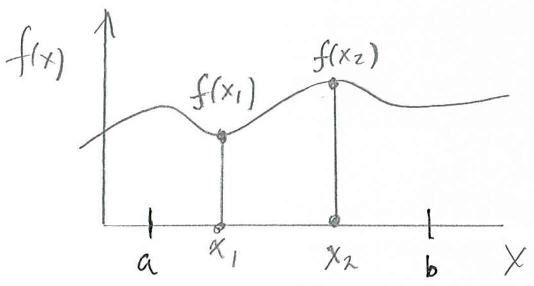
INTRODUCTION TO HILBERT SPACE

TWO (RELATED) VIEWS OF A FUNCTION AS A VECTOR IN AN ∞-DIM SPACE

(I)



In this way a function is represented in the interval $[a, b]$ by an ∞ set of values $(f(x_a), f(x_1), \ldots f(x_b))$ which give the "projections" of $f(x)$ on the "axes" $(x_a, x_1, \ldots x_b)$. This is similar to specifying the components of an n-dimensional vector by giving its components along the n-axes:
$\vec{V} = (V_1, V_2, \ldots V_n)$.

(II) We can also represent a function via a Taylor series in $[a, b]$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow f(x) = [a_0, a_1, \ldots a_n \ldots]$$

A function which is piecewise continuous (continuous in every subinterval in $[a,b]$)
These two are related: A function which is piecewise continuous can be expanded as in (II) via a Taylor series [More later!]

SCALAR PRODUCT: For 2 functions $f(x), g(x)$

$$\boxed{\langle f | g \rangle \equiv \int_a^b dx\, f^*(x)\, g(x)} \qquad (1)$$

NORM OF A FUNCTION $\quad N[f] \equiv \langle f | f \rangle = \int_a^b dx\, |f(x)|^2 \qquad (2)$

Side Comment on Scalar Product:

The definition $\langle f|g\rangle = \int_a^b dx\, f^*(x)\, g(x)$ is the natural extension of the usual finite dimensional result:

$$\langle a|b\rangle = \vec{a}\cdot\vec{b} = \sum_{i=1}^{N} a_i^* b_i = a_1^* b_1 + a_2^* b_2 + \dots + a_N^* b_N$$

Let $a_1 \to a(x_1), \dots, a_N \to a(x_N)$; $b_1 \to b(x_1)$, $b_N \to b(x_N)$

Then $\langle a|b\rangle \to \sum_i a^*(x_i)\, b(x_i) \to \int dx\, a^*(x)\, b(x)$ ✓

The only additional ingredient needed is the interval over which the integral is evaluated.

HILBERT SPACE:

The set of functions for which the norm is finite constitutes Hilbert space. Sometimes to ensure that the norm is finite, especially when the integration limits are ∞, a weighting function $\omega(x)$ is introduced:

$$\boxed{\langle f|g\rangle \longrightarrow \int_a^b dx\, f^*(x)\, g(x)\, w(x)} \quad (3)$$

For example: $w(x) \sim e^{-x}$ (Laguerre) or $e^{-x^2/2}$ (Hermite)

Reisz-Fischer Theorem: (not proved here)

Functions with $N[f] < \infty$ are complete: A Hilbert space is then a complete linear vector space with a complex scalar product.

DIGRESSION ON "∞" [G.F. Cantor – Theory of Transfinite Numbers]

- Denumerable ∞: $\aleph_0$ ["Aleph-Null"] $1, 2, 3, \ldots$ / $2, 4, 6, \ldots$
- Continuum: C [points on a line]

KNOW CANTOR PROOF!! SEE "WORLD OF MATH" or GAMOW "ONE, TWO, THREE... INFINITY"

(I) $\Rightarrow$ Continuum infinity to specify points

(II) $\Rightarrow$ Denumerable infinity " " "

However, for a piecewise continuous function a denumerable ∞ is enough.

Separable Hilbert Space: One having at least one denumerable basis. Otherwise the Hilbert space is non separable:

NOTATION IN HILBERT SPACE

(a) $\langle f | g \rangle = 0 \Rightarrow f, g$ are orthogonal in $[a,b]$

(b) $\langle f | f \rangle = 1 \Rightarrow f$ is normalized in $[a,b]$

(c) $\langle f_i | f_j \rangle = \delta_{ij} \Rightarrow f_i, f_j$ are orthonormal in $[a,b]$

Example: $f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$; $n = 0, \pm 1, \pm 2, \ldots$

The set $\{f_n(x)\}$ is orthonormal in $[-\pi, \pi]$. To see this:

$$\langle f_n | f_m \rangle = \int_{-\pi}^{\pi} f_n^*(x) f_m(x)\, dx = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\pi}^{\pi} (e^{inx})^* e^{imx}\, dx \qquad (1)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, e^{i(m-n)x} = \frac{1}{2\pi i (m-n)} e^{i(m-n)x} \Big]_{-\pi}^{\pi} = 0 \text{ if } m \neq n \qquad (2)$$

However, when $n = m$ (2) gives

$$\langle f_n | f_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \cdot 1 = 1 \checkmark \qquad (3)$$

Hence altogether $\langle f_n | f_m \rangle = \delta_{mn}$ ✓

When a (non-negative) weight function $w(x)$ is used (as previously) then

$$\langle f_n | f_m \rangle \equiv \int_a^b dx\, f_n^*(x) f_m(x) w(x) = \delta_{mn} \Rightarrow \text{orthonormality}$$

Completeness of $\{f_n(x)\}$ & Convergence:

- For the ∞-dim case this is important for the same reasons as in the finite dim case
- Completeness has physical consequences:
  Example: To obtain a complete set of solutions to the DIRAC EQUATION one had to include negative frequency solutions. This lead to the discovery of ANTIMATTER (positrons)

- As noted above there are 2 ways to specify $f(x)$ in $[a,b]$
  a) Specify $f(x_i)$ $a \le x_i \le b$ at an ∞ (c_0) number of points
  b) Choose a basis in $[a,b]$ and expand $f(x) = \sum_{n=0}^{\infty} c_n f_n(x)$.
  This requires a denumerable ∞ ($\aleph_0$) of points. This is sufficient for the continuous functions we are studying.

- In the ∞-dim case the completeness of $\{f_n(x)\}$ – because they have finite norms – ensures that any function formed from them will also have a finite norm i.e. it will belong to the same Hilbert space:

$$f(x) = \sum_n^{\infty} f_n(x) \Rightarrow \langle f|f\rangle < \infty \qquad (1)$$

This is important since there are many examples where a sequence of terms having some property converges to an expression which does not:

Ex: $$(1-x)^{1/2} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \dots \qquad (2)$$

For $x = 1/2$ l.h.s $= \sqrt{1/2} = \sqrt{2}/2 =$ irrational, but each term on the r.h.s is rational

UNDERSTANDING CONVERGENCE

For the finite dim case there is no ambiguity in what it means to add up ~~...~~ a sum of terms. However, in the infinite dim case one must deal with convergence, and the different types of convergence.

Q: What does it mean for a sequence of terms to converge to $f(x)$?

Consider : $$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots \qquad (1)$$

Define : $$f_n(x) \equiv \sum_{i=0}^{n} a_i x^i \qquad (2)$$

We expect that if the series in (1) converges then the difference between successive f_n's should get smaller:

$$\lim_{n\to\infty} |f_{n+1}(x) - f_n(x)| \Rightarrow 0 \qquad (3)$$

Definition : $\{f_n(x)\}$ is called a SEQUENCE

Uniform Convergence : In $[a,b]$ $f_n(x)$ converges uniformly to $f(x)$ if for every x in $a \leq x \leq b$ and for every $\epsilon > 0$ $\exists$ $N(\epsilon)$ such that

$$\boxed{|f(x) - f_n(x)| < \epsilon \quad \text{for } n > N(\epsilon)} \qquad (4)$$

This can be replaced by the CAUCHY CRITERION which does not require knowing $f(x)$ in advance:

$$\boxed{|f_r(x) - f_s(x)| < \epsilon \quad \text{for } r,s > N} \qquad (5)$$

CAUCHY CRITERION FOR UNIFORM CONVERGENCE

(A) COMPLETENESS OF A SET OF FUNCTIONS – UNIFORM CONVERGENCE

Let $g(x)$ = piecewise continuous on $[a,b]$

$\{f_i(x)\} \stackrel{?}{=}$ "basis" set of functions on $[a,b]$. They will be a basis if they are complete.

<u>Define</u>: $g_n(x) = \sum_{i=1}^{n} a_i f_i(x)$

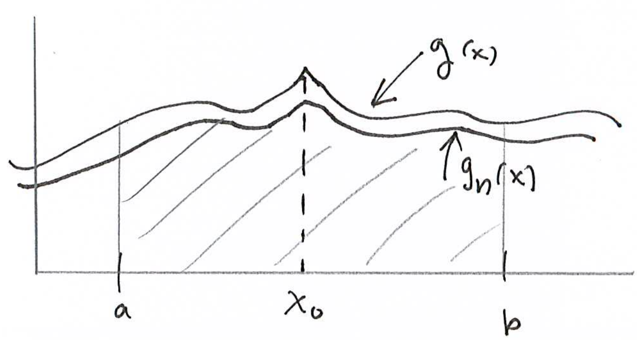
$\{f_i(x)\}$ will be complete in the sense of <u>uniform convergence</u> if

$$\boxed{|g(x) - g_n(x)| = \left|g(x) - \sum_{i=1}^{n} a_i f_i(x)\right| < \epsilon \quad \text{for } n > N(\epsilon)} \qquad (1)$$

For a given set of $\{f_i(x)\}$ this comes down to asking whether a_i can be found satisfying (1).

(B) COMPLETENESS – CONVERGENCE IN THE MEAN

Uniform convergence may be too strong a criterion for some purposes: Suppose we are evaluating an integral over $[a,b]$; then it would not really be necessary for $g_n(x)$ to converge to $g(x)$ at every x:



It may be that the "smooth" approximation to $g(x)$ by $g_n(x)$ is sufficient for evaluating the area under the curve: This leads to the idea of <u>convergence in the mean</u>:

Definition:

A sequence of functions $h_n(x)$ <u>converges in the mean</u> ("on average") to $h(x)$ if

$$\lim_{n\to\infty} \int_a^b dx \, |h(x) - h_n(x)|^2 \to 0 \qquad (1)$$

or

$$\lim_{n\to\infty} \int_a^b dx \, \Big|h(x) - \sum_{i=1}^{n} k_i(x)\Big|^2 \to 0 \qquad (2)$$

COMPLETENESS IN THE SENSE OF CONVERGENCE IN THE MEAN:

Define $g_n(x) = \sum_{i=1}^{n} a_i f_i(x)$ (3)

$\{f_i(x)\}$ are complete in the sense of convergence in the mean if

$$\boxed{\lim_{n\to\infty} \int_a^b dx \, \Big|g(x) - \sum_{i=1}^{n} a_i f_i(x)\Big|^2 \to 0} \qquad (4)$$

NOTES: • When talking of completeness, which form must be specified

• Uniform convergence $\Rightarrow$ convergence in the mean, but not vice versa.

Thus one must be more specific when writing

$$g(x) = \lim_{n\to\infty} \sum_{m=1}^{n} a_m f_m(x) = \sum_{m=1}^{\infty} a_m f_m(x) \qquad (5)$$

to specify which form of convergence is meant by the limiting process

F18/1/85

To Show that Uniform Convergence $\Rightarrow$ Convergence in the Mean:

Assume uniform convergence $\Rightarrow$ $\exists$ an integer N such that for $\forall x$ in $[a,b]$ and all $\epsilon > 0$ then

$$|h(x) - h_n(x)| < \epsilon \quad \text{for } n > N(\epsilon). \tag{1}$$

Then consider

$$\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon^2 \int_a^b dx = \epsilon^2 (b-a) \tag{2}$$

Hence uniform convergence $\Rightarrow$ $\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon^2 (b-a)$ for $n > N(\epsilon)$. (3)

What we now want to show is that $\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon'$, where ϵ' is some pre-assigned number. [for $n > N$]

Since ϵ' can be made arbitrarily small, this is what we mean by $\lim_{n\to\infty} \int_a^b |h(x) - h_n(x)|^2 = 0$. To achieve what we want choose

$$\epsilon^2 (b-a) = \epsilon' \Leftrightarrow \epsilon = \sqrt{\epsilon'/(b-a)}$$

Then if $|h(x) - h_n(x)| < \epsilon = \sqrt{\epsilon'/(b-a)} \Rightarrow$

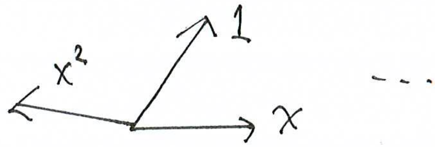
$$\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon^2 (b-a) = \left[\frac{\epsilon'}{b-a}\right] \cdot (b-a) = \epsilon' \checkmark$$

__Corollary:__ Completeness in the sense of uniform convergence $\Rightarrow$ Completeness in the sense of convergence in the mean

THE WEIERSTRASS APPROXIMATION THEOREM

Theorem: The polynomials $1, x, x^2, \dots$ form a complete set in any closed interval $a \le x \le b$, in the (Strong) sense of uniform convergence. [not proved in class]

Schematically:



However, these do not in general form a CON set: For example Consider the basis $\{f_n(x)\} = \{x^n\}$ $n = 0, 1, \dots$. Then

$$\langle x^2 | x^4 \rangle_{-1}^{1} \equiv \int_{-1}^{1} dx\, (x^2)^* x^4 = \frac{1}{7} x^7 \Big|_{-1}^{1} = \frac{2}{7} \neq 0 \leftarrow$$

Also $$\langle x^2 | x^2 \rangle = \int_{-1}^{1} dx\, (x^2)^* x^2 = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5} \neq 1 \leftarrow$$

Solution: Use GRAM-SCHMIDT METHOD to form CON Sets

$1, x, x^2, x^3, \dots$

GRAM-SCHMIDT

YOUR CHOICE!

$[-1,1]$	$[-\infty,\infty]$	$[0,\infty]$	$[-\pi,\pi]$	...
Legendre	Hermite	Laguerre	Fourier	
$P_0(x) = 1$	$H_0(x) = 1$	$L_0 = 1$	$\sin(nx), \cos(nx)$	
$P_1(x) = x$	$H_1(x) = 2x$	$L_1 = 1 - x$		
$P_2(x) = \frac{1}{2}(3x^2 - 1)$	$H_2(x) = -2 + 4x^2$	$L_2 = 2 - 4x + x^2$		
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$	$H_3(x) = -12x + 8x^3$	$L_3 = 6 - 18x + 9x^2 - x^3$		
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$	$H_4(x) = 12 - 48x^2 + 16x^4$			
	$\vdots$			

p.177 ← [Merzbacher, QM, p.56] → p.195 & Morrison/Estle/Lane QM p.55

Legendre Polynomials:

These form a CON set in $[-1,1]$. Note the GRAM-SCHMIDT method produces Legendre polynomials $\equiv \bar{P}_n(x)$ which are normalized as

$$\int_{-1}^{1} dx\, \bar{P}_n(x)\bar{P}_m(x) = \delta_{mn} \qquad (1)$$

The usual "textbook" functions are normalized as: $\int_{-1}^{1} dx\, P_n(x)P_m(x) = \left(\frac{2}{2n+1}\right)\delta_{mn}$ (2)

We return below to discuss these normalization issues: In using G-S we have:

$$\boxed{\bar{P}_0(x) = \text{Const}\cdot 1 = \frac{1}{\sqrt{2}}\cdot 1} \Rightarrow \int_{-1}^{1} dx\, \bar{P}_0(x)\bar{P}_0(x) = \frac{1}{2}\int_{-1}^{1} dx = 1 \checkmark$$

Next consider: $\bar{P}_1(x) = \dfrac{x - \langle \bar{P}_0(x)|x\rangle \bar{P}_0(x)}{|x - \langle \bar{P}_0(x)|x\rangle \bar{P}_0(x)|} \Leftrightarrow$ Recall $y_2 = \dfrac{x_2 - \langle y_1|x_2\rangle y_1}{|x_2 - \langle y_1|x_2\rangle y_1|}$ (3)

$$\langle \bar{P}_0(x)|x\rangle = \int_{-1}^{1} dx \left(\frac{1}{\sqrt{2}}\right)\cdot x = \frac{1}{\sqrt{2}}\frac{1}{2}x^2\Big|_{-1}^{1} = 0 \qquad (3a)$$

Hence: $\bar{P}_1(x) = \dfrac{x}{|x|} = \text{Const}\cdot x$; $\langle \bar{P}_1(x)|\bar{P}_1(x)\rangle = 1 \Rightarrow \text{Const} = \sqrt{\frac{3}{2}}$ (4)

Check $\langle \bar{P}_1(x)|\bar{P}_1(x)\rangle = \left(\sqrt{\frac{3}{2}}\right)^2 \int_{-1}^{1} dx\, x\cdot x = \frac{3}{2}\cdot\frac{x^3}{3}\Big|_{-1}^{1} = 1 \checkmark$ (5)

$$\therefore \boxed{\bar{P}_1(x) = \sqrt{\frac{3}{2}}\, x} \qquad (6)$$

This case was trivial since $\langle \bar{P}_0(x)|x\rangle = 0$ in Eq. (3a).

Next: $\bar{P}_2(x) = \dfrac{x^2 - \langle \bar{P}_0(x)|x^2\rangle \bar{P}_0(x) - \langle \bar{P}_1(x)|x^2\rangle \bar{P}_1(x)}{|x^2 - \dots \quad \dots \quad|}$ (6)

$$\bar{P}_2(x) = \frac{x^2 - \left(\frac{1}{\sqrt{2}}\int_{-1}^{1} x'^2 dx'\right)\cdot\frac{1}{\sqrt{2}} - \overbrace{\left(\sqrt{\tfrac{3}{2}}\int_{-1}^{1} x'^3 dx'\right)}^{\equiv 0}\cdot\sqrt{\tfrac{3}{2}}\,x}{|\quad\quad\quad\quad\quad\quad|} \tag{7}$$

$$= \frac{x^2 - \frac{1}{2}\int_{-1}^{1} dx'\, x'^2}{|\quad\quad\quad|} = \frac{x^2 - \frac{1}{3}}{|x^2 - 1/3|} \;; \quad |x^2 - 1/3|^2 = \int_{-1}^{1} dx'\,(x'^2 - 1/3)(x'^2 - \tfrac{1}{3})$$

$$|x^2 - 1/3|^2 = \int_{-1}^{1} dx' \left\{ x'^4 - \frac{2}{3} x'^2 + \frac{1}{9} \right\} = \frac{8}{45} \Rightarrow \boxed{\bar{P}_2(x) = \frac{x^2 - 1/3}{\sqrt{8/45}} = \sqrt{\frac{5}{8}}\,(3x^2 - 1)} \tag{8}$$

Notes: [1] We see already that $P_n(x)$ contains only even powers of x if $n =$ even, and odd powers of x if $n =$ odd. This could have been deduced at the outset from a PARITY ARGUMENT, which we give later. Here we simply note that this arises from the fact that integrals of odd powers vanish in $[-1,1]$ as in (7).

[2] The expression in (8) can be rewritten as:

$$\bar{P}_2(x) = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \equiv \sqrt{\frac{5}{2}}\,\underbrace{P_2(x)}_{\text{"textbook" expression}} \tag{9}$$

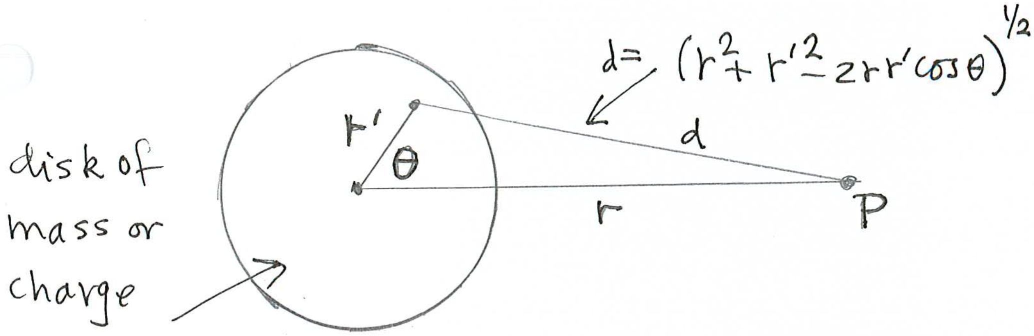
These two conventions are related via

$$\boxed{\bar{P}_n(x) = \sqrt{\frac{2n+1}{2}}\;P_n(x)} \tag{10}$$

The "textbook" $P_n(x)$ are normalized differently for reasons we now discuss.

[3] Apart from an overall normalization, $\bar{P}_n(x)$ or $P_n(x)$ are unique in $[-1,1]$

Other Derivations of $P_n(x)$:



$$\frac{1}{d} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta) \qquad r' < r \qquad (1)$$

In (1) set $r = 1$, $r' \equiv t$ $\quad |t| < 1$, $\quad x = \cos\theta$; (1) $\Rightarrow$

$$(1 - 2tx + t^2)^{-1/2} = \sum_{l=0}^{\infty} t^l P_l(x) \qquad |t| < 1 \qquad (2a)$$

GENERATING FUNCTION FOR $P_l(x)$

or

$$(1 - 2tx + t^2)^{-1/2} = \sum_{l=0}^{\infty} t^{-l} P_l(x) \qquad |t| > 1 \qquad (2b)$$

To show that the generating function generates the usual $P_l(x)$:

$$(1 - 2tx + t^2)^{-1/2} = [1 - (2tx - t^2)]^{-1/2} \equiv [1 - z]^{-1/2}$$

First note that $|z| < 1$: For fixed x $\quad \frac{d}{dt} z(t,x) = 2x - 2t \Rightarrow t = x \qquad (3)$

Also $\frac{d^2}{dt^2} z(t,x) = -2 \Rightarrow$ $t = x$ is a maximum. At $t = x$ we have

$$z(t,x)_{x=t} = (2tx - t^2)_{x=t} = 2t^2 - t^2 = t^2 < 1 \text{ when } |t|^2 < 1 \qquad (4)$$

Hence altogether $|z| < 1 \Rightarrow$ $\boxed{(1-z)^{-1/2} = 1 + \frac{1}{2}z + \frac{3}{8}z^2 + \frac{15}{48}z^3 + \frac{35}{128}z^4 + \ldots}$ (5)

Then:

$$z^2 = 4t^2x^2 + t^4 - 4t^3x$$
$$z^3 = -t^6 + 6t^5x - 12t^4x^2 + 8t^3x^3 \qquad (6)$$
$$z^4 = t^8 - 8t^7x + 24t^6x^2 - 32t^5x^3 + 16t^4x^4$$
$$\vdots$$

Collecting together the coefficients of like powers of t gives:

$$(1-2tx+t^2)^{-1/2} = 1 \cdot P_0 + tP_1(x) + t^2 P_2(x) + t^3 P_3(x) + \ldots \quad (7)$$

$$= 1 + tx - \tfrac{1}{2}t^2 - \tfrac{3}{2}t^3 x + \tfrac{3}{8}t^4 + \tfrac{3}{2}t^2x^2 + \tfrac{5}{2}t^3x^3 + \ldots \quad (8)$$

$$= \underbrace{1}_{P_0(x)} + t\underbrace{[x]}_{P_1(x)} + t^2\underbrace{\left[\tfrac{3}{2}x^2 - \tfrac{1}{2}\right]}_{P_2(x)} + t^3\underbrace{\left[\tfrac{5}{2}x^3 - \tfrac{3}{2}x\right]}_{P_3(x)} + \ldots \quad (9)$$

These are the standard "textbook" expressions for $P_0(x), \ldots, P_3(x)$. Note that they agree with the values of the $P_n(x)$ obtained from the G-S method up to a (trivial!) overall normalization constant.

Rodrigues' Formula for $P_n(x)$:

For various applications it is useful to have several expressions for the $P_n(x)$. Another way of deriving the $P_n(x)$ is via the Rodrigues formula:

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n} \quad (10)$$

Checks: $P_0(x) = \frac{1}{2^0 0!} \cdot 1\,(x^2-1)^0 = 1$ ✓

$$P_1(x) = \frac{1}{2 \cdot 1!}\frac{d}{dx}(x^2-1)^1 = \frac{1}{2} \cdot 2x = x \checkmark$$

$$P_2(x) = \frac{1}{2^2 \cdot 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = \frac{1}{8}\frac{d}{dx}(4x^3 - 4x)$$
$$= \tfrac{1}{2}(3x^2-1) \checkmark$$

Generalization of the Generating Function:

One can define a set of polynomials by

$$\frac{1}{(1-2tx+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)\, t^n \qquad (1)$$

Evidently $C_n^{(1/2)}(x) = P_n(x) =$ Legendre Polynomials

More generally the $C_n^{(\alpha)}(x)$ are called

ultraspherical polynomials ~

Gegenbauer polynomials

These would be of potential use in calculating the potential energy arising from an interaction which decreased with $r = |\vec{r}|$ as $1/r^{2\alpha}$.

Proof of Rodrigues' Formula:

Outline: If we accept the theorem that in any interval there is a unique set of orthogonal polynomials, then if we can show that the Rodrigues formula leads to an orthogonal set of polynomials in [-1, 1] then they must be the $P_n(x)$. Specifically we show that the formula leads to

$$\int_{-1}^{1} dx\, P_n(x) P_m(x) = \frac{2}{2n+1}\,\delta_{nm} \tag{1}$$

Assume to start $n > m$; $d^n \equiv d^n/dx^n$

$$I = \int_{-1}^{1} dx\, P_n(x) P_m(x) = \int_{-1}^{1} dx \left[\frac{1}{2^n n!} d^n(x^2-1)^n\right]\left[\frac{1}{2^m m!} d^m(x^2-1)^m\right] \overset{?}{=} 0 \quad (n \neq m) \tag{2}$$

Dropping constants,

$$I = \int_{-1}^{1} dx\, \underbrace{[d^n(x^2-1)^n]}_{dv}\, \underbrace{[d^m(x^2-1)^m]}_{u} = \tag{3}$$

$$\underbrace{[d^{n-1}(x^2-1)^n]}_{v}\,\underbrace{[d^m(x^2-1)^m]}_{u}\Big|_{-1}^{1} - \int_{-1}^{1} dx\, \underbrace{[d^{n-1}(x^2-1)^n]}_{v}\,\underbrace{[d^{m+1}(x^2-1)^m]}_{du} \tag{4}$$

$\rightarrow$ polynomial $\otimes$ (x^2-1) $\rightarrow$ but $\exists$ one more power of (x^2-1) than there are derivatives on it $\Rightarrow$ at the end we are left with a factor (x^2-1) which vanishes at $x = \pm 1$.

Continuing in this manner we see that

$$\int_{-1}^{1} dx\, [d^n(x^2-1)^n][d^m(x^2-1)^m] = (-1)^n \int_{-1}^{1} dx\, (x^2-1)^n [d^{m+n}(x^2-1)^m] \tag{5}$$

$$= 0 \quad \text{Since } n > m \Rightarrow m+n > 2m \Rightarrow d^{m+n}(x^2-1)^m \equiv 0. \checkmark \tag{6}$$

$$\therefore \boxed{I = 0 \quad \text{when } n \neq m} \tag{7}$$

To complete the proof of uniqueness we consider the case $m=n$. Reinstating the constants we get:

$$\int_{-1}^{1} dx\, P_n(x)\, P_n(x) = \frac{1}{(2^n n!)^2} (-1)^n \int_{-1}^{1} dx\, (x^2-1)^n \Big[d^{2n} \underbrace{(x^2-1)^n}_{\text{polynomial} \sim x^{2n}} \Big] \qquad (8)$$

When d^{2n} acts on x^{2n} there will be one term which does survive: All lower powers will be absent as a result of differentiation. This surviving term is given by

$$d^{2n} x^{2n} = d^{2n-1}(2n x^{2n-1}) = d^{2n-2}\left[(2n)(2n-1)\, x^{2n-2}\right] = \dots (2n)! \qquad (9)$$

Hence altogether $$\int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{1}{2^{2n}(n!)^2} \cdot (2n)!\, \underbrace{(-1)^n \int_{-1}^{1} dx\, (x^2-1)^n}_{\frac{\sqrt{\pi}\; n!\; (-1)^n}{(n+\frac{1}{2})!}} \qquad (10)$$

} Jahnke-Emde p.20

$$(n+\tfrac{1}{2})! = \frac{\sqrt{\pi}\;\; 1\cdot 3\cdot 5 \dots (2n+1)}{2^{n+1}} \qquad (11)$$

} Jahnke-Emde p.11

Collecting these results together we find

$$\int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{1}{2^{n-1}} \; \frac{(2n)!}{n!} \; \frac{1}{1\cdot 3\cdot 5 \dots (2n+1)} \qquad (12)$$

$$(2n)! = (2n)(2n-1)(2n-2)\dots 3\cdot 2\cdot 1 = \left[(2n)(2n-2)(2n-4)\dots\right]\left[(2n-1)(2n-3)\dots 3\cdot 1\right]$$

$$\downarrow \; (2\cdot n)\, 2(n-1)\; 2(n-2)\dots$$

$$= 2^n n! \qquad (13)$$

Hence $$(2n)! = 2^n n! \left[(2n-1)(2n-3)\dots 3\cdot 1\right] \qquad (14)$$

$$\therefore \int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{1}{2^{n-1}} \frac{[2^n \cancel{n!}]\,\cancel{[(2n-1)(2n-3)\cdots 3\cdot 1]}}{\cancel{n!}} \frac{1}{\cancel{[1\cdot 3\cdots(2n-1)]}(2n+1)} \quad (15)$$

Hence finally $$\boxed{\int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{2^n}{2^{n-1}} \frac{1}{(2n+1)} = \frac{2}{2n+1}} \quad (16)$$

This establishes that the Rodrigues formula produces polynomials in $[-1,1]$ which have the same normalization & orthogonality properties as $P_n(x) = \sqrt{\frac{2}{2n+1}}\, \bar{P}_n(x)$, which is what we want.

[Recall that $\int_{-1}^{1} dx\, \bar{P}_n(x)\, \bar{P}_m(x) = \delta_{mn}$] (17)

Return to Normalization Questions:

The $\bar{P}_n(x)$ have a simple normalization as in (17), but $P_n(x)$ have another simple property:

$$\boxed{P_n(x) \xrightarrow{x=1} 1} \quad (18)$$

2 Proofs of (18):

(a) Generating function: $$(1-2tx+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad (19)$$

Set $x=1 \Rightarrow (1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(1) = 1\cdot P_0(1) + t\,P_1(1) + t^2 P_2(1) + \ldots$ (20)

$$\hookrightarrow \frac{1}{\sqrt{(1-t)^2}} = \frac{1}{1-t} = 1+t+t^2+\ldots \quad \Rightarrow P_n(1) = 1 \checkmark \quad (21)$$

(b) Rodrigues Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [\underbrace{(x^2-1)}_{(x+1)(x-1)}]^n \quad \leftarrow \text{Differentiate each factor separately:} \qquad (22)$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} \left\{ n(x-1)^{n-1} \cdot (x+1)^n + (x-1)^n \, n(x+1)^{n-1} \right\} \qquad (23)$$

Among the many terms which survive the differentiations, the only one which will survive in the very end when we set $x=1$ is this one since the differentiations eventually remove the factor $(x-1)$ completely.

Thus at the end this term gives

$$P_n(x) = \frac{1}{2^n n!} \left\{ n! \, (x-1)^0 (x+1)^n + \ldots \right\} \overset{x=1}{=} \frac{1}{2^n n!} \left\{ n! \, 1 \cdot 2^n \right\} = 1 \checkmark \qquad (24)$$

THE LEGENDRE EQUATION

This is usually the starting point, since this equation can be obtained by solving the angular part of $\nabla^2 \phi(\vec{x}) = 0$.

Here we reverse the process, by deriving the Legendre equation:

We show that the polynomials defined by the Rodrigues formula [and which are unique for $-1 \leq x \leq +1$] also solve the Legendre equation

$$\boxed{(x^2-1) P_n''(x) + 2x P_n'(x) - n(n+1) P_n(x) = 0} \qquad (1)$$

Begin with the identity:

$$(x^2-1)\frac{d}{dx}(x^2-1)^n \equiv (x^2-1)\,d(x^2-1)^n = (x^2-1)\,n(x^2-1)^{n-1}2x = 2nx\,(x^2-1)^n \qquad (2)$$

Differentiate both sides w.r.t. x $(n+1)$ times: Using the "binomial expansion" of the product rule:

$$d^m(uv) = u\,d^m v + m(du)(d^{m-1}v) + \frac{m(m-1)}{2!}(d^2u)(d^{m-2}v) + \dots$$

$$+ \frac{m!}{(m-k)!\,k!}(d^k u)(d^{m-k}v) + \dots + (d^m u)v \qquad (3)$$

Take $(n+1)$ derivatives of the l.h.s. of (2) [so that $m \to n+1$]

$$d^{n+1}[\underbrace{(x^2-1)}_{u}\ \underbrace{d(x^2-1)^n}_{v}] = (x^2-1)\,d^{n+2}(x^2-1)^n + (n+1)[\underbrace{d(x^2-1)}_{2x}]\,d^{n+1}(x^2-1)^n \qquad (4)$$

$$+ \frac{(n+1)(n+1-1)}{2!}[\underbrace{d^2(x^2-1)}_{2}][d^n(x^2-1)^n] \qquad (5)$$

$$+ \dots \otimes [\underbrace{d^3(x^2-1)}_{\overset{|||}{0}}][d^{n-1}(x^2-1)^n] + \dots$$

$\leftarrow$ also for remaining terms

Collecting terms d^{n+1} (l.h.s. of (2)) $\Rightarrow$

$$d^{n+1}[(x^2-1)\,d(x^2-1)^n] = (x^2-1)\,d^{n+2}(x^2-1)^n + (n+1)\,2x\,d^{n+1}(x^2-1)^n \qquad (6)$$

$$+ n(n+1)\,d^n(x^2-1)^n$$

Fourier Series:

Fourier series expansions can also be obtained from a generalized (2-dim) version of Weierstrass' theorem:

$$\boxed{f(x,y) = \sum_{n,m=0}^{\infty} C_{nm}\, x^n y^m} \qquad (1)$$

~~Similarly~~ Changing to polar coordinates with $r=1$,

$$x^n = (r\cos\theta)^n \to \cos^n\theta \qquad (2)$$
$$y^m = (r\sin\theta)^m \to \sin^m\theta$$

$$\therefore\ f(x,y) \Rightarrow f(\theta) = \sum_{n,m=0}^{\infty} C_{nm} \cos^n\theta \sin^m\theta \qquad (3)$$

$$\cos^n\theta = \left[\tfrac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right]^n \ ; \ \sin^m\theta = \left[\tfrac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\right]^m \qquad (4)$$

Eqs. (3) & (4) $\Rightarrow$ that we can also expand $f(\theta)$ in terms of $e^{\pm in\theta}$

$$f(\theta) = \sum_{n=-\infty}^{\infty} A_n'\, e^{in\theta} \qquad (5)$$

} note from (4) that negative powers of n will appear

To connect to the usual Fourier expansions:

$$\text{Eq. (5)} \Rightarrow f(\theta) = \sum_{n=-\infty}^{\infty} A_n' (\cos n\theta + i\sin n\theta)$$

$$= A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + i\sum_{n=1}^{\infty} A_n' \sin n\theta \qquad (6)$$

$$+ \sum_{n=-\infty}^{-1} A_n' \cos n\theta + i\sum_{n=-\infty}^{-1} A_n' \sin n\theta$$

in this line rename $\boxed{m \equiv -n}$

$$(6b) \Rightarrow \sum_{-\infty}^{-1} A_n' \cos n\theta + i \sum_{-\infty}^{-1} A_n' \sin n\theta \qquad (7)$$

$$= \sum_{m=1}^{\infty} A_{-m}' \underbrace{\cos(-m\theta)}_{\cos(m\theta)} + i \sum_{m=1}^{\infty} A_{-m}' \underbrace{\sin(-m\theta)}_{-\sin(m\theta)}$$

Since m is just a dummy summation variable rename $m \to n$:

$$(7) = \sum_{n=1}^{\infty} A_{-n}' \cos n\theta + \sum_{n=1}^{\infty} (-iA_{-n}') \sin n\theta \qquad (8)$$

Hence: $$f(\theta) = \underbrace{A_0'}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \underbrace{(A_n' + A_{-n}')}_{a_n} \cos n\theta + \sum_{n=1}^{\infty} \underbrace{i(A_n' - A_{-n}')}_{b_n} \sin n\theta \qquad (9)$$

Finally: $$\boxed{f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \qquad (10)}$$ FOURIER EXPANSION

We note that since $\cos(n\pi) = \cos(-n\pi)$ and $\sin(n\pi) = 0$

Eq.(10) holds only for functions for which $f(\pi) = f(-\pi)$.

Summary to this Point:

(a) We have shown previously that $\left\{ f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \right\}$ are orthonormal in $[-\pi, \pi]$

(b) Now we have shown that $\frac{1}{\sqrt{2\pi}} e^{inx}$ are complete, via Weierstrass theorem.

If we combine these statements $\Rightarrow$ $\frac{1}{\sqrt{2\pi}} e^{inx}$ form a CON set on $[-\pi, \pi]$ in the sense of uniform convergence.

Determining the Fourier Coefficients

For any vector expansion

$$|f\rangle = \sum_n \underbrace{|f_n\rangle\langle f_n|}_{I}\overset{\leftarrow c_n \rightarrow}{f\rangle} \Leftrightarrow f(x) = \sum_n c_n f_n(x) \quad (1)$$

$$\therefore c_n = \langle f_n|f\rangle = \int_{-\pi}^{\pi} dx\, f_n^*(x)\, f(x) \qquad f_n = \left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\} \quad (2)$$

Since the f_n are complete $\Rightarrow$ Bessel's equality: $\sum_n |c_n|^2 = \sum_n |\langle f_n|f\rangle|^2 = |f|^2$ (3)

From (1) & (2): $$f(x) = \sum_{n=-\infty}^{\infty} A_n \frac{e^{inx}}{\sqrt{2\pi}} \equiv \sum_{n=-\infty}^{\infty} \bar{A}_n e^{inx} \quad (4)$$

Then: $$\boxed{A_n = \langle f_n|f\rangle = \int_{-\pi}^{\pi} dx \left\{\frac{1}{\sqrt{2\pi}} e^{-inx}\right\} f(x)} \quad (5)$$

If we wish to carry out the expansion in terms of $\cos(nx)$ & $\sin(nx)$

Then from the previous results:

$$\boxed{a_0 = 2\,(A_0/\sqrt{2\pi}) = \sqrt{\frac{2}{\pi}} \int_{-\pi}^{\pi} dx \cdot \left(\frac{1}{\sqrt{2\pi}} f(x)\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)} \quad (6)$$

$$a_n = \frac{1}{\sqrt{2\pi}}(A_n + A_{-n}) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\pi}^{\pi} dx \left\{e^{-inx} + e^{inx}\right\} f(x) \quad (7)$$

$$\boxed{\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, \cos(nx)\, f(x)} \quad (8)$$

$$b_n = \frac{i}{\sqrt{2\pi}}(A_n - A_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} dx\, \underbrace{(-e^{+inx} + e^{-inx})}_{-2i\sin(nx)} f(x) \quad (9)$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, \sin(nx)\, f(x)} \quad (10)$$

FROM R. W. FULLER

F-205

Table of Orthogonal Functions Arising from Sturm - Liouville Systems

Name and Physical Application	Rodrique's Formula	Generating Function	Differential Equation	S -L Form of D. E.	Orthonormality
Legendre Polynomials 1) Multiple Expansion 2) ∇^2 in sph. coor.	$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$	$\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}}$ $(0 < t < 1)$	$(x^2-1)P_n'' + 2xP_n' - n(n+1)P_n = 0$	$\frac{d}{dx}\left((1-x^2)P_n'\right) + n(n+1)P_n = 0$	$\int_{-1}^{1} P_n P_m \, dx = \delta_{nm} \frac{2}{2n+1}$
Hermite Polynomials Quantum Oscillator	$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}$	$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{-t^2+2tx}$ $(t > 0)$	$H_n'' - 2xH' + 2nH_n = 0$	$\frac{d}{dx}(e^{-x^2}H_n') + 2ne^{-x^2}H_n = 0$	$\int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = \delta_{nm} \sqrt{\pi}\, 2^n$
Laguerre Polynomials H - atom	$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$	$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n = \frac{e^{\frac{-xt}{1-t}}}{1-t}$ $(0 < t < 1)$	$xL_n'' + (1-x)L_n' + nL_n = 0$	$\frac{d}{dx}(xe^{-x}L_n') + ne^{-x}L_n = 0$	$\int_0^{\infty} L_n L_m e^{-x} dx = \delta_{nm}(n!)^2$
	Series Presentation				
Bessel's Function (of integral order) ∇^2 in cylindrical coordinates	$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m+n}}{(n+m)!\, m!}$	$\sum_{n=-\infty}^{\infty} J_n(x) t^n = e^{\frac{1}{2}x(t-\frac{1}{t})}$ $(t > 0)$	$x^2 J_n'' + xJ_n' + (x^2-n^2)J_n = 0$	$\frac{d}{dx}(xJ_n') + (x - \frac{n^2}{x})J_n = 0$	
Trigonometric Functions Classical Oscillator	$f_n = \sin nx = \sum_{m=0}^{\infty} (-1)^m \frac{(nx)^{2m+1}}{(2m+1)!}$ $g_n = \cos nx = \sum_{m=0}^{\infty} (-1)^m \frac{(nx)^{2m}}{(2m)!}$		$f_n'' + n^2 f_n = 0$ $g_n'' + n^2 g_n = 0$	$\frac{d}{dx}(f_n') + n^2 f_n = 0$ $\frac{d}{dx}(g_n') + n^2 g_n = 0$	$\int_{-\pi}^{\pi} f_n f_m \, dx = \delta_{nm}\pi$ $\int_{-\pi}^{\pi} g_n g_m \, dx = \delta_{nm}\pi$ $\int_{-\pi}^{\pi} f_n g_m \, dx = 0$