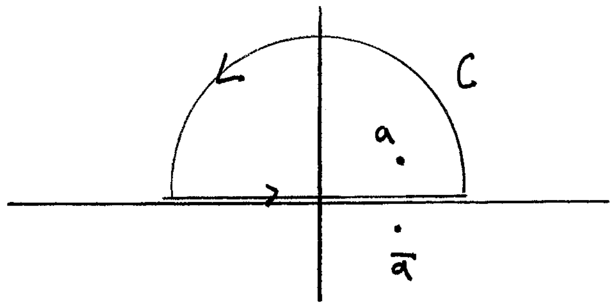
a) HILBERT TRANSFORMS & DISPERSION RELATIONS

This formalism allows a complex function $f(z)$ to be expressed as a real integral over its real and imaginary parts. [Applications to follow!!]



$\bar{a} = a^*$

Consider:

$$I = \frac{1}{2\pi i}\oint_C dz \left\{ \frac{f(z)}{z-a} + \frac{f(z)}{z-\bar{a}} \right\} \qquad (1)$$

Cauchy's Integral Formula $\Rightarrow$ only the singularity inside C counts so that

$$I = \frac{1}{2\pi i}\oint_C dz\, \frac{f(z)}{z-a} = f(a) \qquad (2)$$

Suppose now that $f(z)$ is a function like e^{iz} which vanishes along (as $z \to \infty$) the semicircle: At any point along the semi-circle

$$e^{iz} = e^{i(x+iy)} = e^{ix} e^{-y} \xrightarrow{y \to \infty} 0 \text{ in upper half-plane (uhp)}$$

Then

$$I = \oint_C dz \ldots = \int_{-\infty}^{\infty} dx + \underbrace{\int_{\text{semi-circle}} dz}_{\to 0} \longrightarrow \int_{-\infty}^{\infty} dx \ldots \qquad (3)$$

Hence

$$f(a) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx \left\{ \frac{f(x)}{x-a} + \frac{f(x)}{x-\bar{a}} \right\} \qquad (4)$$

Side Comment: Note the technique that we used:

1) First we evaluate I by using Cauchy to get $\oint$ for the whole contour. This gives $I = f(a)$

2) Next we break up $\oint$ into 2 pieces, one going $\to 0$. This allows us to evaluate real integrals via complex integration.

Returning to Eq. (4) write $a = \alpha + i\beta \qquad \bar{a} = \alpha - i\beta$

$$\frac{f(x)}{x-a} + \frac{f(x)}{x-\bar{a}} = f(x)\left\{\frac{(x-\bar{a})+(x-a)}{(x-a)(x-\bar{a})}\right\} = f(x)\left\{\frac{2x-(\bar{a}+a)}{(x-a)(x-\bar{a})}\right\} \qquad (5)$$

$$a + \bar{a} = 2\alpha$$

$$(x-a)(x-\bar{a}) = [(x-\alpha)+i\beta][(x-\alpha)-i\beta] = (x-\alpha)^2+\beta^2 \qquad (6)$$

$$\therefore \; f(x)\{\ldots\} = f(x)\cdot\frac{2(x-\alpha)}{(x-\alpha)^2+\beta^2} \qquad (7)$$

$\to u + iv$

Hence altogether: $$I = f(a) = \frac{1}{\pi i}\int_{-\infty}^{\infty} dx\, f(x)\, \frac{(x-\alpha)}{(x-\alpha)^2+\beta^2} \qquad (8)$$

$\to u(\alpha,\beta) + iV(\alpha,\beta)$

$$u(\alpha,\beta) + iV(\alpha,\beta) = \frac{1}{\pi i}\int_{-\infty}^{\infty} dx\, [u(x,y=0) + i\,V(x,y=0)]\cdot\frac{(x-\alpha)}{(x-\alpha)^2+\beta^2} \qquad (9)$$

Equating real and imaginary parts,

$$u(\alpha,\beta) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{(x-\alpha)V(x)}{(x-\alpha)^2+\beta^2}$$

$$V(\alpha,\beta) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{(x-\alpha)u(x)}{(x-\alpha)^2+\beta^2} \qquad (10)$$

HILBERT TRANSFORM PAIR

$u(\alpha,\beta)$ & $V(\alpha,\beta)$

When $\beta = 0$: $$u(\alpha) = u(\alpha,0) = P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{V(x)}{x-\alpha}$$

$$V(\alpha) = V(\alpha,0) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{u(x)}{x-\alpha} \qquad (11)$$

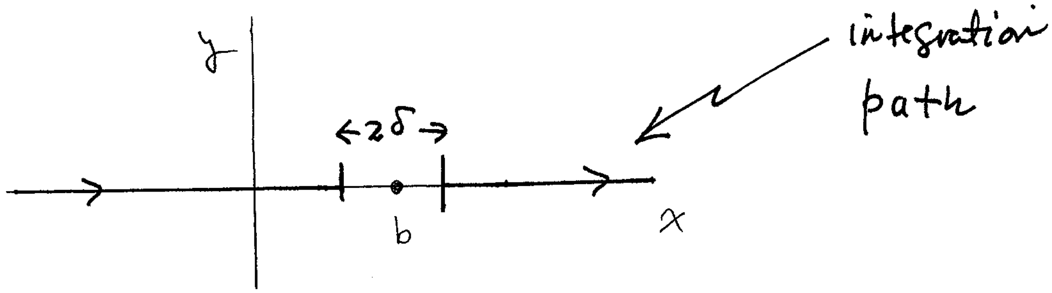
$P \equiv$ Principal Value Integration

PRINCIPAL VALUE INTEGRATION

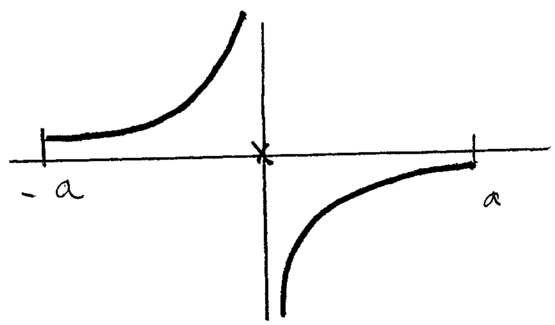
This is a formal technique for making sense of integrals such as those on p. CV-48.2 where there is a singularity (pole) along the path of integration.

$$P\int_a^c dx f(x) \equiv \lim_{\delta\to 0}\left[\int_a^{b-\delta} dx\, f(x) + \int_{b+\delta}^{c} dx\, f(x)\right] \qquad (1)$$

Pictorially:



Examples:



(a)

Consider $I = \int_{-a}^{a} dx \frac{1}{x}$ } not well defined since ∃ a singularity at $x=0$.

However, by symmetry we expect to find $I=0$. Doing a Principal Value integration we find:

$$I = P\int_{-a}^{a}\frac{dx}{x} = \lim_{\delta\to 0}\left[\int_{-a}^{0-\delta}\frac{dx}{x} + \int_{0+\delta}^{a}\frac{dx}{x}\right] \qquad (2)$$

$$= \lim_{\delta\to 0}\left\{\ln(-\delta) - \ln(-a) + \ln(a) - \ln(\delta)\right\} = \lim_{\delta\to 0} \ln\left(\frac{-\delta a}{-a\delta}\right) = \ln(1) = 0 \checkmark \qquad (3)$$

(b) For $0 < b < a$ Consider

CV-49,50

$$I = P\int_{-a}^{a} dx \, \frac{f(x)}{x-b}$$ ← this is often encountered in QM

To evaluate: $$I = P\left[\int_{-a}^{a} dx \, \frac{f(b)}{x-b} + \int_{-a}^{a} dx \, \frac{f(x)-f(b)}{x-b}\right] \qquad (4)$$

$$= f(b)\, P\int_{-a}^{a} dx \, \frac{1}{x-b} + \int_{-a}^{a} dx \, \frac{f(x)-f(b)}{x-b} \qquad (5)$$

P not needed here since $\int$ is well-behaved at $x=b$

$$\boxed{I = f(b) \ln\left(\frac{b-a}{b+a}\right) + \int_{-a}^{a} dx \, \frac{f(x)-f(b)}{x-b}} \qquad 6)$$

known

well behaved

(c) DIRAC'S FORMULA

Symbolically:
$$\boxed{\lim_{\epsilon\to 0}\left(\frac{1}{\omega \pm i\epsilon}\right) = P\left(\frac{1}{\omega}\right) \mp i\pi\delta(\omega)} \qquad (1)$$

- As with every formula involving $\delta(\omega)$, this is understood as holding under an integral sign.
- This formula is very widely used in QM!

Proof!
$$\frac{1}{\omega \pm i\epsilon} = \frac{1}{\omega \pm i\epsilon}\,\frac{\omega \mp i\epsilon}{\omega \mp i\epsilon} = \frac{\omega}{\omega^2+\epsilon^2} \mp \frac{i\epsilon}{\omega^2+\epsilon^2} \qquad (2)$$

At the beginning of the semester we established the following representation for $\delta(x)$:
$$\lim_{\epsilon\to 0}\left(\frac{1}{\pi}\,\frac{\epsilon}{x^2+\epsilon^2}\right) = \delta(x) \qquad (3)$$

Consider then Eq.(2) appearing (as it should!) in an integral:
$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega\,\frac{f(\omega)}{\omega \pm i\epsilon} = \underbrace{\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega\,\frac{\omega f(\omega)}{\omega^2+\epsilon^2}}_{I} \mp \lim_{\epsilon\to 0} i\int_{-\infty}^{\infty} d\omega\, f(\omega)\,\frac{\epsilon}{\omega^2+\epsilon^2} \qquad (4)$$

$$= I \mp i\pi\int_{-\infty}^{\infty} d\omega\, f(\omega)\,\delta(\omega) = I \mp i\pi f(0) \qquad (5)$$

Next evaluate
$$I = \lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\delta} d\omega\,\frac{f(\omega)\,\omega}{\omega^2+\epsilon^2} + \int_{\delta}^{\infty} d\omega \ldots + \int_{-\delta}^{\delta} d\omega \ldots\right\} \qquad (6)$$

The reason for introducing δ is to make the integrals well-behaved when the limit $\epsilon\to 0$ is taken. NOTE: (6) is an identity.

In Eq. (6) we take the limit as $\delta \to 0$ after first taking the limit $\epsilon \to 0$ in the first two terms:

$$I = \underbrace{\lim_{\delta \to 0} \left[\int_{-\infty}^{-\delta} d\omega \frac{f(\omega)}{\omega} + \int_{\delta}^{\infty} d\omega \frac{f(\omega)}{\omega} \right]}_{\equiv P \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega}} + \underbrace{\lim_{\substack{\epsilon \to 0 \\ \delta \to 0}} \int_{-\delta}^{\delta} d\omega \frac{\omega f(\omega)}{\omega^2 + \epsilon^2}}_{f(0) \underbrace{\int_{-\delta}^{\delta} d\omega \frac{\omega}{\omega^2+\epsilon^2}}_{\equiv 0}} \qquad (7)$$

(8)

$\equiv 0$ (odd function over a symmetric interval)

Hence $$I = P \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega} \qquad (9)$$

Combining Eqs. (5), (6), and (9) then gives:

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\omega\, f(\omega) \left\{ \frac{1}{\omega \pm i\epsilon} \right\} = P \int_{-\infty}^{\infty} d\omega\, f(\omega) \left\{ \frac{1}{\omega} \right\} \mp i\pi \int_{-\infty}^{\infty} d\omega\, f(\omega) \left\{ \delta(\omega) \right\} \qquad (10)$$

or symbolically: $$\boxed{\lim_{\epsilon \to 0} \left\{ \frac{1}{\omega \pm i\epsilon} \right\} = P \left\{ \frac{1}{\omega} \right\} \mp i\pi\, \delta(\omega)} \qquad (11)$$

<u>Related Identities:</u> $$\frac{d}{dx} \log x = \frac{1}{x} - i\pi\, \delta(x) \qquad (12)$$

$$= \lim_{\epsilon \to 0} \frac{1}{x + i\epsilon} \qquad (13)$$

To justify these formulas consider

$$\int_{-\epsilon}^{\epsilon} dx \left\{ \frac{d}{dx} \log x \right\} \stackrel{?}{=} \underbrace{\int_{-\epsilon}^{\epsilon} dx \left(\frac{1}{x} \right)}_{\equiv 0} \qquad (14)$$

$\equiv 0$ by symmetry (as before!)

However, the l.h.s. of (14) gives

$$\int_{-\epsilon}^{\epsilon} dx \left\{ \frac{d}{dx} \log x \right\} = \log x \Big|_{-\epsilon}^{\epsilon} = \log\left(\frac{\epsilon}{-\epsilon}\right) = \log(-1) = -i\pi \quad \checkmark \quad (15)$$

Hence to obtain a correct identity we should write – as in (12) –

$$\frac{d}{dx} \log x = \frac{1}{x} - i\pi\,\delta(x) \qquad (16)$$

Since $\displaystyle\int_{-\epsilon}^{\epsilon} dx\,(-i\pi\,\delta(x)) = -i\pi$ ✓ (17)

<u>Another Identity:</u> Start with $\displaystyle \delta(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$ (18)

$$\therefore \boxed{\delta(x) = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\left(\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon}\right)} \qquad (19)$$

From the previous results we have yet another identity

$$\lim_{\epsilon\to 0}\left(\frac{1}{\omega+i\epsilon}\right) + \lim_{\epsilon\to 0}\left(\frac{1}{\omega-i\epsilon}\right) = \left[P\left(\frac{1}{\omega}\right) - i\pi\,\delta(\omega)\right] + \left[P\left(\frac{1}{\omega}\right) + i\pi\,\delta(\omega)\right] \qquad (20)$$

$$\therefore \quad \boxed{P\left(\frac{1}{\omega}\right) = \lim_{\epsilon\to 0}\left[\frac{1}{\omega+i\epsilon} + \frac{1}{\omega-i\epsilon}\right]} \qquad (21)$$

Return to Hilbert Transform Pairs:

From p. CV-48.2 Eq.(1):

$$u(\alpha) = P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\ \frac{V(x)}{x-\alpha} \quad ; \quad V(\alpha) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\ \frac{u(x)}{x-\alpha} \qquad (1)$$

$$= \underbrace{P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\ \frac{V(\alpha)}{x-\alpha}}_{I_1} + \underbrace{P\frac{1}{\pi}\int_{-\infty}^{\infty} dx\ \frac{V(x)-V(\alpha)}{x-\alpha}}_{I_2} \qquad (2)$$

$$I_1 = \frac{V(\alpha)}{\pi} P\int_{-\infty}^{\infty} \frac{dx}{x-\alpha} = \frac{V(\alpha)}{\pi} \lim_{\delta\to 0}\left\{\int_{-\infty}^{\alpha-\delta} dx \cdots + \int_{\alpha+\delta}^{\infty} dx \cdots\right\} \qquad (3)$$

$$\{\ldots\} = \log(x-\alpha)\Big|_{-\infty}^{\alpha-\delta} + \ln(x-\alpha)\Big|_{\alpha+\delta}^{\infty} = \lim_{L\to\infty}\left\{\begin{array}{l}\log(\cancel{\alpha}-\delta-\cancel{\alpha}) - \log(-L-\alpha) \\ +\log(L-\alpha) - \log(\cancel{\alpha}+\delta-\cancel{\alpha})\end{array}\right\} \qquad (4)$$

$$= \lim_{L\to\infty}\left\{\log(-\delta) - \log(-L) + \log(L) - \log(\delta)\right\} \qquad (5)$$

$$\{\ldots\} = \log(-\delta L) - \log(-L\delta) = \log\left(\frac{-\delta L}{-L\delta}\right) = \log(\delta/\delta) = \log(1) = 0 \qquad (6)$$

Hence $I_1 \equiv 0$.

In I_2 the integral is well behaved so that P can be dropped. This allows (1) to be rewritten as:

$$\boxed{u(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx\ \frac{V(x)-V(\alpha)}{x-\alpha} \quad ; \quad V(\alpha) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx\ \frac{u(x)-u(\alpha)}{x-\alpha}} \qquad (7)$$

Application of Hilbert Transform Pairs:

Consider the function $f(z) = i e^{iz} \equiv u(x,y) + i V(x,y)$

$$= e^{-y}(-\sin x + i\cos x) \qquad (1)$$

$\therefore u(x,0) \equiv u(x) = -\sin x \quad ; \quad V(x,0) \equiv V(x) = \cos x$

Since these are the real and imaginary parts of $f(z)$ they form a Hilbert transform pair. Formally, $f(z) \to 0$ as $y \to \infty$ so that the integral we previously considered in the derivation of the Hilbert transform pair vanishes along the semi-circle in the u.h.p. Then we can Write:

$$V(\alpha) = \cos\alpha = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx \, \frac{u(x) - u(\alpha)}{x-\alpha} = \frac{-1}{\pi}\int_{-\infty}^{\infty} dx \left[\frac{-\sin x + \sin\alpha}{x-\alpha}\right] \qquad (2)$$

Taking $\alpha = 0$ then gives:

$$\cos(0) = 1 = +\frac{1}{\pi}\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} \Rightarrow \boxed{\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} = \pi} \qquad (3)$$

This demonstrates how Hilbert transform pairs can be used to evaluate real integrals. We will return to derive (3) by the more conventional techniques of contour integration. These techniques show the power of <u>using complex variables to evaluate real integrals</u> in an elegant way!

DISPERSION RELATIONS

A dispersion relation (as we use it) is an integral relation between two observable quantities where the integration is restricted to values of the argument that are <u>physically</u> <u>meaningful</u>.

Consider the Hilbert transform pair in Eq. (11) p. CV-48.2

$$u(\alpha) = P\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{v(x)}{x-\alpha} \; ; \quad v(\alpha) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} dx \; \frac{u(x)}{x-\alpha} \qquad (1)$$

Let us rename variables to make a connection with real problems:

$$u(\omega) = P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \; \frac{v(\omega')}{\omega'-\omega} \; ; \quad v(\omega) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \; \frac{u(\omega')}{\omega'-\omega} \qquad (2)$$

Now if ω and ω' are actually frequencies then negative frequencies are not meaningful, so that (2) is not really a dispersion relation.

However, u, v may be the real and imaginary parts of a function $f(z)$ which is the FOURIER TRANSFORM of a <u>real</u> ~~XXXXXX~~ <u>function</u> $G(t)$:

$$f(z) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dt \; G(t)\, e^{izt} \qquad (3)$$

Then $$f^*(z) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dt \; \underbrace{G^*(t)}_{G(t)}\, e^{-iz^*t} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} dt \; G(t)\, e^{-iz^*t} = f(-z^*) \qquad (4)$$

If z is real ($z=\omega$) then

$$\boxed{f^*(\omega) = f(-\omega)} \qquad (5)$$

$$[u(\omega)+iv(\omega)]^* = u(-\omega)+iv(-\omega)$$

$$\hookrightarrow u(\omega)-iv(\omega) = u(-\omega)+iv(-\omega)$$

$\Rightarrow$ REALITY CONDITIONS

$$\boxed{\begin{aligned} u(\omega) &= u(-\omega) \Rightarrow \text{EVEN} \\ v(-\omega) &= -v(\omega) \Rightarrow \text{ODD} \end{aligned}} \qquad (6)$$

To use the REALITY CONDITIONS, return to Eq.(2) on the previous page:

$$u(\omega) = P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \, \frac{v(\omega')}{\omega'-\omega} = P\frac{1}{\pi}\int_{-\infty}^{0} d\omega' \, \frac{v(\omega')}{\omega'-\omega} + P\frac{1}{\pi}\int_{0}^{\infty} d\omega' \, \frac{v(\omega')}{\omega'-\omega} \qquad (7)$$

In the first $\int$ let $\omega' \to -\omega'$:

$$P\frac{1}{\pi}\int_{-\infty}^{0}\cdots \longrightarrow P\frac{1}{\pi}\int_{+\infty}^{0}(-d\omega')\,\frac{\overset{\longrightarrow\, -v(\omega)}{v(-\omega')}}{-\omega'-\omega} = P\frac{1}{\pi}\int_{0}^{\infty} d\omega' \, \frac{v(\omega')}{\omega'+\omega} \qquad (8)$$

Combining (7) & (8): $$u(\omega) = P\frac{1}{\pi}\int_{0}^{\infty} d\omega' \, v(\omega')\left\{\underbrace{\frac{1}{\omega'+\omega} + \frac{1}{\omega'-\omega}}_{\frac{2\omega'}{\omega'^2-\omega^2}}\right\} \qquad (9)$$

Hence finally:

$$\boxed{u(\omega) = \frac{2}{\pi}P\int_{0}^{\infty} d\omega' \, \frac{\omega' v(\omega')}{\omega'^2-\omega^2}} \qquad (10)$$

Note that this integral is a true dispersion relation: It expresses the real function $u(\omega)$ as an integral over the imaginary part $v(\omega')$, but restricted to physical frequencies $v(\omega')$ where $0 \le \omega' \le \infty$.

In a similar manner we have from (2):

$$v(\omega) = -P\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \, \frac{u(\omega')}{\omega'-\omega} \longrightarrow \cdots\int_{-\infty}^{0}\cdots + \int_{0}\cdots \qquad (11)$$

$$\int_{-\infty}^{0} d\omega' \, \frac{u(\omega')}{\omega'-\omega} = \int_{\infty}^{0}(-d\omega')\,\frac{u(-\omega')}{-\omega'-\omega} \xrightarrow{(u\to +u)} -\int_{0}^{\infty} d\omega' \, \frac{v(\omega')}{\omega'+\omega} \qquad (12)$$

$$\therefore\ v(\omega) = \frac{-P}{\pi}\int_{0}^{\infty} d\omega' \, u(\omega')\left\{\frac{-1}{\omega'+\omega} + \frac{1}{\omega'-\omega}\right\} \qquad (13)$$

$$\therefore\ \boxed{v(\omega) = -\frac{2\omega}{\pi}P\int_{0}^{\infty} d\omega' \, \frac{u(\omega')}{\omega'^2-\omega^2}} \qquad (14)$$

EQS. (10) & (14) ARE THE KRAMERS-KRONIG DISPERSION RELATIONS