TENSOR ANALYSIS

$g_{\mu\nu}$ & all that!!

WHY TENSORS?

Example: Exploring for oil by measuring the local acceleration of gravity $\vec{g}(\vec{x})$ and its derivatives:

We can form a tensor from the 9 quantities $\partial g_i/\partial x_j$

$$\gamma_{ij} \equiv \begin{pmatrix} \partial \bar{g}_x/\partial x & \partial \bar{g}_x/\partial y & \partial \bar{g}_x/\partial z \\ \partial \bar{g}_y/\partial x & \partial \bar{g}_y/\partial y & \partial \bar{g}_y/\partial z \\ \partial \bar{g}_z/\partial x & \partial \bar{g}_z/\partial y & \partial \bar{g}_z/\partial z \end{pmatrix} \equiv \begin{pmatrix} \bar{g}_{xx} & \bar{g}_{xy} & \bar{g}_{xz} \\ \bar{g}_{yx} & \bar{g}_{yy} & \bar{g}_{yz} \\ \bar{g}_{zx} & \bar{g}_{zy} & \bar{g}_{zz} \end{pmatrix}$$

$\longrightarrow$ Such an object is called a 2ND rank tensor and has 9 components in 3-dimensions (not all of which are necessarily independent).

TENSORS

ds = distance from $\vec{x}$ to $\vec{x}+d\vec{x}$

3 dimensions

$$ds^2 = dx^2 + dy^2 + dz^2 \equiv g_{ij}\, dx^i dx^j$$

a) CARTESIAN

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij} \quad ; \; i,j = 1,2,3$$

$dx^1 = dx$
$dx^2 = dy$
$dx^3 = dz$

$$g \equiv \det g_{ij} = +1$$

b) SPHERICAL

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$\equiv g_{ij}\, dx^i dx^j$$

$dx^1 = dr$
$dx^2 = d\theta$
$dx^3 = d\phi$

$$; \; g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$g_{rr} = 1$
$g_{\theta\theta} = r^2$
$g_{\phi\phi} = r^2 \sin^2\theta$

$$g \equiv \det g_{ij} = r^4 \sin^2\theta$$

3-dim volume element

$$\int d\text{Volume} = \int \sqrt{g}\, dx^1 dx^2 dx^3 \;\checkmark \equiv \int \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3$$

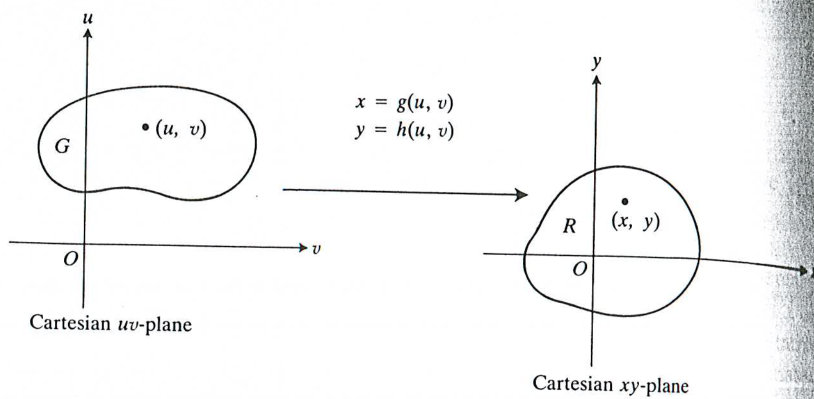
$$= \int 1\, dx^1 dx^2 dx^3 = \int dx\,dy\,dz \quad \text{CARTESIAN}$$

$$= \int (r^2 \sin\theta)\, dr\, d\theta\, d\phi \quad \text{SPHERICAL}$$

$$= \int (r^2 dr)(\sin\theta\, d\theta)\, d\phi \;\checkmark$$

FINNEY: CALCULUS AND ANALYTIC GEOMETRY

43b



18.28 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to write an integral over a region R in the xy-plane as an integral over a region G in the uv-plane.

The factor $J(u, v)$, whose absolute value appears in this formula, is the determinant

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}. \tag{9}$$

It is called the **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$, named after the mathematician Carl Gustav Jacob Jacobi (1804–1851). The alternative notation $\partial(x, y)/\partial(u, v)$ may help you to remember how the determinant is constructed from the partial derivatives of x and y. The derivation of Eq. (8) is intricate and we shall not give it here.

For polar coordinates, we have r and θ in place of u and v. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\[2ex] \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r. \checkmark$$

In 2-dimensions

$ds^2 = 1 \cdot dr^2 + r^2 \cdot d\theta^2$

$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$g = \det g_{ij} = r^2$

$\sqrt{g} = r$ ✓

Hence, Eq. (8) becomes

$$\begin{aligned} \iint_R F(x, y)\,dx\,dy &= \iint_G F(r\cos\theta, r\sin\theta)\,|r|\,dr\,d\theta \\ &= \iint_G F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta, \end{aligned} \tag{10}$$

which is Eq. (7).

Figure 18.29 shows how the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1$, $0 \le \theta \le \pi/2$ into the quarter-circle R bounded by $x^2 + y^2 = 1$ in the first quadrant of the xy-plane.

Notice that the integral on the right-hand side of Eq. (10) is not the integral of $F(r\cos\theta, r\sin\theta)$ over a region in the polar coordinate plane. It is the integral of the product of $F(r\cos\theta, r\sin\theta)$ and r over a region G in the *Cartesian* $r\theta$-plane.

Here is an example of another substitution.

3+1 dimensions (relativity)

time

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \quad (\text{Sum on } \mu, \nu = 1, 2, 3, 0)$$

Minkowski ("flat" space) $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

can't travel to past!!

Coordinates

$dx^1 = dx$; $dx^2 = dy$; $dx^3 = dz$; $dx^0 = c\,dt$

$\therefore\ ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$

TRANSFORMATION OF VECTORS & TENSORS

43.3

For an arbitrary transformation $x^\mu \to x'^\mu$

We define the following objects:

a) CONTRAVARIANT VECTOR $\quad V'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$

example: $\quad dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} dx^\nu$

b) COVARIANT VECTOR $\quad U'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} U_\nu(x)$

example (the gradient)

$$\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi}{\partial x^\nu}$$

c) MIXED TENSOR

$$T^{\mu\lambda}{}_{\nu}(x) \sim V^{\mu}(x)\, W^{\lambda}(x)\, U_{\nu}(x)$$

$\boxed{\text{3rd RANK TENSOR (3 indices)}}$

Can always view a mixed tensor this way. This helps keep track of indices.

example: $T'^{\mu\lambda}{}_{\nu}(x') = \left(\dfrac{\partial x'^{\mu}}{\partial x^{\alpha}}\right)\left(\dfrac{\partial x'^{\lambda}}{\partial x^{\beta}}\right)\left(\dfrac{\partial x^{\gamma}}{\partial x'^{\nu}}\right) T^{\alpha\beta}{}_{\gamma}(x)$

d) MOST IMPORTANT TENSOR = metric tensor

(defines the coordinate system we are in)

$$\boxed{g_{\mu\nu}(x) \equiv \left(\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\right)\left(\frac{\partial \xi^{\beta}}{\partial x^{\nu}}\right)\eta_{\alpha\beta}(\xi)}$$

flat space (Minkowski coods)

any other coordinates

To verify that $g_{\mu\nu}$ is a tensor note that:

$$g'_{\mu\nu}(x') \equiv \frac{\partial \xi^{\alpha}}{\partial x'^{\mu}}\frac{\partial \xi^{\beta}}{\partial x'^{\nu}}\eta_{\alpha\beta} = \left(\frac{\partial \xi^{\alpha}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\left(\frac{\partial \xi^{\beta}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right)\eta_{\alpha\beta}(\xi)$$

$$= \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right)\underbrace{\left(\frac{\partial \xi^{\alpha}}{\partial x^{\rho}}\frac{\partial \xi^{\beta}}{\partial x^{\sigma}}\eta_{\alpha\beta}(\xi)\right)}_{g_{\rho\sigma}(x)} = \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) g_{\rho\sigma}(x)$$

✓

CONTRAVARIANT = COVARIANT FOR CARTESIAN VECTORS

$$x_i' = a_{ij} x_j \Rightarrow a_{ij} = \frac{\partial x_i'}{\partial x_j}$$ old notation (1)

$$x^{i'} = a^i_j x^j \Rightarrow a^i_j = \frac{\partial x^{i'}}{\partial x^j}$$ new notation (2)

Invert (2): $a^m_i x^{i'} = \underbrace{a^m_i a^i_j}_{\delta^m_j} x^j = x^m$ $\boxed{\Rightarrow x^m = a^m_i x^{i'} \quad \Rightarrow a^m_i = \frac{\partial x^m}{\partial x^{i'}}}$ (3)

In (3) replace m by j $\Rightarrow$ $\boxed{a^j_i = \frac{\partial x^j}{\partial x^{i'}}}$ (4)

Compare this to (2): $a^i_j = \frac{\partial x^{i'}}{\partial x^j} \overset{i \leftrightarrow j}{\Longrightarrow} \boxed{a^j_i = \frac{\partial x^{j'}}{\partial x^i}}$ (5)

Finally, comparing (4) & (5) we see that

$$\boxed{\frac{\partial x^j}{\partial x^{i'}} = \frac{\partial x^{j'}}{\partial x^i}} \quad (6)$$

From Eq. (6) we see that it makes no difference whether the primed coordinates are in the numerator or denominator, which is to say that CONTRAVARIANT = COVARIANT

MANIPULATING TENSORS

a) Contraction ("generalized trace")

$$T'^{\mu\rho} = T'^{\mu\rho\nu}_{\ \ \ \ \ \nu} \quad \text{(sum over repeated index } \nu) \qquad (1)$$

↳ Show that this is a 2ND rank tensor

$$\cancel{T'^{\mu\nu\rho}_{\ \ \ \ \ \lambda}(x')} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\rho}}{\partial x^{\beta}} \frac{\partial x'^{\nu}}{\partial x^{\gamma}} \frac{\partial x^{\epsilon}}{\partial x'^{\lambda}} T^{\alpha\beta\gamma}_{\ \ \ \ \ \epsilon}(x) \equiv T'^{\mu\rho\nu}_{\ \ \ \ \ \lambda}(x') \qquad (2)$$

Set $\lambda = \nu$ and sum:

$$T'^{\mu\rho\nu}_{\ \ \ \ \ \nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\rho}}{\partial x^{\beta}} \underbrace{\frac{\partial x'^{\nu}}{\partial x^{\gamma}} \frac{\partial x^{\epsilon}}{\partial x'^{\nu}}}_{\frac{\partial x^{\epsilon}}{\partial x^{\gamma}} \equiv \delta^{\epsilon}_{\gamma} \text{ (key step)}} T^{\alpha\beta\gamma}_{\ \ \ \ \ \epsilon}(x) \qquad (3), (4)$$

Hence:

$$\boxed{T'^{\mu\rho\nu}_{\ \ \ \ \ \nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\rho}}{\partial x^{\beta}} T^{\alpha\beta\gamma}_{\ \ \ \ \ \gamma}(x)} \qquad (5)$$

This shows that $T'^{\mu\rho\nu}_{\ \ \ \ \ \nu}(x')$ transforms as a 2ND rank contravariant tensor.

b) Raising & Lowering Indices

- Define $S^{\mu\rho}_{\nu\lambda\sigma} \equiv g_{\nu\lambda} T^{\mu\rho}_{\sigma}$ = 5TH RANK MIXED TENSOR

- Then set $\mu = \nu$ & sum $\Rightarrow S^{\mu\rho}_{\mu\lambda\sigma} = \boxed{S^{\rho}_{\lambda\sigma} = g_{\mu\lambda} T^{\mu\rho}_{\sigma}}$

Hence effect of $g_{\mu\lambda}$ is to <u>lower</u> an index of the original tensor $T^{\mu\rho}_{\sigma}$. This is important when keeping track of indices (an example will follow later — generalized Laplacian)

- Indices can also be <u>raised</u> using the inverse tensor to $g_{\mu\nu} \equiv g^{\lambda\mu}$:

$$g^{\lambda\mu}(x)\, g_{\mu\nu}(x) = \delta^{\lambda}_{\nu} = \begin{cases} 1 & \nu = \lambda \\ 0 & \nu \neq \lambda \end{cases}$$

ANOTHER SPECIAL TENSOR

Levi-Civita tensor $\epsilon^{\mu\nu\lambda\kappa}$

$$\epsilon^{\mu\nu\lambda\kappa} = \begin{cases} +1 & \text{even permutation of } 1234 \ (x^4 = ict) \\ -1 & \text{odd} \quad '' \quad '' \quad '' \\ 0 & \text{2 indices equal} \end{cases}$$

This symbol plays an important role in formulating Maxwell's equations in 4-dimensional notation. It allows $\vec{E}$ & $\vec{B}$ to be expressed in terms of a single tensor $F_{\mu\nu}$.

Notation:

$F_{\mu\nu} = F_{\nu\mu}$ (Symmetric tensor)

$F_{\mu\nu} = -F_{\nu\mu}$ (antisymmetric tensor)

Let $A_{\mu\nu}$ be any tensor. Then

$$\boxed{A_{\mu\nu} = \underbrace{\tfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu})}_{\text{SYMMETRIC}} + \underbrace{\tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu})}_{\text{ANTISYMMETRIC}}}$$

BACK BY POPULAR DEMAND!!

TENSOR DENSITIES!!

TENSOR DENSITIES

We have previously defined $g(x) = \det g_{\mu\nu}(x)$ (1)

Since $g_{\mu\nu}(x)$ is a 2ND rank covariant tensor it transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x) \tag{2}$$

We can regard $\partial x^\rho/\partial x'^\mu \equiv f^\rho_\mu$ as the matrix relating x' and x in which case Eq. (2) can be written in the form

$$g'_{\mu\nu} = f^\rho_\mu\, g_{\rho\sigma}\, f^\sigma_\nu \tag{3}$$

Taking the determinant of both sides of (3) we get

$$\underbrace{\det g'_{\mu\nu}}_{g'} = \det\left(f^\rho_\mu\, g_{\rho\sigma}\, f^\sigma_\nu\right) = (\det f^\rho_\mu)\underbrace{(\det g_{\rho\sigma})}_{g}(\det f^\sigma_\nu) \tag{4}$$

where we have used $\det(ABC) = (\det A)(\det B)(\det C)$ (5)

Hence from (4):

$$\boxed{g'(x') = (\det f)^2\, g(x) \equiv \left|\frac{\partial x}{\partial x'}\right|^2 g(x)} \tag{6}$$

JACOBIAN OF TRANSFORMATION FROM $x \to x'$

Since $g(x)$ is a scalar [it has no indices] the "expected" transformation law would be

$$g'(x') = g(x) \tag{7}$$

without the additional factor $(\det f)^2 = \left|\frac{\partial x}{\partial x'}\right|^2$.

The presence of the additional factor $\left|\frac{\partial x}{\partial x'}\right|^2$ makes $g(x)$ not just a scalar but a Scalar density.

More generally a quantity which transforms as a tensor except for additional factors of $|\partial x'/\partial x|^{W}$ is a tensor density of weight W.

Note that: $$\left|\frac{\partial x}{\partial x'}\right| = \left|\frac{\partial x'}{\partial x}\right|^{-1} \qquad (8)$$

Combining (8) & (6) we have

$$\boxed{g'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-2} g(x)} \qquad (9)$$

Hence $g(x)$ is a scalar density of weight $W=-2$.

We can show that any tensor (density) of weight W can be expressed as a product of an ordinary tensor multiplied by a factor $g^{-W/2}$.

To see this let $\mathcal{T}^{\mu}_{\nu}(x)$ be a tensor density of weight W so that

$$\mathcal{T}'^{\mu}_{\nu}(x') = \left|\frac{\partial x'}{\partial x}\right|^{W} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \mathcal{T}^{\rho}_{\sigma}(x) \qquad (10)$$

$$\therefore \left[g'^{W/2}\right]\mathcal{T}'^{\mu}_{\nu}(x') = \underbrace{\left[\left|\frac{\partial x'}{\partial x}\right|^{-2} g(x)\right]^{W/2} \left|\frac{\partial x'}{\partial x}\right|^{W}}_{g(x)^{W/2}} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \mathcal{T}^{\rho}_{\sigma}(x) \qquad (11)$$

$$\therefore \left\{g'^{W/2}_{(x')}\, \mathcal{T}'^{\mu}_{\nu}(x')\right\} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \left\{g^{W/2}(x)\, \mathcal{T}^{\rho}_{\sigma}(x)\right\} \Rightarrow \qquad (12)$$

$$g^{W/2}_{(x)}\, \mathcal{T}^{\rho}_{\sigma}(x) \sim T^{\rho}_{\sigma}(x) \rightarrow \text{an ordinary tensor}$$

Transformation of the Levi-Civita Symbol $\epsilon^{\mu\nu\lambda\kappa}$:

Recall: $\epsilon^{1234} = +1$ = even permutations of 1234

$\epsilon^{2134} = -1$ = even permutations of 2134 = odd permutations of 1234

$\epsilon^{\mu\nu\lambda\kappa} = 0$ when 2 indices are equal

To show that $\epsilon^{\mu\nu\lambda\kappa}$ is a tensor density consider the expression

$$X^{\rho\sigma\eta\xi} \equiv \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\eta}}{\partial x^{\lambda}} \frac{\partial x'^{\xi}}{\partial x^{\kappa}} \epsilon^{\mu\nu\lambda\kappa} \tag{1}$$

We can see that $X^{\rho\sigma\eta\xi}$ must be proportional to $\epsilon^{\rho\sigma\eta\xi}$ by noting that (for example) $X^{\rho\rho\eta\xi} = 0$. This follows immediately by noting from (1) that setting $\rho = \sigma$ on the r.h.s. of (1) has the effect of making the coefficient of $\epsilon^{\mu\nu\lambda\kappa}$ <u>SYMMETRIC</u> in $\mu \leftrightarrow \nu$, whereas $\epsilon^{\mu\nu\lambda\kappa}$ itself is <u>ANTISYMMETRIC</u>, and hence their product vanishes. We can thus write

$$X^{\rho\sigma\eta\xi} = \underset{\text{constant}}{C} \cdot \epsilon'^{\rho\sigma\eta\xi} \tag{2}$$

To evaluate C write:

$$X^{1234} \equiv C\,\epsilon'^{1234} = \frac{\partial x'^1}{\partial x^1} \frac{\partial x'^2}{\partial x^2} \frac{\partial x'^3}{\partial x^3} \frac{\partial x'^4}{\partial x^4} \overset{+1}{\epsilon^{1234}} \tag{3}$$

$$+ \frac{\partial x'^1}{\partial x^2} \frac{\partial x'^2}{\partial x^1} \frac{\partial x'^3}{\partial x^3} \frac{\partial x'^4}{\partial x^4} \overset{-1}{\epsilon^{2134}} + \ldots \text{ all remaining permutations of } 1234$$

It can be seen that the expression on the r.h.s. of (3) is just the determinant of the transformation from $x \to x'$!

$$\text{r.h.s.} = \begin{vmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^3}{\partial x^1} & \frac{\partial x'^4}{\partial x^1} \\ \frac{\partial x'^1}{\partial x^2} & \frac{\partial x'^2}{\partial x^2} & \frac{\partial x'^3}{\partial x^2} & \frac{\partial x'^4}{\partial x^2} \\ \cdots & \cdots & \frac{\partial x'^3}{\partial x^3} & \frac{\partial x'^4}{\partial x^3} \\ \cdots & \cdots & \cdots & \frac{\partial x'^4}{\partial x^4} \end{vmatrix} \qquad (4)$$

From (3) & (4) we thus see that

$$\mathfrak{X}^{1234} \equiv C \underbrace{\epsilon'^{1234}}_{+1} = \left|\frac{\partial x'}{\partial x}\right| \cdot 1 \quad \Rightarrow \quad \boxed{C = \left|\frac{\partial x'}{\partial x}\right| \quad (5)}$$

Combining the previous results we find

$$\mathfrak{X}^{\rho\sigma\eta\xi} = \left|\frac{\partial x'}{\partial x}\right| \epsilon'^{\rho\sigma\eta\xi} = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \frac{\partial x'^\eta}{\partial x^\lambda} \frac{\partial x'^\xi}{\partial x^\kappa} \epsilon^{\mu\nu\lambda\kappa} \qquad (6)$$

Hence

$$\boxed{\epsilon'^{\rho\sigma\eta\xi} = \left|\frac{\partial x'}{\partial x}\right|^{-1} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} \frac{\partial x'^\eta}{\partial x^\lambda} \frac{\partial x'^\xi}{\partial x^\kappa} \epsilon^{\mu\nu\lambda\kappa}} \qquad (7)$$

This establishes that $\epsilon^{\mu\nu\lambda\kappa}$ is a 4th rank tensor density of weight $W = -1$