LINEAR ALGEBRA

TERMINOLOGY:

Ⓐ Group: A system $(G, \cdot)$ with elements $a, b, \ldots \in G$ and a closed operation $\cdot$ such that

1) $a \cdot b = c \in G \quad \forall a, b$
2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ — associativity *
3) $\exists\, i : \; i \cdot a = a \cdot i = a \;\; \forall a$ — identity element
4) For $\forall a \;\exists\, a^{-1} \ni \; a \cdot a^{-1} = a^{-1} \cdot a = i$ — inverse element

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5) If $a \cdot b = b \cdot a$ the group is commutative (Abelian)

Examples

a) Real numbers (excluding 0) $\quad \cdot = \times \quad i = 1$, $x^{-1} = 1/x$

b) Integers with $\cdot = +$ (Abelian)

c) Rotations of a sphere (non-Abelian)

* d) An example of a non-associative operation is

$$a \cdot b = a \times b + a + b$$

(check for yourself!) $\quad (a \cdot b) \cdot c - a \cdot (b \cdot c) = 2 \times (c - a)$

(B) Field : A system $\{F, +, \cdot\}$ satisfying the following axioms:

a) $\{F, +\}$ is an Abelian Group ; identity $= 0$

b) $\{F_0, \cdot\}$ is an Abelian Group ; identity $= i$

↳ all x except $x = 0$

c) For $a, b, c \in F$ $\quad a \cdot (b + c) = a \cdot b + a \cdot c$

distributivity

Examples:

rational numbers, real numbers, complex numbers with $+ =$ addition and $\cdot =$ multiplication

There are other structures we can define such as RINGS, but for our purposes GROUPS & FIELDS suffice.

VECTOR SPACES

A Vector Space is defined over a field F: See below

Then a vector space is a set of elements $\{V\}$ and an operation $+$ such that

a) $\{V, +\}$ = Abelian group

Also:

b) If $\beta, \alpha \in F$ and $X \in V$ then αX is also in V, and

b1) $\alpha(\beta X) = (\alpha\beta) X$

b2) $1X = X \quad \forall X$

c) distributivity

c1) $\alpha(X+y) = \alpha X + \alpha y$

c2) $(\alpha+\beta) X = \alpha X + \beta X$

Terminology: F = real numbers $\Rightarrow$ Real Vector Space

F = complex " $\Rightarrow$ Complex " "

Examples of Vector Spaces: (Prototype Vector space)

1) Set of all n-tuples: $X = (x_1, x_2, x_3, \ldots, x_n)$

$\alpha X = (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$; $X+y = (x_1+y_1, \ldots x_n+y_n)$

$0 = (0, 0, \ldots 0)$

2) Set of all complex numbers

The numbers themselves can be viewed as vectors

$$X = a + ib$$

3) The set of all polynomials in a variable t

$$x_1 = P_1(t) = a_1 + b_1 t + c_1 t^2 + \dots$$
$$x_2 = P_2(t) = a_2 + b_2 t + c_2 t^2 + \dots$$

This space is infinite-dimensional, as we discuss later when we develop the theory of Hilbert Space

LINEAR INDEPENDENCE

(l.i.)

A finite set of vectors $\{x_i\}$ is linearly-independent if and only if

$$\boxed{\sum_i \alpha_i x_i = 0 \Rightarrow \alpha_i = 0 \;\; \forall i}$$

To understand this suppose that $a_j \neq 0$. Then one can solve for x_j:

$$x_j = -\frac{1}{\alpha_j}(\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n)$$

This expresses x_j in terms of the other vectors, so x_j is not linearly independent. (Note that the vector $x = 0$ is linearly-dependent since one can express it in terms of any other vector x: $0 = 0 \cdot x$)

EXAMPLES: (1) $x_1(t) = 1-t \qquad x_2(t) = t-t^2 \qquad x_3(t) = 1-t^2$

These 3 vectors are linearly-dependent since

$$\sum_{i=1}^{3} \alpha_i x_i(t) = 0$$

can be satisfied with $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$, so $\alpha_i \neq 0$.

(2) $x_0(t) = 1 \quad x_1(t) = t, \ \ldots \ x_n(t) = t^n$

Form $\displaystyle\sum_{i=1}^{n} \alpha_i x_i(t) = 0$

In this case the only way that this equation can hold is if all $\alpha_i \equiv 0$ [consequence of a theorem of algebra]

BASES AND DIMENSIONALITY

A basis in V is a set $\{x_i\}$ for $x_i \in V$ such that every $x \in V$ is a linear combination of the x_i:

$$\boxed{x = \sum_{i=1}^{n} \alpha_i x_i} \qquad \forall x$$

If the $\{x_i\}$ are finite in number then V is a finite-dimensional vector space. Otherwise it is infinite-dimensional

Examples:

1) In 3-dim $\{x_i\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ where

$\hat{\imath} = (1,0,0) \quad \hat{\jmath} = (0,1,0) \quad \hat{k} = (0,0,1)$

2) For the polynomials the basis can be taken to be $\{x_n(t)\}$ where $x_n(t) = t^n$. This is an infinite basis. We will later see that this is the starting point of our discussion of HILBERT SPACE

Theorem: For $\forall\, x \in V$ there is a unique representation in terms of a given basis $\{x_i\}$.

Proof: Assume the contrary, then

$$x = \sum_i \alpha_i x_i \quad \text{and also} \quad x = \sum_i \beta_i x_i$$

Then $$0 = \sum_i (\alpha_i - \beta_i) x_i \equiv \sum_i \gamma_i x_i$$

But the $\{x_i\}$ are linearly independent (by assumption)

$\Rightarrow \gamma_i \equiv 0 \Rightarrow \boxed{\alpha_i = \beta_i} \leftarrow$ uniqueness

Definition: Dimension of a vector space = (unique) number of basis vectors

ISOMORPHISM OF VECTOR SPACES: ("ISO" = SAME ; "MORPH" = FORM)

The space spanned by the n-tuples in n-dimensions $\equiv F^n$

Ex: F^3 is spanned by $(1,0,0)$ $(0,1,0)$ $(0,0,1)$.

We can then show that every finite n-dimensional vector space is isomorphic to F^n.

Definition: Two vector spaces (U and V) are isomorphic if

a) U and V are over the same field

b) There exists a 1-1 correspondence between $x \in U$ and $y \in V$ such that $y = T(x)$ where

$$T(x_1 + x_2) = T(x_1) + T(x_2) = y_1 + y_2$$

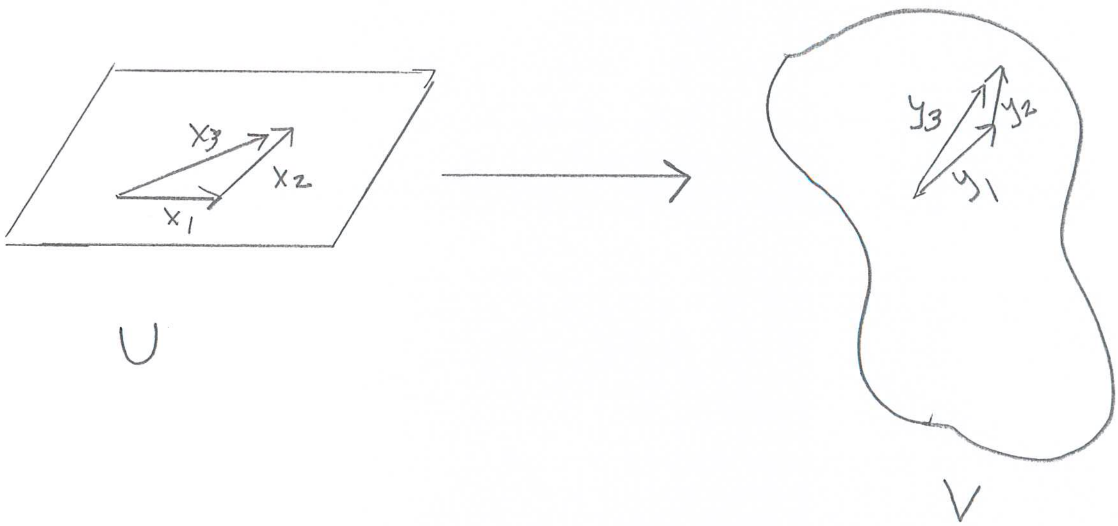
$$T(\alpha x_1) = \alpha T(x_1) = \alpha y_1$$

These relations can be grouped into a single equation

$$\boxed{T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) = \alpha_1 y_1 + \alpha_2 y_2}$$

"preserves algebraic structure"

Schematically this can be visualized as follows:



Simply stated: If $x_1 \to y_1$, and $x_2 \to y_2$, and $x_3 \to y_3$, then if the effect of adding x_1 and x_2 in U is to give x_3, then it must be the case also that the effect of adding y_1 and y_2 in V gives the corresponding vector y_3.

ISOMORPHISMS MORE GENERALLY

Consider 2 systems $S = \{E, *\}$ and $S' = \{E', \star\}$

We wish to establish an isomorphism between S & S':

E	f: 1-1 mapping	E'
a	$\longrightarrow$	$f(a) \equiv a'$
b	$\longrightarrow$	$f(b) \equiv b'$
$a * b = c$	$\longrightarrow$	$f(a) \star f(b) \stackrel{?}{=} f(c) = f(a*b)$ $a' \star b' \stackrel{?}{=} c'$
c	$\longrightarrow$	$f(c) = c'$

If the ? turns out correctly then S and S' are <u>isomorphic</u>.

The two systems S and S' can be isomorphic even if the operations $*$ and $\star$ have nothing to do with each other. This is illustrated by the following examples where the elements of S are rotations, whereas the elements of S' are permutations:

isomorphic systems that at first glance look very different. Consider the six 2x2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

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$$C = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \qquad D = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \qquad E = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

These six matrices form a group which can be verified. We give the multiplication table for this group which tells us everything about it. Note that this group is not Abelian.

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	D	E	B	C
B	B	E	I	D	C	A
C	C	D	E	I	A	B
D	D	C	A	B	E	I
E	E	B	C	A	I	D

AB = D

First, it is clear that the system is closed. The unit element of the group is the matrix I. To determine the inverse of a given element (say C) we look at the multiplication table to see what element X is such that CX = XC = I. Answer $C^{-1} = C$. Also, $D^{-1} = E$. Matrix multiplication is associative (to be proven later), so we have a group. Let us refer to this system as $\{M, \cdot\}$, M for the set of six matrices and $\cdot$ for matrix multiplication.

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Now we consider the set of six permutations of three numbers as another system:

$$\underset{I'}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}} \qquad \underset{A'}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}} \qquad \underset{B'}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}}$$

$$\underset{C'}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}} \qquad \underset{D'}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}} \qquad \underset{E'}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}$$

The product of two permutations is the single permutation that accomplishes what two applied successively would accomplish, e.g.

i.e. the map that $1 \to 3$, $2 \to 1$, $3 \to 2$

1) $$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \odot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$B' \odot D' = C'$$

2) $$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \odot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$A' \odot B' = D'$$

3) $$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \odot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$I' \odot C' = C'$$

4) $$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$E' \odot C' = A'$$

5) $$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \odot \begin{pmatrix} 1 & 2 & 3 \\ x & y & z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ z & x & y \end{pmatrix}$$

ANGULAR MOMENTUM AND ISOMORPHISMS

A common example of the application of the ideas of isomorphisms is the theory of angular momentum in quantum mechanics. The angular momentum operator $\vec{J}$ for spin-1/2 particles is given by

$$\vec{J} = \frac{\hbar}{2}\vec{\sigma} \qquad (1)$$

KNOW THESE!!

PAULI MATRICES

$$\boxed{\sigma_x \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \qquad (2)$$

By direct computation we can establish that these operators satisfy the commutation relations

$$\boxed{[J_x, J_y] = i\hbar J_z , \quad \underline{\text{etc}} \text{ cycl.}} \qquad (3)$$

$|\vec{J}| = \frac{1}{2}\hbar$ is the smallest angular momentum that any object can have, and the representation of $\vec{J}$ as in (1) & (2) is called the <u>FUNDAMENTAL REPRESENTATION</u>, and this defines the algebra of angular momentum operators as in (3).

However, once the abstract relations in (3) are obtained, they can be realized by other operators: 3X3, 4X4, ... matrices, which are then said to be isomorphic to $\frac{\hbar}{2}\vec{\sigma}$. Moreover, the commutation relations in (3) can be realized by other structures, such as differential operators; these are then said to be isomorphic to the matrices in (2):

$$J_x = \left(y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}\right); \quad J_y = \left(z\frac{\hbar}{i}\frac{\partial}{\partial x} - x\frac{\hbar}{i}\frac{\partial}{\partial z}\right); \quad J_z = \left(x\frac{\hbar}{i}\frac{\partial}{\partial y} - y\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$

Theorem:

Every n-dimensional vector space V over a field F is isomorphic to the space F^n of the n-tuples of F.

Proof: Let $\{x_1, \dots x_n\}$ be any basis in V. Hence for $x \in V$

$$x = \sum_i \alpha_i x_i \qquad (\alpha_i \text{ are unique}) \qquad (1)$$

The proposed isomorphism is

$$V \longleftrightarrow F^n \qquad (2)$$

$$x \in V \longleftrightarrow (\alpha_1, \alpha_2, \dots \alpha_n) \qquad (3)$$

Symbolically $\boxed{T(x) = (\alpha_1, \alpha_2, \dots \alpha_n)}$ (4)

[T(x) picks out the coefficients of x_i in $x = \sum_i \alpha_i x_i$]*

(1)-(4) $\Rightarrow$ If $y = \sum_i \beta_i x_i$ then $(cx+dy) = c\sum_i \alpha_i x_i + d\sum_i \beta_i x_i$

$$= \sum_i (c\alpha_i + d\beta_i) x_i \qquad (4)$$

*Since $T(\dots)$ picks out the coeff. of x_i $\Rightarrow$

$$T(cx+dy) = c\alpha_i + d\beta_i = (c\alpha_1 + d\beta_1, c\alpha_2 + d\beta_2, \dots) \Longleftarrow \text{OK} \qquad (5)$$

To establish an isomorphism we want to show that this is the same as

$cT(x) + dT(y)$: $\quad cT(x) = c(\alpha_1, \alpha_2, \dots \alpha_n) = (c\alpha_1, c\alpha_2, \dots c\alpha_n)$

$\quad dT(y) = d(\beta_1, \beta_2, \dots \beta_n) = (d\beta_1, d\beta_2, \dots d\beta_n)$

$\therefore cT(x) + dT(y) = (c\alpha_1 + d\beta_1, c\alpha_2 + d\beta_2, \dots c\alpha_n + d\beta_n) \Longleftarrow$ OK QED

Corollary: Any two n-dim vector spaces over the same F are isomorphic to each other [Since they are both isomorphic to F^n].

LINEAR TRANSFORMATIONS (l.t.)

A l.t. $\underline{\underline{A}}$ on V assigns to every $x \in V$ another vector $\underline{\underline{A}} x \in V$

such that

$$\boxed{A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)} \quad (1)$$

$$\text{The product: } Px = (AB)x \equiv A(Bx) \quad (2)$$

Examples:

1) Rotations in a ~~plane~~ plane: $A \equiv R =$ Rotation operator (3)

$$\vec{x} \to R\vec{x} = R(\theta)\vec{x}$$

2)
$$0x = 0 \quad (4)$$
$$Ix = x$$

3) $D =$ differentiation $\Rightarrow Df(x) \equiv \frac{df}{dx}$ (5)

$X =$ multiplication by $x \Rightarrow Xf(x) = xf(x)$

Then: $(DX)f(x) = D(Xf(x)) = \frac{d}{dx}(xf(x)) = f(x) + x\frac{df(x)}{dx}$ (6)

$(XD)f(x) = X\left(\frac{df(x)}{dx}\right) = x\frac{df(x)}{dx}$ (7)

$\therefore [DX - XD]f(x) = f(x) \Leftrightarrow \boxed{[D, X] = I}$ (8)

This Commutation relation is at the heart of quantum mechanics!

Let $p_x \to -i\hbar\frac{d}{dx} = -i\hbar D$. Then (8) $\Rightarrow$ $\boxed{[X, p_x] = i\hbar}$

4) In the space of polynomials of degree m, P_m define the linear transformation D via

$$D P_m(x) = \frac{d}{dx} P_m(x)$$

$$D^1 \equiv D, \quad D^2 \equiv DD \quad ; \quad D^n = \underbrace{DD\ldots}_{n \text{ factors}}$$

Evidently, if $n>m$ then

$$D^n P_m = 0 \quad \Rightarrow \quad D^n = 0$$

Hence $\exists$ nonzero l.t.s which have the property that even though $D \neq 0$ $D^n = 0$. [We will later see that typically such operators do not have inverses.]

Linear Transformations obey usual algebraic relations:

a) $A0 = 0A = 0$

b) $AI = IA = A$

c) $A(B+C) = AB + AC$

d) $A(BC) = (AB)C$

Inverse of a Linear Transformation

This subject is important because the solution of many algebraic problems requires finding the inverse, or at least knowing when an inverse exists.

A l.t. A has an inverse A^{-1} if

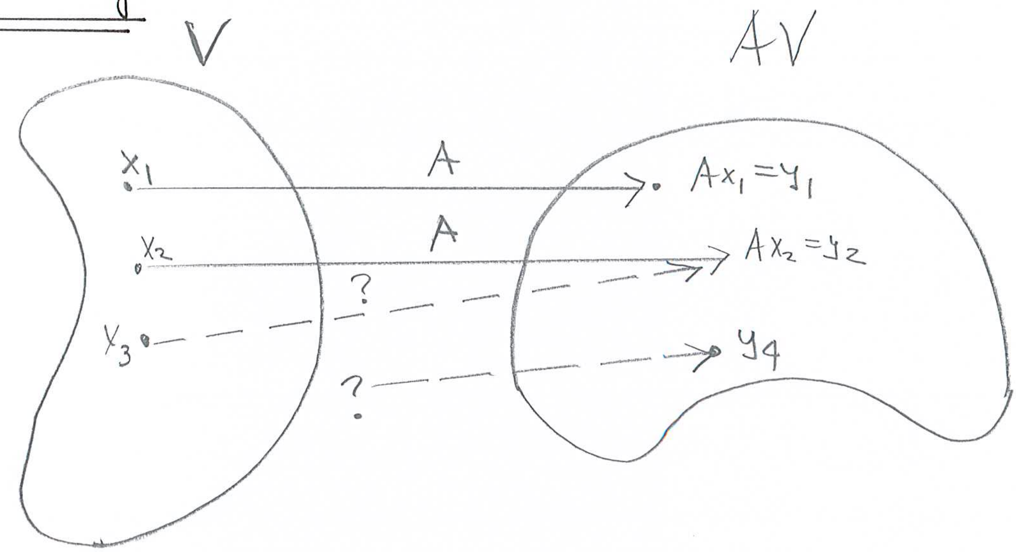
1) $x_1 \neq x_2 \Rightarrow Ax_1 \neq Ax_2$ "uniqueness"

or

$Ax_1 = Ax_2 \Rightarrow x_1 = x_2$

2) For $\forall\, y \in V$ there exists at least one $x \in V \ni$ $Ax = y$. "parentage"

Pictorially



If the situations described by the ---- lines hold then no inverse of A exists.

The picture explains intuitively why both "uniqueness" and "parentage" are necessary for A^{-1} to exist.

Definition of A^{-1}

If y_0 is any vector in AV then condition 2) $\Rightarrow \exists$ an x_0 in V such that $y_0 = Ax_0$. x_0 is unique by condition 1). Then A^{-1} is defined by $\boxed{A^{-1}y_0 = x_0}$ (1)

If A^{-1} exists then $AA^{-1} = A^{-1}A = I$ (2)

Theorem: A, B, C are l.t.s, such that

$$AB = CA = I \tag{3}$$

then A^{-1} exists and $A^{-1} = B = C$

Proof: To prove that A^{-1} exists we have to show that conditions 1) & 2) on p. F106 hold.

1) if $AX_1 = AX_2$ and $CAX_1 = X_1$; $CAX_2 = X_2$ (4)

$\therefore$ $AX_1 = AX_2 \Rightarrow CAX_1 = CAX_2 \Rightarrow X_1 = X_2$ ✓

2) Let y be any vector in AV, and define $x = By$.
Then $Ax = ABy = Iy = y$. This assigns to every y a vector Ax in $V \Rightarrow$ every y has a "parent" ✓

This establishes that A^{-1} exists. To find A^{-1},

$$AB = I \Rightarrow \underbrace{A^{-1}A}_{I}B = A^{-1}I \Rightarrow \boxed{A^{-1}I = A^{-1} = B} \tag{5}$$

$$CA = I \Rightarrow C\underbrace{AA^{-1}}_{I} = IA^{-1} = A \Rightarrow \boxed{C = IA^{-1} = A^{-1}} \quad \text{QED} \tag{6}$$

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For a finite dimensional V either condition $AB=I$ or $CA=I$ is sufficient to prove that A^{-1} exists.

However, for an infinite dimensional V both conditions are needed.

Examples: Let $V=$ infinite-dim vector space of polynomials $P(x)$

Then define

$$D\,P(x) = \frac{d}{dx} P(x)$$

$$S\,P(x) = \int_0^x P(t)\,dt$$

Even though $DS\,P(x) = \frac{d}{dx}\int_0^x P(t)\,dt = P(x)$ [fundamental thm of calculus]

$$\therefore\; DS = I$$

Nonetheless neither D nor S is invertible:

① D violates condition 1), since $D(x^2+3) = D(x^2+17)$ etc.

② S violates condition 2), since for $y = x^2+1$ there is no $x(t)$ such that $x^2+1 = \int_0^x x(t)\,dt$ [Hint: Try $x(t) = at+b$]

FUNDAMENTAL THEOREM ON INVERSES A^{-1}:

Thm: If $Ax=0 \Rightarrow x=0$, then a l.t. on a finite dimensional V is invertible. [Also if A^{-1} exists then $Ax=0 \Rightarrow x=0$.]

Proof: If $Ax_1 = Ax_2 \Rightarrow Ax_1 - Ax_2 = 0 = A\underbrace{(x_1-x_2)}_{x} = 0 \Rightarrow Ax=0$

By assumption this implies $x = x_1 - x_2 = 0$ or $x_1 = x_2$

Hence defining $x = x_1 - x_2$ it follows that $Ax=0 \Rightarrow x=0$ satisfies condition 1) on p. F106

To prove condition 2) ["parentage"] from these assumptions let $\{x_1 \dots x_n\}$ be a finite basis in V. If it can be shown that $\{Ax_1 \dots Ax_n\}$ is also a basis then any y can be written as

$$y = \sum_i \alpha_i \underbrace{(Ax_i)}_{\text{in } AV} = A \sum_i \underbrace{\alpha_i x_i}_{\text{in } V} = AX \quad = \text{Condition 2}$$

Q: is $\{Ax_1, \dots Ax_n\}$ a basis?

A: yes since there are n of them (n-dim space) – provided they are linearly independent. Form

$$\sum_i \alpha_i Ax_i = 0 \quad = A \underbrace{\sum_i \alpha_i x_i}_{x_i \text{ are l.i}} = 0 \Rightarrow \alpha_i = 0$$

$$\therefore \sum_i \alpha_i Ax_i = 0 \Rightarrow \alpha_i = 0 \Rightarrow \quad Ax_i \text{ lin. indep.}$$

This completes the proof QED.

Theorem: If A & B have inverses ~~...~~ then AB has an inverse and $(AB)^{-1} = B^{-1}A^{-1}$. Also $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and $(A^{-1})^{-1} = A$.

Proof: It is sufficient to prove that [see earlier theorem p. 107]

$$\left.\begin{aligned} (AB)B^{-1}A^{-1} &= A\underbrace{BB^{-1}}_{I}A^{-1} = AA^{-1} = I \\ B^{-1}A^{-1}(AB) &= B^{-1}A^{-1}AB = B^{-1}B = I \end{aligned}\right\} \text{trivial}$$

Rest is obvious.

MATRICES AS LINEAR TRANSFORMATIONS

By choosing an appropriate basis in a finite dim vector space, a linear transformation A can be expressed in terms of a matrix. (The same txfmtion will have a different representation as a matrix w.r.t. another basis)

Let $\{x_i\}$ be a basis for an n-dim V. Thus any vector Ax_j is some (other) vector in V and can be expressed in terms of $\{x_i\}$

$$A x_j = \sum_i a_{ij} x_i \qquad (1)$$

For example: Let $j=7$: $A x_7 = \sum_{i=1}^{N} a_{i7} x_i = a_{17} x_1 + a_{27} x_2 + \ldots a_{N7} x_N$ (2)

If we write the coefficients a_{ij} as a matrix, then

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{17} & a_{18} \\ a_{21} & & & & a_{27} & a_{28} \\ a_{31} & & & & a_{37} & a_{38} \\ \vdots & & & & \vdots & \vdots \end{pmatrix}$$

Hence the coefficients of x_i in expressing Ax_7 in terms of $\{x_i\}$ form the 7th column in this convention. So Ax_j for $j=7$ is another vector, represented by a column vector, as x_7 itself would be.

NOTE!! Different authors use different conventions!!
BE CONSISTENT!!!

Side Comment on $Ax=0$:

If $Ax=0 \Rightarrow x=0$ then A^{-1} exists. Why is this important?

(I) Eigenvalue Problem:

$$\underset{\uparrow \text{ eigenvalue}}{H x = \lambda} \overset{\text{eigenvector}}{x} = \lambda I x \qquad I = \text{unit matrix} \qquad (1)$$

Then $(1) \Rightarrow (H - \lambda I) x = 0 \equiv Ax$

$\to x = 0 \Leftrightarrow A^{-1}$ exists

$\to x \neq 0 \Leftrightarrow A^{-1}$ does not exist

For the eigenvalue problem $x=0$ is trivial, and is not the solution we want. Hence $x \neq 0 \Rightarrow A^{-1}$ does not exist. We show later that

$$A^{-1} = \frac{\text{Adj } A \leftarrow \text{a matrix}}{\det A \leftarrow \text{a number}} \qquad (3)$$

When A^{-1} does not exist it is because $\det A = 0$, which becomes the condition which leads to a solution for the eigenvalues λ:

$$\boxed{\underbrace{\det(H - \lambda I) = 0}_{\text{characteristic equation}} = \text{polynomial in } \lambda}$$

(II) Linear Independence of the Solutions of a Differential Equation:

A 2ND order diff. eqn. has 2 lin. indep. solutions.

How do we know whether the solutions we have found are lin. indep.?

Consider more generally a set of functions $f_i(x)$: The condition for lin. indep. is

$$\boxed{\sum_{i=1}^{N} \alpha_i f_i(x) = 0 \Rightarrow \alpha_i \equiv 0 \quad \forall i}$$

Q: Given a set of functions $f_i(x)$ how do we know whether nonzero α_i can be found?

A: Rather than focus on α_i, focus on solutions

Then
$$\sum_i \alpha_i f_i(x) = 0 \quad \Rightarrow \qquad (1)$$
$$\sum_i \alpha_i f_i'(x) = 0 \qquad (2)$$
$$\sum_i \alpha_i f_i'' = 0 \qquad (3)$$
$$\vdots$$
$$\sum_i \alpha_i f_i^{(N-1)} = 0 \qquad (4)$$

Writing these explicitly:

$$f_1\alpha_1 + f_2\alpha_2 + \dots + f_N\alpha_N = 0 = m_{11}\alpha_1 + m_{12}\alpha_2 + \dots m_{1N}\alpha_N \qquad (5)$$
$$f_1'\alpha_1 + f_2'\alpha_2 + \dots + f_N'\alpha_N = 0 = m_{21}\alpha_1 + m_{22}\alpha_2 + \dots + m_{2N}\alpha_N \qquad (6)$$
$$\vdots \qquad\qquad \vdots$$
$$f_1^{(n-1)}\alpha_1 + f_2^{(n-1)}\alpha_2 + \dots + f_N^{(N-1)}\alpha_N = 0 = m_{N1}\alpha_1 + m_{N2}\alpha_2 + \dots + m_{NN}\alpha_N \qquad (7)$$

Define $M = \begin{pmatrix} m_{11} & \dots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \dots & m_{NN} \end{pmatrix} \qquad \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \qquad (8)$

$$(5)-(7) \Rightarrow \boxed{M\alpha = 0} \Leftrightarrow Ax = 0 \qquad (9)$$

From our discussion of A^{-1} we again note:

$$M\alpha = 0 \begin{cases} \rightarrow \alpha = 0 \Leftrightarrow M^{-1} \text{ exists} \\ \rightarrow \alpha \neq 0 \Leftrightarrow M^{-1} \text{ does not exist} \end{cases} \qquad (10)$$

The condition for lin. indep. of $f_i(x)$ is that $\alpha_i = 0 \Rightarrow \alpha = 0 \Rightarrow M^{-1}$ exists.

As before $M^{-1} = \text{Adj}\, M / \det M \Rightarrow$ lin. indep. $\Rightarrow \det M \neq 0 \qquad (11)$

Define $W(x) = \det M(x) =$ WRONSKIAN

$$\boxed{\therefore\ W(x) \neq 0 \Rightarrow f_i(x) \text{ are linearly independent}} \qquad (12)$$

Applications:

[1] $y''(x) + w^2 y(x) = 0$ has 2 lin. indep. solutions

$$y_1(x) = \sin wx \qquad (13)$$
$$y_2(x) = \cos wx$$

To show that these are infact linearly independent:

$$M(x) = \begin{pmatrix} \sin wx & \cos wx \\ w\cos wx & -w\sin wx \end{pmatrix} \qquad (14)$$

$$W(x) = \det M(x) = -w\sin^2 wx - w\cos^2 wx = -w \neq 0 \checkmark$$

[2] For any 2 solutions $y_1(x)$ & $y_2(x)$

$$M(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \Rightarrow W(x) = y_1 y_2' - y_2 y_1' \qquad (15)$$

We show next semester that one can determine $W(x)$ without completely knowing the 2 solutions $y_1(x)$ & $y_2(x)$.

It can then be shown that

$$\boxed{\left.\begin{array}{l} y_1(x) \\ W(x) \end{array}\right\} \Rightarrow y_2(x)} \qquad (16)$$

Hence, knowing $W(x)$ and one solution $y_1(x)$ we can find a second solution $y_2(x)$.

PROPERTIES OF MATRICES

[1] Matrices of same form can be added.
[2] Matrices can be multiplied by scalars
} trivial

[3] Matrix multiplication:

Theorem: $A=(\alpha_{ij})$ $\quad B=(\beta_{ij}) \Rightarrow C=AB\equiv\gamma_{ij}$ where

$$\boxed{\gamma_{ij}=\sum_k \alpha_{ik}\beta_{kj}}$$

basis vector

Proof: $Cx_j = A(Bx_j) = A\sum_k \beta_{kj}x_k = \sum_k \beta_{kj}(Ax_k)$

$$=\sum_k \beta_{kj}\sum_i \alpha_{ik}x_i = \sum_i\left(\sum_k \alpha_{ik}\beta_{kj}\right)x_i$$

But by definition $Cx_j \equiv \sum_i \gamma_{ij}x_i$

$\Rightarrow \sum_i\left(\sum_k \alpha_{ik}\beta_{kj}\right)x_i = \sum_i \gamma_{ij}x_i$

Hence $\sum_i\left(\gamma_{ij}-\sum_k \alpha_{ik}\beta_{kj}\right)x_i = 0$

Since $x_i \in \{x_i\}$ = basis, x_i are lin. indep. $\Rightarrow (\ldots)=0$

Hence $\gamma_{ij}=\sum_k \alpha_{ik}\beta_{kj}$ Q.E.D

Comment: $Ax_j = \sum_i \alpha_{ij}x_i$ is an isomorphism

$$A \Leftrightarrow (\alpha_{ij})$$

SPECIAL MATRICES

[1] PAULI MATRICES:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Along with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for a basis for any 2x2 matrix M:

$$\boxed{M = aI + \vec{b} \cdot \vec{\sigma}} \qquad a, \vec{b} = \text{complex (in general)}$$

a) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

b) $[\sigma_x, \sigma_y] = 2i\,\sigma_z$; More generally $[\sigma_a, \sigma_b] = 2i\,\epsilon_{abc}\,\sigma_c$

c) $\{\sigma_x, \sigma_y\} = \sigma_x \sigma_y + \sigma_y \sigma_x = 0$, etc.

d) $\sigma_i^{-1} = \sigma_i^{\dagger}$

[2] Diagonal Matrices: $D = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{33} \end{pmatrix}$

a) Any 2 nxn diag. matrices commute, and their product is also diagonal

b) The ~~eigenvalues~~ diagonal elements of the matrix are its eigenvalues.
[This is why we speak about "diagonalizing the Hamiltonian" in quantum mechanics]

[3] Idempotent Matrices

$\hookrightarrow \; A^2 = A$

Example: $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$; $A^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \checkmark$

Similarly: $A^3 = A \cdot A^2 = A \cdot A = A^2 = A$ etc.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are also idempotent

[4] Non-Singular Matrices = Invertible Matrices:

A is non-singular if $\exists B \ni AB = I$ (recall p. F106/107)

If $\det A = 0$ it is singular (no inverse), e.g.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

We will later show that in general if 2 rows or columns of a matrix are lin. dep. then A is singular.

[5] Transpose of a matrix $\equiv A^T = \tilde{A}$

$A = (\alpha_{ij}) \Rightarrow \quad A^T = \alpha_{ji}$ (rows $\Leftrightarrow$ columns)

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(AB)^T = B^T A^T \rightarrow$ Proof: $(A) = \alpha_{ij} \updownarrow [A]_{ij} = \alpha_{ij}$ $\quad [B]_{ij} = \beta_{ij}$

Then $[A^T]_{ij} = \alpha_{ji}$; $[B^T]_{ij} = \beta_{ji}$

$$\left.\begin{aligned} [AB]_{ij} &= \sum_k \alpha_{ik}\beta_{kj} \Rightarrow [(AB)^T]_{ij} = \sum_k \alpha_{jk}\beta_{ki} \\ [B^T A^T]_{ij} &= \sum_k [B^T]_{ik}[A^T_{kj}] = \sum_k \beta_{ki}\alpha_{jk} = \sum_k \alpha_{jk}\beta_{ki} \end{aligned}\right\} \Rightarrow (AB)^T = B^T A^T$$

[6] Symmetric & Antisymmetric Matrices

$S = (a_{ij})$ is symmetric if $S = S^T \Rightarrow a_{ij} = a_{ji}$

$A = (a_{ij})$ is antisymmetric if $A = -A^T \Rightarrow a_{ij} = -a_{ji}$

in this case diagonal elements $\equiv 0 \Rightarrow \boxed{\mathrm{Tr}\, A = 0}$

Any square matrix can be decomposed into a sum of a symmetric and an antisymmetric matrix:

$$M = \underbrace{\tfrac{1}{2}(M + M^T)}_{\equiv S} + \underbrace{\tfrac{1}{2}(M - M^T)}_{\equiv A} \qquad (1)$$

For an $n \times n$ matrix there are n^2 entries in the original matrix M. These break up into

$$\underbrace{M}_{n^2} = \underbrace{S}_{\frac{1}{2}n(n+1)} + \underbrace{A}_{\frac{1}{2}n(n-1)} \qquad (2)$$

Example: $n = 3$

$$\underset{3\times 3 = 9}{M} = \underset{\frac{1}{2}(3\cdot 4) = 6}{S} + \underset{\frac{1}{2}(3\cdot 2) = 3}{A} \qquad (3)$$

The decomposition in (1) & (2) is useful because it allows some simplifications from identities such as:

$$\epsilon_{ijk}[A]_{jk} \neq 0 \quad \text{but} \quad \epsilon_{ijk}[S]_{jk} \equiv 0 \qquad (4)$$

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where the sum is over all permutations P of the integers 1, 2, ...n and where the + or - sign is affixed to each product according to whether P is even or odd.

Examples:

1) 2 x 2 $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

P		Sign P
P(1)	P(2)	
1	2	+
2	1	-

$\therefore$ $\det A = +a_{11}\,a_{22} - a_{21}\,a_{12}$

2) 3 x 3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

P			Sign P
1	2	3	+
P(1)	P(2)	P(3)	
1	3	2	-
2	1	3	-
2	3	1	+
3	1	2	+
3	2	1	-

$$\therefore \det A = \begin{aligned} &+a_{11}\,a_{22}\,a_{33} \\ &+a_{21}\,a_{32}\,a_{13} \\ &+a_{31}\,a_{12}\,a_{23} \\ &-a_{11}\,a_{32}\,a_{23} \\ &-a_{21}\,a_{12}\,a_{33} \\ &-a_{31}\,a_{22}\,a_{13} \end{aligned}$$

do this as an example

We now summarize some of the properties of the determinant function. These properties can all be proven from our definition but we will not go into the proofs. If these properties are not familiar satisfy yourself that they are valid for 2 x 2 and 3 x 3 matrices.

DETERMINANTS:

A function which acts on matrices $\rightarrow$ scalars

Definition: $\det A = \sum_{\substack{\text{permutations} \\ P}} \left\{ \pm\, a_{P(1)1}\, a_{P(2)2} \cdots a_{P(n)n} \right\}$

even (+), odd permutation (−)

$$= \sum_{P} \left\{ \pm \prod_{i=1}^{n} a_{P(i)i} \right\}$$

Properties of Determinants:

1) A common factor of a row or column can be factored out.
2) $\det A = 0$ if any row or column $= 0$.
3) $\det A = 0$ if 2 rows or columns are lin. dependent.
4) $\det I = 1$
5) $\det A$ is unchanged if a scalar multiple of one row or column is added to another row or column [Useful in computations].
6) $\det A \rightarrow -\det A$ when 2 rows are interchanged (or 2 columns)
7) $\det A^T = \det A$

★ 8) $\det A = 0 \Leftrightarrow$ the row or column vectors are lin. dep.

The determinant as a volume:

$$\det \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix} = \begin{array}{l} A_x B_y C_z + B_x C_y A_z + C_x B_z A_y \\ -A_z B_y C_x - B_z C_y A_x - C_z B_x A_y \end{array}$$

$$= A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z) + A_z (B_x C_y - B_y C_x) = \vec{A} \cdot (\vec{B} \times \vec{C})$$

= Solid (parallelopiped) volume determined by $\vec{A}, \vec{B}, \vec{C}$

The geometric picture can be used as a mnemonic for the properties of a determinant: F118/119

Properties [from previous page]

1) 2) obvious ✓

3) $\vec{A}\cdot(\vec{B}\times\vec{C}) \to \vec{A}\cdot(\vec{B}\times\vec{B}) = 0$ etc. ✓

4) volume of unit cube $= 1$

5) $\vec{A}\cdot(\vec{B}\times\vec{C}) \to (\vec{A}+\lambda\vec{B})\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C}) + \lambda \underbrace{\vec{B}\cdot(\vec{B}\times\vec{C})}_{=0}$ etc. ✓

6) $\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$

7) The volume is the same whether A, B, C are rows or columns ✓

★ 8) a) If $\vec{C} = \lambda\vec{B}$ then $\vec{A}\cdot(\vec{B}\times\vec{C}) \to \lambda\vec{A}\cdot(\vec{B}\times\vec{B}) = 0$

Hence if 2 rows (or columns) are lin. dep $\Rightarrow$ volume $\to 0$

b) (Converse) if $\det A = 0$ this means that the vectors $\vec{A}, \vec{B}, \vec{C}$ have 0 3-dim volume. This means that these vectors must "live" in a lower dimensional space, such as a plane $(d=2)$ or a line $(d=1)$. But 3 vectors $\vec{A}, \vec{B}, \vec{C}$ in $d=2$ or $d=1$ must be lin. dependent.

$\therefore$ $\boxed{\det A = 0 \Rightarrow \vec{A}, \vec{B}, \vec{C} \text{ are lin. dependent}}$

This geometric picture generalizes to higher dimension matrices & their determinants.

FINDING THE INVERSE OF A MATRIX:

The "Cofactor" Rule for det A:

Definition: Let $A = (a_{ij})$. The cofactor $|A_{ij}|$ of a_{ij} is a number given by $(-1)^{i+j} \otimes \det (n-1)\times(n-1)$ matrix formed by deleting the row i and column j in A.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow |A_{12}| = (-1)^{1+2}(a_{21}a_{33} - a_{23}a_{31})$$

$$= a_{23}a_{31} - a_{21}a_{33}$$

Cofactor Rule for det A:

$$\boxed{\begin{aligned} \det A &= \sum_{j=1}^{n} a_{ij}|A_{ij}| \quad \text{for a fixed } i \text{ (row)} \\ &= \sum_{i=1}^{n} a_{ij}|A_{ij}| \quad \text{for a fixed } j \text{ (column)} \end{aligned}}$$

Example: 3X3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det A = a_{11}|A_{11}| + a_{12}|A_{12}| + a_{13}|A_{13}|$$

$$|A_{11}| = (-1)^{1+1}(a_{22}a_{33} - a_{23}a_{32}) \; ; \; |A_{12}| = (-1)^{1+2}(a_{21}a_{33} - a_{23}a_{31})$$

$$|A_{13}| = (-1)^{1+3}(a_{21}a_{32} - a_{22}a_{31})$$

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

✓ ✓ F117

F12 0.1

CONSTRUCTING THE INVERSE MATRIX:

We start with

$$\det A = \sum_{j=1}^{n} a_{ij}\,|A_{ij}| \qquad \text{(fixed row; sum over columns)} \qquad (1)$$

Form the matrix $\sum_{j=1}^{n} a_{ij}\,|A_{kj}|$ (2)

Claim: This matrix is just $= \delta_{ik} \det A$ (3)

Proof: For $i=k$ (2) = (3) obviously. For $i \neq k$ $\delta_{ik} \det A = 0$

So we have to prove that this is true for $\sum_{j=1} a_{ij}\,|A_{kj}|$. This can be done generally by showing that this yields the det of a matrix with 2 equal columns or rows. Here we simply illustrate this for 3x3:

$$\delta_{12} \det A = 0 \overset{?}{=} \sum_{j=1}^{n} a_{ij}\,|A_{2j}| = \sum_{j=1}^{n=3} a_{ij}\,|A_{2j}| = a_{11}|A_{21}| + a_{12}|A_{22}| + a_{13}|A_{23}| \qquad (4)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \begin{aligned} |A_{21}| &= (-)(a_{12}a_{33} - a_{13}a_{32}) \\ |A_{22}| &= (+)(a_{11}a_{33} - a_{13}a_{31}) \\ |A_{23}| &= (-)(a_{11}a_{32} - a_{12}a_{31}) \end{aligned} \qquad (5)$$

Hence

$$\sum_{j=1}^{n=3} a_{ij}\,|A_{2j}| = (-)a_{11}(a_{12}a_{33} - a_{13}a_{32}) + a_{12}(a_{11}a_{33} - a_{13}a_{31}) - a_{13}(a_{11}a_{32} - a_{12}a_{31}) = 0 \qquad (6)$$

Hence we can write

$$\boxed{\delta_{ik} \det A = \sum_{j=1}^{n} a_{ij}\,|A_{kj}|} \qquad (7)$$

To construct the inverse matrix we next introduce the <u>classical adjoint matrix</u> (to be distinguished later from the <u>Hermitian adjoint</u>. Classical adjoint of $A \equiv \mathrm{Adj}\, A$:

$$(\mathrm{Adj}\, A)_{ij} = |A_{ji}| \qquad \text{ex: } (\mathrm{Adj}\, A)_{12} = |A_{21}| \tag{8}$$

Then consider: $$(A \cdot \mathrm{Adj}\, A)_{ij} = \sum_{k=1}^{n} a_{ik}\, (\mathrm{Adj}\, A)_{kj} = \sum_{k=1}^{n} a_{ik}\, |A_{jk}| \tag{9}$$

$$= \delta_{ij} \det A \leftarrow \text{using (7)}$$

Hence $$(A \cdot \mathrm{Adj}\, A)_{ij} = \delta_{ij} \det A \tag{10}$$

Since this holds for all elements ij on both sides of the equation, we can write (10) as a matrix equation:

$$\underset{\substack{\uparrow \\ \text{matrix}}}{A} \cdot \underset{\substack{\uparrow \\ \text{matrix}}}{\mathrm{Adj}\, A} = \underset{\substack{\uparrow \\ \text{number}}}{\det A} \cdot \underset{\substack{\uparrow \\ \text{matrix}}}{I} \tag{11}$$

$$\Rightarrow \quad \cancel{A^{-1} A} \cdot \mathrm{Adj}\, A = A^{-1} \det A\, I \Rightarrow \boxed{A^{-1} = \frac{\mathrm{Adj}\, A}{\det A}} \tag{12}$$

As we have noted previously, $\mathrm{Adj}\, A$ is a matrix we can compute for any A. Hence the question of whether A^{-1} exists comes down to the question of whether $\det A = 0$.