# THE AFFINE CONNECTION

[See WEINBERG "Gravitation & Cosmology"]

- The affine connection $\Gamma^{\lambda}_{\mu\nu}(x)$ is NOT a tensor. In fact its importance derives precisely from the fact that this is so.
- The affine connection is part of the formalism for COVARIANT DIFFERENTIATION:

$$\underbrace{\frac{\partial}{\partial x^\lambda} V^\mu(x)}_{\text{not a tensor}} \longrightarrow \underbrace{\underbrace{\frac{\partial}{\partial x^\lambda} V^\mu(x)}_{\substack{\text{not a}\\ \text{tensor}}} + \underbrace{\Gamma^{\mu}_{\lambda\nu}}_{\text{not a tensor}} V^\nu(x)}_{\text{a tensor}} = D_\lambda V^\mu(x) \qquad (1)$$

We show that the "bad" parts of $\frac{\partial}{\partial x^\lambda} V^\mu$ which make it not a tensor, exactly cancel against the "bad" parts of $\Gamma^{\mu}_{\lambda\nu} V^\nu$ so that their sum is indeed a tensor.

## MOTIVATING THE AFFINE CONNECTION:

From elementary physics we know that (Newton 2ND Law)

$$\text{no forces} \Rightarrow \underbrace{\frac{d^2\vec{x}}{dt^2}}_{\text{3-dim}} = 0 \longrightarrow \frac{d^2\xi^\alpha}{d\tau^2} \qquad d\tau \Leftrightarrow dt \qquad (2)$$

$\Updownarrow$

"inertial" coordinate system

$$c^2 d\tau^2 = c^2 dt^2 - (d\vec{x})^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

QUESTION: What does "Newton II" look like in a non-inertial coordinate system?

ANSWER: New "forces" arise (centrifugal, coriolis ...)

The affine connection describes these new "forces"

Define

$$\xi^\alpha = \xi^\alpha(x^\mu) \quad \text{or} \quad x^\mu = x^\mu(\xi^\alpha) \tag{3}$$

$\xi^\alpha$: inertial, $x^\mu$: non-inertial

$$(2) \Rightarrow \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau}\right) = 0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu} \cdot \frac{dx^\mu}{d\tau}\right) \tag{4}$$

introduces new cood system

$$= \underbrace{\left(\frac{d}{d\tau}\frac{\partial \xi^\alpha}{\partial x^\mu}\right)}\left(\frac{dx^\mu}{d\tau}\right) + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} \tag{5}$$

what we want: acceleration in new cood system

$$\frac{\partial}{\partial x^\mu}\frac{d\xi^\alpha}{d\tau} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \xi^\alpha}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\nu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\nu}\underbrace{\frac{\partial}{\partial x^\mu}\left(\frac{dx^\nu}{d\tau}\right)}_{\frac{d}{d\tau}\left(\underbrace{\frac{\partial x^\nu}{\partial x^\mu}}_{\delta^\nu_\mu}\right) = 0} \tag{6}$$

Altogether: (4)–(6) $\Rightarrow$

$$0 = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\left(\frac{dx^\mu}{d\tau}\right)\frac{dx^\nu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} \tag{7}$$

what we want

Multiply each term in this equation by $\partial x^\lambda / \partial \xi^\alpha$ using

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial \xi^\alpha} = \frac{\partial x^\lambda}{\partial x^\mu} = \delta^\lambda_\mu \Rightarrow \tag{8}$$

$$\boxed{\begin{aligned} 0 &= \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \delta^\lambda_\mu \frac{d^2 x^\mu}{d\tau^2} \\ 0 &= \frac{d^2 x^\lambda}{d\tau^2} + \underbrace{\left(\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\right)}_{\equiv \Gamma^\lambda_{\mu\nu} = \text{AFFINE CONNECTION}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \end{aligned}} \tag{9}$$

Eq. (9) can be written in the form:

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} = -\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \qquad (10)$$

Compare this to the usual formulation of Newton II

$$\frac{d^2\vec{x}}{dt^2} = \frac{\vec{F}}{m} \quad \leftrightarrow \quad \boxed{\frac{d^2x^i}{dt^2} = \frac{F^i}{m}} \qquad (11)$$

We see that $\Gamma^\lambda_{\mu\nu}$ plays the role of the external forces acting on a system: Later we will see that since $\Gamma^\lambda_{\mu\nu}$ enters into the formula for the covariant derivative, <u>We can interpret covariant differentiation as introducing forces in a natural way</u> into a free particle Hamiltonian or Lagrangian. This is one reason why $\Gamma^\lambda_{\mu\nu}$ is important.

## <u>Relation Between $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$:</u>

Start with an earlier equation: $\begin{pmatrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix}$ $x^4 = ict$

$$g_{\mu\nu}(x) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}(\xi) \qquad (12)$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2\xi^\alpha}{\partial x^\lambda\partial x^\mu}\cdot\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial^2\xi^\beta}{\partial x^\lambda\partial x^\nu}\eta_{\alpha\beta} \qquad (13)$$

$$\frac{\partial^2\xi^\alpha}{\partial x^\lambda\partial x^\mu} \rightarrow \Gamma^\rho_{\lambda\mu}\frac{\partial\xi^\alpha}{\partial x^\rho} \qquad\qquad \frac{\partial^2\xi^\beta}{\partial x^\lambda\partial x^\nu} \rightarrow \Gamma^\rho_{\lambda\nu}\frac{\partial\xi^\beta}{\partial x^\rho}$$

Hence $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu} \underbrace{\left( \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \right)}_{g_{\rho\nu}}$ (14)

$$+ \Gamma^\rho_{\lambda\nu} \underbrace{\left( \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \eta_{\alpha\beta} \right)}_{g_{\mu\rho}} \qquad (15)$$

$$\therefore \boxed{\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\mu\rho}} \qquad (16)$$

For H.W.: You will show that proceeding similarly,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2\, g_{\rho\nu} \Gamma^\rho_{\lambda\mu} \qquad (17)$$

Hence finally, using $g^{\nu\sigma} g_{\rho\nu} = \delta^\sigma_\rho$ (18)

"METRIC COMPATIBILITY CONDITION"

$$\boxed{\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left( \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right)} \qquad (9)$$

not a tensor

a tensor

not tensors

NOTES: (a) $\Gamma^\sigma_{\lambda\mu} = \Gamma^\sigma_{\mu\lambda}$ [See Eq. (9) above] (10)

(b) In an inertial coord system

$$g_{\mu\nu} \to \eta_{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & & 1 \end{pmatrix} \qquad (11)$$

$$\Rightarrow \Gamma^\sigma_{\lambda\mu} \equiv 0$$

# Behavior of the Affine Connection Under Coordinate Transformations

We have noted that the significance of $\Gamma^{\sigma}_{\lambda\mu}$ in part stems from the fact that it is not a tensor, which means that it does not transform properly under a change of coords. We now show this: Start with

$$\Gamma'^{\lambda}_{\mu\nu}(x') \equiv \left(\frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\right] = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial}{\partial x'^{\mu}}\cdot\frac{\partial \xi^{\alpha}}{\partial x'^{\nu}}\right] \qquad (12)$$

We want to end up with terms like this which eventually give $\Gamma$ in the $x$ cood system

$$\Gamma'^{\lambda}_{\mu\nu}(x') = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial}{\partial x'^{\mu}}\left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right)\right] \qquad (13)$$

$$= (\;\;)\left[\frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu}\partial x^{\sigma}}\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) + \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\cdot\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}\right] \qquad (14)$$

$$\hookrightarrow \frac{\partial^2 \xi^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}\cdot\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \qquad (15)$$

$$\hookrightarrow \frac{\partial \xi^{\alpha}}{\partial x^{\epsilon}}\Gamma^{\epsilon}_{\sigma\tau}$$

Collecting everything together gives

$$\Gamma'^{\lambda}_{\mu\nu}(x') = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial \xi^{\alpha}}{\partial x^{\epsilon}}\Gamma^{\epsilon}_{\sigma\tau}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} + \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}\right] \quad (16)$$

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial \xi^{\alpha}}{\partial x^{\epsilon}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\epsilon}_{\sigma\tau} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\sigma\tau} \quad (17)$$

$$\frac{\partial x^{\rho}}{\partial x^{\epsilon}} = \delta^{\rho}_{\epsilon}$$

Hence $\Gamma'^{\lambda}_{\mu\nu}(x') = \dfrac{\partial x'^{\lambda}}{\partial x^{\rho}}\dfrac{\partial x^{\sigma}}{\partial x'^{\nu}}\dfrac{\partial x^{\tau}}{\partial x'^{\mu}}\Gamma^{\rho}_{\sigma\tau} + \dfrac{\partial x'^{\lambda}}{\partial x^{\rho}}\dfrac{\partial x^{\rho}}{\partial \xi^{\alpha}}\dfrac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\dfrac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}$ (18)

$$\frac{\partial x^{\rho}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma}$$

Finally!! $$\boxed{\Gamma'^{\lambda}_{\mu\nu} = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\right)\left(\frac{\partial x^{\tau}}{\partial x'^{\mu}}\right)\Gamma^{\rho}_{\sigma\tau} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}}} \quad (19)$$

"expected" "extra piece"

The presence of the "extra piece" demonstrates why $\Gamma'^{\lambda}_{\mu\nu}$ is not a tensor.

<u>A Useful Identity</u>: Start with $\dfrac{\partial x'^{\lambda}}{\partial x^{\rho}}\dfrac{\partial x^{\rho}}{\partial x'^{\mu}} = \delta^{\lambda}_{\mu}$ $(\mu \to \nu)$ (20)

and differentiate w.r.t. $x'^{\mu}$:

$$\frac{\partial^2 x'^{\lambda}}{\partial x'^{\mu}\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x'^{\nu}} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} = \frac{\partial}{\partial x'^{\mu}}\left(\delta^{\lambda}_{\nu}\right) = 0 \quad (21)$$

$$\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\rho}}{\partial x'^{\nu}}$$

"extra piece" in (19)

It follows that we can replace the "extra piece" in C(19) by the first term in (21) giving (note (−) sign)

$$\boxed{\Gamma'^{\lambda}_{\mu\nu}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \Gamma^{\rho}_{\sigma\tau} - \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\nu}}} \qquad (22)$$

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## COVARIANT DIFFERENTIATION

To show the need for covariant differentiation we show that conventional partial derivatives do not produce tensors:

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x) \qquad (23)$$

$$\frac{\partial}{\partial x'^{\lambda}} V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \left( \frac{\partial V^{\nu}(x)}{\partial x'^{\lambda}} \right) + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x'^{\lambda}} V^{\nu}(x) \qquad (24)$$

change in vector    change in coord system

$$\boxed{\therefore \frac{\partial}{\partial x'^{\lambda}} V'^{\mu}(x') = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left( \frac{\partial V^{\nu}(x)}{\partial x^{\rho}} \right)}_{\text{"expected"}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x^{\rho}} \left( \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}(x) \right)}_{\text{"extra piece"}}} \qquad (25)$$

# CONSTRUCTING THE COVARIANT DERIVATIVE

Having shown that $\frac{\partial}{\partial x'^\lambda} V'^{\mu}(x')$ and $\Gamma'^{\lambda}_{\mu\nu}(x')$ both have "extrapieces" left over that prevent each from behaving as proper tensors, we seek to combine these to make these extra pieces "go away". Consider:

$$\left(\Gamma'^{\mu}_{\lambda\kappa}\right)\left[V'^{\kappa}\right] = \left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^{\nu}_{\rho\sigma} - \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\nu}{\partial x'^\kappa}\right)\left[\frac{\partial x'^\kappa}{\partial x^\eta}V^\eta\right] \quad (5)$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^{\nu}_{\rho\sigma}\frac{\partial x'^\kappa}{\partial x^\eta}V^\eta - \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\nu}{\partial x'^\kappa}\frac{\partial x'^\kappa}{\partial x^\eta}V^\eta \quad (6)$$

$$\frac{\partial x^\rho}{\partial x^\eta} = \delta^\rho_\eta \qquad\qquad \frac{\partial x^\nu}{\partial x^\eta} = \delta^\nu_\eta$$

$$\boxed{\text{Hence: } \Gamma'^{\mu}_{\lambda\kappa}(x')V'^{\kappa}(x') = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^{\nu}_{\rho\sigma}(x)V^\rho(x) - \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\left(\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu(x)\right)} \quad (7)$$

If we combine this result with Eq. (25) p. 59 we find

$$\frac{\partial}{\partial x'^\lambda}V'^\mu(x') + \Gamma'^{\mu}_{\lambda\kappa}(x')V'^{\kappa}(x') = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\left(\frac{\partial V^\nu(x)}{\partial x^\rho}\right) + \cancel{\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\left(\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu(x)\right)}$$

$$+ \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\left(\Gamma^{\nu}_{\rho\sigma}(x)V^\rho(x)\right) - \cancel{\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\left(\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu(x)\right)}$$

NOTE! The extrapieces have cancelled!!

Collecting together the previous results:

$$\boxed{\left(\frac{\partial}{\partial x'^{\lambda}} V'^{\mu}(x') + \Gamma'^{\mu}_{\lambda\kappa}(x')\, V'^{\kappa}(x')\right) = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left(\frac{\partial}{\partial x^{\rho}} V^{\nu}(x) + \Gamma^{\nu}_{\rho\sigma}(x)\, V^{\sigma}(x)\right)}$$

2ND RANK TENSOR $x'$

CORRECT TRANFORMATION MATRICES

2ND RANK TENSOR IN $x$

The expressions in ( ) are defined as the COVARIANT DERIVATIVE OF THE CONTRAVARIANT VECTOR $V'^{\mu}$

TERMINOLOGY: "COVARIANT" means 2 things in tensor analysis:

a) Refers to a vector such as $U_{\mu} = \partial\phi/\partial x^{\mu}$

b) Refers to a quantity which transforms properly when going from one coordinate system to another

COVARIANT DERIVATIVE OF A COVARIANT VECTOR $U_{\mu}(x)$:

$$\boxed{\left(\frac{\partial U'_{\mu}(x')}{\partial x'^{\lambda}} - \Gamma'^{\nu}_{\mu\lambda}(x')\, U'_{\nu}(x')\right) = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} \left(\frac{\partial U_{\rho}(x)}{\partial x^{\sigma}} - \Gamma^{\tau}_{\sigma\rho}(x)\, U_{\tau}(x)\right)}$$

NOTATION: $\left(\frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa}\right) \equiv D_{\lambda} V^{\mu} \equiv V^{\mu}_{;\lambda} = V^{\mu}_{\|\lambda}$ ; $\frac{\partial V^{\mu}}{\partial x^{\lambda}} \equiv V^{\mu}_{,\lambda} \equiv V^{\mu}_{|\lambda} \equiv \partial_{\lambda} V^{\mu}$

$\left(\frac{\partial U_{\mu}}{\partial x^{\lambda}} - \Gamma^{\kappa}_{\mu\lambda} U_{\kappa}\right) \equiv D_{\lambda} U_{\mu} \equiv U_{\mu;\lambda} = U_{\mu\|\lambda}$ ; $\frac{\partial U_{\mu}}{\partial x^{\lambda}} \equiv U_{\mu,\lambda} = U_{\mu|\lambda} \equiv \partial_{\lambda} U_{\mu}$

The latter notation helps as a mnemonic for the indices.

# COVARIANT DERIVATIVE OF AN ARBITRARY TENSOR

$$D_\rho T^{\mu\sigma}_{\lambda} \sim D_\rho (V^\mu W^\sigma U_\lambda) = (D_\rho V^\mu) W^\sigma U_\lambda + V^\mu (D_\rho W^\sigma) U_\lambda + V^\mu W^\sigma (D_\rho U_\lambda) \tag{18}$$

$$= (\partial_\rho V^\mu + \Gamma^\mu_{\rho\kappa} V^\kappa) W^\sigma U_\lambda + (\partial_\rho W^\sigma + \Gamma^\sigma_{\rho\kappa} W^\kappa) V^\mu U_\lambda + V^\mu W^\sigma (\partial_\rho U_\lambda - \Gamma^\kappa_{\rho\lambda}) U_\kappa \tag{19}$$

$$= \underbrace{\left\{ \partial_\rho V^\mu \cdot W^\sigma U_\lambda + \partial_\rho W^\sigma \cdot V^\mu U_\lambda + \partial_\rho U_\lambda V^\mu W^\sigma \right\}} \longrightarrow = \partial_\rho (V^\mu W^\sigma U_\lambda) = \partial_\rho T^{\mu\sigma}_{\lambda} \equiv T^{\mu\sigma}_{\lambda,\rho}$$

$$+ \Gamma^\mu_{\rho\kappa} \underbrace{V^\kappa W^\sigma U_\lambda}_{T^{\kappa\sigma}_{\lambda}} + \Gamma^\sigma_{\rho\kappa} \underbrace{V^\mu W^\kappa U_\lambda}_{T^{\mu\kappa}_{\lambda}} - \Gamma^\kappa_{\rho\lambda} \underbrace{V^\mu W^\sigma U_\kappa}_{T^{\mu\sigma}_{\kappa}} \tag{20}$$

$$\boxed{\text{Hence: } D_\rho T^{\mu\sigma}_{\lambda} \equiv T^{\mu\sigma}_{\lambda;\rho} = T^{\mu\sigma}_{\lambda,\rho} + \Gamma^\mu_{\rho\kappa} T^{\kappa\sigma}_{\lambda} + \Gamma^\sigma_{\rho\kappa} T^{\mu\kappa}_{\lambda} - \Gamma^\kappa_{\rho\lambda} T^{\mu\sigma}_{\kappa}} \tag{21}$$

## An Application : (The "real" Metric Compatibility Condition")

Consider $g_{\mu\nu;\lambda} \equiv D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa_{\nu\lambda} g_{\mu\kappa} = 0^*$

* p. 55 Eq. (16)

Alternatively: $g'_{\mu\nu;\lambda} = \dfrac{\partial \xi^\alpha}{\partial x'^\mu} \dfrac{\partial \xi^\beta}{\partial x'^\nu} \dfrac{\partial x^\gamma}{\partial x'^\lambda} \underbrace{\eta_{\alpha\beta;\gamma}}_{\equiv 0} = 0$

# Covariant Derivatives do NOT commute!

This is part of the reason why they are so interesting!

Consider $(D_\alpha D_\beta)V^\mu - (D_\beta D_\alpha)V^\mu \equiv [D_\alpha, D_\beta]V^\mu$ (1)

$$= D_\alpha(D_\beta V^\mu) - D_\beta(D_\alpha V^\mu)$$

↑ differentiates affine connection in $D_\beta$ etc.

Thus: $[D_\alpha, D_\beta]V^\mu =$

$$\partial_\alpha(\partial_\beta V^\mu + \Gamma^\mu_{\lambda\beta}V^\lambda) + \Gamma^\mu_{\rho\alpha}(\partial_\beta V^\rho + \Gamma^\rho_{\lambda\beta}V^\lambda) - \Gamma^\sigma_{\beta\alpha}(\partial_\sigma V^\mu + \Gamma^\mu_{\lambda\sigma}V^\lambda)$$

$$- \partial_\beta(\partial_\alpha V^\mu + \Gamma^\mu_{\lambda\alpha}V^\lambda) - \Gamma^\mu_{\rho\beta}(\partial_\alpha V^\rho + \Gamma^\rho_{\lambda\alpha}V^\lambda) + \Gamma^\sigma_{\alpha\beta}(\partial_\sigma V^\mu + \Gamma^\mu_{\lambda\sigma}V^\lambda)$$

(4)

$$= (\partial_\alpha \Gamma^\mu_{\lambda\beta})V^\lambda + \cancel{\Gamma^\mu_{\lambda\beta}\partial_\alpha V^\lambda} - (\partial_\beta \Gamma^\mu_{\lambda\alpha})V^\lambda - \cancel{\Gamma^\mu_{\lambda\alpha}\partial_\beta V^\lambda}$$

$$+ \cancel{\Gamma^\mu_{\lambda\alpha}\partial_\beta V^\lambda} + \Gamma^\mu_{\rho\alpha}\Gamma^\rho_{\lambda\beta}V^\lambda - \cancel{\Gamma^\mu_{\lambda\beta}\partial_\alpha V^\lambda} - \Gamma^\mu_{\rho\beta}\Gamma^\rho_{\lambda\alpha}V^\lambda$$

(6)

Hence $[D_\alpha, D_\beta]V^\mu = \left\{\partial_\alpha \Gamma^\mu_{\lambda\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\rho\alpha}\Gamma^\rho_{\lambda\beta} - \Gamma^\mu_{\rho\beta}\Gamma^\rho_{\lambda\alpha}\right\}V^\lambda$

$$\equiv R^\mu_{\ \lambda\beta\alpha}V^\lambda \quad \neq 0 \text{ (in general)}$$ (7)

→ Riemann-Christoffel Curvature Tensor

# Comments About $R^{\mu}_{\ \lambda\beta\alpha}$:

1) $R^{\mu}_{\ \lambda\beta\alpha} \neq 0 \Rightarrow$ Space has intrinsic curvature

2) It is convenient to express $R^{\cdot}_{\ \cdots}$ in terms of all Covariant (lower) indices by lowering the index [redacted] $\mu$. Then:

$$\boxed{R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\nu\lambda}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\kappa\lambda}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right) + g_{\eta\sigma}\left(\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu}\right)}$$

3) In terms of $R_{\lambda\mu\nu\kappa}$ the following relations hold:

a) $R_{(\lambda\mu)(\nu\kappa)} = R_{(\nu\kappa)(\lambda\mu)}$ (symmetry)

b) $R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu}$

(antisymmetry)

c) $R_{\lambda(\mu\nu\kappa)} + R_{\lambda(\kappa\mu\nu)} + R_{\lambda(\nu\kappa\mu)} = 0$

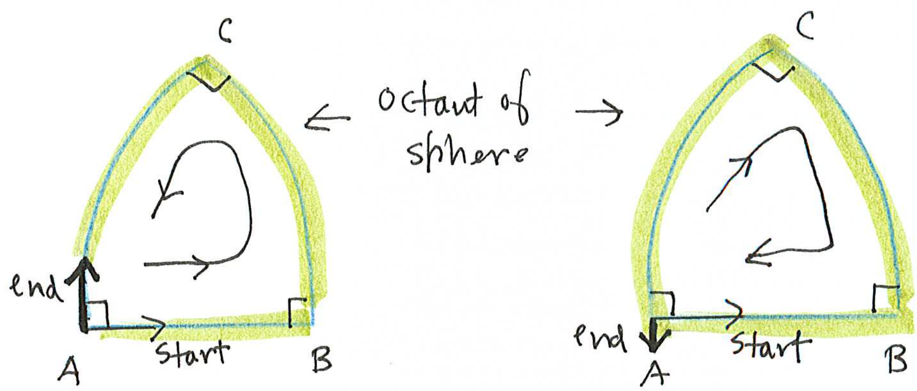
(cyclicity)

# Physical Picture of Non-Commutativity

Recall Taylor series formula:

$$e^{a\frac{\partial}{\partial x}}\psi(x) = \psi(x+a) \quad \Big\} \text{ derivatives} \Rightarrow \text{translations}$$

Consider translations on a sphere:




$D_\alpha$ = translation from $A \to C$ along curve ABC

$D_\beta$ = translation from $A \to C$ along the curve AC

The fact that $[D_\alpha, D_\beta] \neq 0$ then reflects the fact that translations along an intrinsically curved surface do not commute.

# Covariant Expressions for Grad, Curl, Divergence

**Curl:** Recall that $V_{\mu;\nu} = \dfrac{\partial V_\mu}{\partial x^\nu} - \Gamma^{\rho}_{\mu\nu} V_\rho \equiv \partial_\nu V_\mu - \Gamma^{\rho}_{\mu\nu} V_\rho$ (1)

$$\text{Covariant Curl} \equiv V_{\mu;\nu} - V_{\nu;\mu} = \left(\partial_\nu V_\mu - \cancel{\Gamma^{\rho}_{\mu\nu} V_\rho}\right) - \left(\partial_\mu V_\nu - \cancel{\Gamma^{\rho}_{\nu\mu} V_\rho}\right) \quad (2)$$

$$= \partial_\nu V_\mu - \partial_\mu V_\nu \equiv V_{\mu,\nu} - V_{\nu,\mu} \quad (3)$$

Hence the covariant curl is the same as the usual expression.

## Divergence

Start with $V^{\mu}_{;\nu} = V^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\rho} V^\rho \Rightarrow V^{\mu}_{;\mu} = \underbrace{V^{\mu}_{,\mu}}_{\text{usual expression}} + \Gamma^{\mu}_{\mu\rho} V^\rho$ (4)

We will simplify this expression which eventually leads to the covariant (or generalized) LAPLACIAN: Recall that

$$\nabla^2 \phi = \vec{\nabla} \cdot (\vec{\nabla} \phi) \rightarrow \text{covariant divergence}$$

Return to (4): (See p.55(9)) $\Gamma^{\sigma}_{\lambda\mu} = \dfrac{1}{2} g^{\nu\sigma} \left\{ \dfrac{\partial g_{\mu\nu}}{\partial x^\lambda} + \dfrac{\partial g_{\lambda\nu}}{\partial x^\mu} - \dfrac{\partial g_{\mu\lambda}}{\partial x^\nu} \right\} = \Gamma^{\sigma}_{\mu\lambda}$ (5)

Hence
$$\Gamma^{\mu}_{\mu\rho} = \frac{1}{2} g^{\nu\mu} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\rho}}{\partial x^\nu} \right\}$$

$$= \frac{1}{2} g^{\nu\mu} \frac{\partial g_{\mu\nu}}{\partial x^\rho} + \underbrace{\frac{1}{2} \underbrace{g^{\nu\mu}}_{\substack{\text{symm} \\ \text{in } \nu \Leftrightarrow \mu}} \underbrace{\left[ \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\rho\mu}}{\partial x^\nu} \right]}_{\substack{\text{anti-symm} \\ \text{in } \nu \Leftrightarrow \mu}}}_{0} = \frac{1}{2} g^{\nu\mu} \frac{\partial g_{\mu\nu}}{\partial x^\rho}$$

Let us focus on $\Gamma^{\mu}_{\mu\rho} = \frac{1}{2}\, g^{\nu\mu}_{(x)} \dfrac{\partial g_{\mu\nu}(x)}{\partial x^{\rho}}$ (1)

Since $g^{\nu\mu}_{(x)} g_{\mu\lambda}(x) = \delta^{\nu}_{\lambda}$, $g^{\nu\mu}$ is the matrix inverse of $g_{\mu\lambda}$

To Simplify $\Gamma^{\mu}_{\mu\rho}$ we prove the following identity for a matrix M:

$$\boxed{\mathrm{Tr}\left\{ M^{-1}(x) \frac{\partial}{\partial x^{\rho}} M(x) \right\} = \frac{\partial}{\partial x^{\rho}} \ln \det M(x)} \quad (2)$$

First prove the following identity for a matrix A:

$$\boxed{\det e^{A} = e^{\mathrm{Tr}A}} \quad (3)$$

We prove this for the case that A can be diagonalized by a matrix U:

$$U^{-1} A U = B = \text{diagonal} \quad (4)$$

Hence: $\mathrm{Tr}\, B = \mathrm{Tr}\,(U^{-1}AU) = \mathrm{Tr}\,(UU^{-1}A) = \mathrm{Tr}\,A$ (5)

Also: $\det B = \det(U^{-1}AU) = \det U^{-1} \cdot \det A \cdot \det U = \det(U^{-1}U) \cdot \det A$

$= \det A$ (6)

Consider next $\det e^{B} = \det\left\{ \mathbb{1} + B + \frac{1}{2!} B^{2} + \dots \right\}$ (7)

$= \det\left\{ U^{-1}U + U^{-1}AU + \frac{1}{2!} U^{-1}AU\, U^{-1}AU + \dots \right\}$ (8)

$= \det\left\{ U^{-1}\left[ \mathbb{1} + A + \frac{1}{2!} A^{2} + \dots \right] U \right\} = \det\left\{ U^{-1} e^{A} U \right\} = \det e^{A}$ (9)

$$\therefore \boxed{\det e^{B} = \det e^{A}} \quad (10)$$

Since $B$ is diagonal we have:

$$\det e^{A} = \det e^{B} = \det\left\{ \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix} + \begin{pmatrix} b_{11} & & \\ & b_{22} & \\ & & b_{33} \ddots \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} b_{11}^2 & & \\ & b_{22}^2 & \\ & & \ddots \end{pmatrix} \cdots \right\}$$

$$= \det\left\{ \begin{pmatrix} (1+b_{11}+\frac{1}{2!}b_{11}^2+\ldots) & 0 & 0 & \\ 0 & (1+b_{22}+\frac{1}{2!}b_{22}^2+\ldots) & 0 & \cdots \\ 0 & 0 & 1+b_{33}+\frac{1}{2!}b_{33}^2 & \\ \vdots & \vdots & \vdots & \end{pmatrix} \right\} \quad (11)$$

$$= \det\left\{ \begin{pmatrix} e^{b_{11}} & 0 & 0 & \cdots \\ 0 & e^{b_{22}} & 0 & \\ 0 & 0 & e^{b_{33}} & \\ \vdots & \vdots & \vdots & \end{pmatrix} \right\} = e^{b_{11}} e^{b_{22}} e^{b_{33}} \cdots$$

$$= e^{b_{11}+b_{22}+b_{33}+\ldots}$$

$$= e^{\mathrm{Tr}\, B} \quad (12)$$

Hence $\det e^{B} = e^{\mathrm{Tr}\, B}$, with $\det e^{B} = \det e^{A}$ and $e^{\mathrm{Tr}\, B} = e^{\mathrm{Tr}\, A}$ $\Rightarrow$ $\boxed{\det e^{A} = e^{\mathrm{Tr}\, A}}$ (13)

To apply this to $\Gamma^{\mu}_{\mu\rho}$ let $\boxed{B = \ln M}$ (14)

(12), (13) & (14) $\Rightarrow$ $\underbrace{\det e^{\ln M}}_{\det M} = e^{\mathrm{Tr} \ln M}$ $\Rightarrow$ $\boxed{\det M = e^{\mathrm{Tr} \ln M}}$ (15)

This leads to another useful identity: Take $\ln$ of both sides:

$$\det M = e^{\mathrm{Tr} \ln M} \Rightarrow \boxed{\ln \det M = \mathrm{Tr} \ln M} \quad (16)$$

This identity applies even when $M = M(x)$. So, differentiate with respect to $x$:

$$\frac{\partial}{\partial x^\rho} \ln \det M(x) = \frac{\partial}{\partial x^\rho} \mathrm{Tr} \ln M = \mathrm{Tr} \frac{\partial}{\partial x^\rho} \ln M$$

$$= \mathrm{Tr}\left\{ M^{-1}(x) \frac{\partial}{\partial x^\rho} M(x) \right\} \qquad (17)$$

Recall that for an ordinary function $f(x)$

$$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} \frac{df(x)}{dx} \; ; \text{ for a matrix } \frac{1}{f} \to M^{-1} \qquad (18)$$

Returning to p.70(1):

$$\Gamma^\mu_{\mu\rho} = \frac{1}{2} \mathrm{Tr}\left\{ (g_{\mu\nu})^{-1} \frac{\partial}{\partial x^\rho} \overset{\equiv M}{g_{\mu\nu}} \right\} = \frac{1}{2} \frac{\partial}{\partial x^\rho} \ln \underbrace{\det(g_{\mu\nu})}_{\equiv g(x)} \qquad (19)$$

$$\boxed{\Gamma^\mu_{\mu\rho} = \frac{\partial}{\partial x^\rho} \frac{1}{2} \ln g(x) = \frac{\partial}{\partial x^\rho} \ln \sqrt{g(x)} = \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^\rho} \sqrt{g(x)}} \qquad (20)$$

$g(x) = \det g_{\mu\nu}(x)$

Return to the covariant divergence:

$$V^\mu_{;\mu}(x) = \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\rho} V^\rho = \frac{\partial V^\mu}{\partial x^\mu} + \left( \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \sqrt{g} \right) V^\mu \qquad (21)$$

$$\boxed{\therefore \; V^\mu_{;\mu}(x) = \frac{1}{\sqrt{g(x)}} \frac{\partial}{\partial x^\mu} \left( \sqrt{g(x)}\, V^\mu(x) \right)} \qquad (22)$$

NOTE! This expression is covariant even though it is expressed in terms of a conventional partial derivative $\partial/\partial x^\mu$.

# Application: Laplacian in 3-dimensional Spherical coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \quad (1)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \; ; \; g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$$

$$ds^2 = (dr, d\theta, d\phi)\begin{pmatrix} & \downarrow & \end{pmatrix}\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = g_{\mu\nu} dx^\mu dx^\nu \quad (2)$$

$$g = \det g_{\mu\nu} = r^4 \sin^2\theta \; ; \; \sqrt{g} = r^2 \sin\theta \quad (3)$$

<u>Laplacian</u> $\nabla^2 \Phi = \vec{\nabla}\cdot(\vec{\nabla}\Phi) \equiv D_\lambda (D^\lambda \Phi) = \frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g} D^\lambda \Phi)$ (4)

<u>Key Step</u>: $D^\lambda \Phi = g^{\lambda\nu} D_\nu \Phi = g^{\lambda\nu} \partial_\nu \Phi$ (5)

$\hookrightarrow$ conventional partial derivatives are covariant vectors

Also: $D_\nu \Phi = \partial_\nu \Phi$ since $\Gamma^\lambda_{\mu\nu}$ has no way to enter

Hence $D^\lambda \Phi$ has the following components:

$$D^\lambda \Phi = \left(1\cdot\partial_r \Phi, \; \frac{1}{r^2}\partial_\theta \Phi, \; \frac{1}{r^2\sin^2\theta}\partial_\phi \Phi\right) \quad (6)$$

From (4): $\nabla^2 \Phi = D_\lambda (D^\lambda \Phi) = \frac{1}{\sqrt{g}} \partial_\lambda (\sqrt{g}\, D^\lambda \Phi)$

(7)

$$= \frac{1}{r^2\sin\theta}\left\{\partial_r\left(r^2\sin\theta \cdot \partial_r \Phi\right) + \partial_\theta\left(\cancel{r^2}\sin\theta \cdot \frac{1}{\cancel{r^2}}\partial_\theta \Phi\right) + \partial_\phi\left(\cancel{r^2}\cancel{\sin\theta}\cdot\frac{1}{\cancel{r^2}\sin^{\cancel{2}}\theta}\partial_\phi \Phi\right)\right\}$$

Hence finally: $$\boxed{\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}}$$

(8)