to physical frequencies $v(\omega')$ where $0 \leq \omega' \leq \infty$.

In a similar manner we have from (2):

$$v(\omega) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \frac{u(\omega')}{\omega'-\omega} \longrightarrow \cdots\int_{-\infty}^{0}\cdots + \int_{0}^{\infty}\cdots \quad (11)$$

$$\int_{-\infty}^{0} d\omega' \frac{u(\omega')}{\omega'-\omega} = \int_{\infty}^{0} (-d\omega') \frac{u(-\omega')}{-\omega'-\omega} \xrightarrow{(u\to+u)} -\int_{0}^{\infty} d\omega' \frac{v(\omega')}{\omega'+\omega} \quad (12)$$

$$\therefore v(\omega) = \frac{-P}{\pi}\int_{0}^{\infty} d\omega' u(\omega') \left\{\frac{-1}{\omega'+\omega} + \frac{1}{\omega'-\omega}\right\} \quad (13)$$

$$\therefore \boxed{v(\omega) = -\frac{2\omega}{\pi} P\int_{0}^{\infty} d\omega' \frac{u(\omega')}{\omega'^2-\omega^2}} \quad (14)$$

EQS. (10) & (14) ARE THE KRAMERS-KRONIG DISPERSION RELATIONS

The K-K dispersion relations were originally derived for the electric susceptibility $\chi(z) \to \chi_R(\omega) + i\chi_I(\omega)$. However, they are very widely applied in many areas including condensed matter and high-energy physics.

CV-54.3

Subtracted Dispersion Relations:

The Kramers-Kronig dispersion relations would appear to be useless in cases where $u(\omega')$ or $v(\omega')$ does not vanish sufficiently fast at ∞. In this case the integrals in (10) and (14) might not converge.

Even if the integrals formally converge, we may want them to converge faster for computational purposes. Subtracted dispersion relations is a technique for increasing the rate of convergence:

Suppose we know the value of $u(\omega)$ at some $\omega = \omega_0$:

$$u(\omega_0) = \frac{2}{\pi} P \int_0^\infty d\omega' \frac{\omega' v(\omega')}{\omega'^2 - \omega_0^2} \tag{15}$$

Then

$$u(\omega) = u(\omega_0) \underbrace{- \frac{2}{\pi} P \int_0^\infty d\omega' \frac{\omega' v(\omega')}{\omega'^2 - \omega_0^2}}_{0} + \frac{2}{\pi} P \int_0^\infty d\omega' \frac{\omega' v(\omega')}{\omega'^2 - \omega^2} \tag{16}$$

$$u(\omega) = u(\omega_0) + \frac{2}{\pi} P \int_0^\infty d\omega' \, \omega' v(\omega') \left\{ \frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - \omega_0^2} \right\} \tag{17}$$

$$\{\cdots\} = \frac{\omega^2 - \omega_0^2}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)} \Rightarrow \boxed{u(\omega) = u(\omega_0) + \frac{2}{\pi}(\omega^2 - \omega_0^2) \int_0^\infty d\omega' \frac{\omega' v(\omega')}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)}} \tag{18}$$

Convergence of $\int_0^\infty d\omega'$ is improved because the denominator now goes as $(1/\omega')^4$ for large ω', rather than $(1/\omega')^2$; This process can be repeated.