Details: $f(z)$ is analytic (at z_0) if the following limit exists: | CV-8/9/10

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} \qquad (1)$$

"Exists" $\Rightarrow$ SAME limit however $z \to z_0$

Notation: Analytic = differentiable = regular = holomorphic

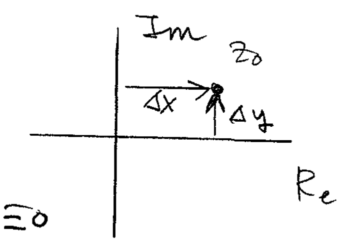
Examples: Start with $f'(z_0)=\lim_{\Delta z\to 0}\dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ (2)

(a) Consider $f(z)=z^2 \Rightarrow f'(z_0)=\dfrac{(z_0+\Delta z)^2-z_0^2}{\Delta z}\cong\dfrac{\cancel{z_0^2}+2z_0\Delta z-\cancel{z_0^2}}{\Delta z}$ (3)

$$\boxed{\therefore f'(z_0)=\lim_{\Delta z\to 0}\frac{2z_0\cancel{\Delta z}}{\cancel{\Delta z}}=2z_0 \text{ (independent of } \Delta z!)} \qquad (4)$$

(b) Next consider $f(z)=\bar{z} \Rightarrow f'(z_0)=\lim_{\Delta z\to 0}\dfrac{\overline{(z_0+\Delta z)}-\bar{z}_0}{\Delta z}$ (5)

$$\therefore f'(z_0)=\lim_{\Delta z\to 0}\left(\frac{\overline{\Delta z}}{\Delta z}=\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}\right) \qquad (6)$$



If the limit is taken in the x-direction then $\Delta y\equiv 0$ and

$$f'(z_0)=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1 \leftarrow \qquad (7a)$$

However, if the limot is taken in the y-direction then $\Delta x=0$ and

$$f'(z_0)=\lim_{\Delta y\to 0}\frac{-i\Delta y}{i\Delta y}=-1 \leftarrow \qquad (7b)$$

Since the limit depends on the path, $f(z)=\bar{z}$ is not analytic.

Derivation of Cauchy-Riemann Conditions:

The preceding examples of path-independence (or not!) lead to the formal proof of the C-R conditions:

(a) First assume that $W(z)$ is analytic; Then we show neccesity of C-R:

$$W'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta W}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\Delta u + i\Delta v}{\Delta z} \right) = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z} \right) \quad (1)$$

Multiply numerators & denominators by $\frac{\Delta x - i\Delta y}{\Delta x - i\Delta y}$

$$W'(z_0) = \lim_{\Delta z \to 0} \left[\frac{\Delta u(\Delta x - i\Delta y)}{(\Delta x)^2 + (\Delta y)^2} + i\frac{\Delta v(\Delta x - i\Delta y)}{(\Delta x)^2 + (\Delta y)^2} \right] \quad (2)$$

Collecting real & imaginary terms gives

$$W'(z_0) = \lim_{\Delta z \to 0} \left[\frac{\Delta u \Delta x + \Delta v \Delta y}{(\Delta x)^2 + (\Delta y)^2} + i\frac{\Delta v \Delta x - \Delta u \Delta y}{(\Delta x)^2 + (\Delta y)^2} \right] \quad (3)$$

Since $W(z)$ is assumed to be analytic we must obtain the same derivative independent of how $\Delta z \to 0$ is taken. Take $\underline{\Delta y = 0}$ initially; Then

$$W'(z_0) = \lim_{\Delta x \to 0} \left[\frac{\Delta u \Delta x}{(\Delta x)^2} + i\frac{\Delta v \Delta x}{(\Delta x)^2} \right] = \lim_{\Delta x \to 0} \underbrace{\left[\frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x} \right]} \quad \text{when } \Delta y \equiv 0 \quad (4)$$

Next take $\underline{\Delta x = 0}$ so that $\Delta z = i\Delta y \Rightarrow$

$$W'(z_0) = \lim_{i\Delta y \to 0} \left[\frac{\Delta v \Delta y}{(\Delta y)^2} - i\frac{\Delta u \Delta y}{(\Delta y)^2} \right] = \lim_{i\Delta y \to 0} \underbrace{\left[\frac{\Delta v}{\Delta y} - i\frac{\Delta u}{\Delta y} \right]} \quad \text{when } \Delta x \equiv 0 \quad (5)$$

Since $W(z)$ is analytic the expressions ~~~ in (4),(5) must be equal. Equating real and imaginary parts, and going to the limit gives:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \quad \text{C-R CONDITIONS} \tag{6}$$

Hence if $W(z)$ is analytic then the C-R conditions hold.

(b) Next we prove the converse: If the C-R conditions hold then $W(z)$ is analytic: (Sufficiency of C-R) [$f(z)$ is assumed continuous]

$$f(z) = u(x,y) + iv(x,y) \Rightarrow \Delta f = f(z_0+\Delta z) - f(z_0) = \Delta u + i\Delta v \tag{7}$$

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \text{, where } \epsilon_{1,2} \to 0 \text{ as } \Delta x, \Delta y \to 0$$

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y \text{, " } \epsilon_{3,4} \to 0 \ldots \tag{8}$$

$$\therefore \Delta f = \Delta u + i\Delta v = \frac{\partial u}{\partial x}\Delta x + \overbrace{\frac{\partial u}{\partial y}}^{-\partial v/\partial x}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \quad \leftarrow \text{C-R} \tag{9}$$

$$+ i\left(\frac{\partial v}{\partial x}\Delta x + \underbrace{\frac{\partial v}{\partial y}}_{\partial u/\partial x}\Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y\right) \quad \leftarrow \text{C-R} \tag{10}$$

Hence:
$$\Delta f = \frac{\partial u}{\partial x}\Delta x + \left(-\frac{\partial v}{\partial x}\Delta y\right) + i\left[\frac{\partial v}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y\right] + \text{terms} \to 0 \tag{11}$$

$$= \frac{\partial u}{\partial x}\underbrace{(\Delta x + i\Delta y)}_{\Delta z} + i\frac{\partial v}{\partial x}\underbrace{(\Delta x + i\Delta y)}_{\Delta z} \tag{12}$$

Dividing by $\Delta z \Rightarrow$
$$\boxed{\frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{INDEPENDENT OF } \Delta z} \tag{13}$$

This establishes that the C-R conditions are sufficient to ensure the analyticity of $f(z)$: the fact that the derivative is independent of path.

Examples: ① $f(z) = e^z = e^x e^{iy} = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$ (14)

$$\frac{\partial u}{\partial x} = e^x \cdot \cos y \overset{?}{=} \frac{\partial v}{\partial y} = e^x \cdot \cos y \checkmark \quad (15)$$

$$\frac{\partial v}{\partial x} = e^x \cdot \sin y \overset{?}{=} -\frac{\partial u}{\partial y} = -e^x(-\sin y) \checkmark \quad (16)$$

Note that for $f(z) = e^z$ the C-R conditions hold everywhere as an identity: Such a function is said to be "entire".

Since $f(z) = e^z$ is analytic everywhere its derivative can be computed along any path:

(a) $f(z) = e^x \cos y + i e^x \sin y$ (17)

$$\Delta z = \Delta x \Rightarrow \frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y)$$

$$= e^x e^{iy} = e^{x+iy} = e^z \checkmark \quad (18)$$

(b) $f(z) = e^x \cos y + i e^x \sin y$

$$\Delta z = i\Delta y \Rightarrow \frac{df}{dz} = \frac{\partial u}{i\partial y} + i\frac{\partial v}{i\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = e^x(\cos y) - i e^x(-)\sin y)$$

$$= e^x(\cos y + i \sin y) = e^z \checkmark \quad (19)$$

(c) $f(z) = e^z$ $\quad \frac{df}{dz} = e^z \checkmark$ (20)

↳ any path $\Delta z \to 0$

Note that Eqs. (18), (19), (20) give the same result!

Examples (continued)

Consider next $f(z) = |z|^2 = z\bar{z} = (x+iy)(x-iy) = x^2+y^2$ (21)

$= u(x,y) + i\,v(x,y) \Rightarrow u(x,y) = x^2+y^2 \,;\, v(x,y) \equiv 0$

$$\frac{\partial u}{\partial x} = 2x \Leftrightarrow \frac{\partial v}{\partial y} = 0 \quad ; \quad \frac{\partial v}{\partial x} = 0 = -\frac{\partial u}{\partial y} = -2y \tag{22}$$

Hence the C-R conditions hold only at the origin $(x=y=0)$; [We would not call a function analytic if C-R hold only at 1 point.]

Note for later: Since $z = x+iy$ and $\bar{z} = x-iy$ we have

$$\boxed{x = \frac{1}{2}(z+\bar{z}) \quad ; \quad y = \frac{1}{2i}(z-\bar{z})} \tag{23}$$

Hence any function $f = u(x,y) + i\,v(x,y) \to f(z,\bar{z})$. We will later show that any function $f = f(x,y)$ which depends on $\bar{z}$ (in addition to z) when use is made of (23) is not analytic. $f(z) = |z|^2 = z\bar{z}$ is an example.

General Rules on Analytic Functions

a) a constant is analytic

b) z^n is analytic

c) the sum, or product of 2 analytic functions is analytic

d) the quotient of 2 analytic functions is analytic, provided that the denominator $\neq 0$

c) an analytic function of an analytic function is analytic (CHAIN RULE);

Example:

$f(z) = z^2 \qquad g(z) = e^z \quad \Rightarrow \quad g(f(z)) = e^{z^2} = \text{analytic}$

Side Comment: Consider $f = u + iv \overset{C-R}{\Rightarrow} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$

Compare to $\underset{f'}{\underset{\parallel}{if}} = iu + i(iv) = \underbrace{-v}_{u'} + \underbrace{iu}_{v'}$

for f' C-R $\Rightarrow \frac{\partial u'}{\partial x} = \frac{\partial v'}{\partial y} \; ; \; \frac{\partial v'}{\partial x} = -\frac{\partial u'}{\partial y}$

$$\underset{\displaystyle -\frac{\partial v}{\partial x} \overset{?}{=} \frac{\partial u}{\partial y} \checkmark}{\underset{\parallel}{}} \; ; \; \frac{\partial u}{\partial x} = -\frac{(-\partial v)}{\partial y} = \frac{\partial v}{\partial y} \checkmark$$

Hence if f is analytic the function $i \otimes f$ is also analytic since the factor of i interchanges u and v with the right phases.

Connection To Physics: Harmonic Functions

$$CR \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (1)$$

$$\Downarrow$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ; \quad \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

Hence
$$\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} \equiv 0 \qquad (3)$$

$$\boxed{\nabla^2 u = 0 \qquad \text{also } \nabla^2 v = 0} \qquad (4)$$

$u(x,y)$ and $v(x,y)$ are harmonic functions. If $f(z) = u + iv$ is analytic then $u(x,y)$ and $v(x,y)$ are harmonic, and are called Conjugate harmonic functions. Given $u(x,y)$ or $v(x,y)$ we can find the other one using the C-R conditions:

Ex! (a) Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic
(b) find $v(x,y)$ its harmonic conjugate

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2 - 3x^2 + 3y^2 \quad ; \quad \frac{\partial^2 u}{\partial x^2} = -6x \\ \frac{\partial u}{\partial y} &= 6xy \quad ; \quad \frac{\partial^2 u}{\partial y^2} = +6x \end{aligned}\right\} \Rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0 \checkmark \qquad (5)$$

$$\Rightarrow u(x,y) \text{ is harmonic}$$

To find $v(x,y)$: $\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \overset{C-R}{=} \frac{\partial v}{\partial y} \Rightarrow v(x,y) = 2y - 3x^2y + y^3 + \varphi(x)$

To fix $\varphi(x)$ use the other C-R relation: $\partial v/\partial x = -\partial u/\partial y$; $\partial v/\partial x = -6xy + \varphi'(x)$ (6)

But $\partial v/\partial x = -\partial u/\partial y = -6xy \Rightarrow \varphi'(x) = 0 \Rightarrow \varphi(x) = \text{const}$

Hence $$\boxed{v(x,y) = 2y - 3x^2y + y^3 + \text{const}} \qquad (7)$$

We can use this result to illustrate an important theorem: $\boxed{\text{If } W(z) = u(x,y) + i\,v(x,y) \text{ is analytic iff } \frac{dW}{d\bar{z}} \equiv 0}$

Note: When we use the notation $f(z)$ or $W(z)$ for an function of a complex variable, our notation is a bit sloppy: As already noted, any function $f = u(x,y) + i\,v(x,y)$ can be expressed in terms of z AND $\bar{z}$ using Eq. (23) p.13:

$$\boxed{x = \frac{1}{2}(z+\bar{z}) \;;\; y = \frac{1}{2i}(z-\bar{z})} \qquad (8)$$

When we write $f(z)$ we are not necessarily saying that f does not also depend on $\bar{z}$. However, what the theorem says is that if f (or W) is analytic, then in fact it does not depend on $\bar{z}$, but only on z.

Returning to the previous example we have

$$f = u(x,y) + i\,v(x,y) = [2x - x^3 + 3xy^2] + i\,[2y - 3x^2y + y^3 + \text{const}] \qquad (9)$$

Substituting for x & y using (8) above we find

$$\boxed{f(x,y) \rightarrow f(z,\bar{z}) = 2z - z^3 + C} \qquad (10)$$

Hence, even though f could have depended on $\bar{z}$ as well as on z, in fact it only depends on z. This is what the theorem tells us! We know that f must be analytic because $u(x,y)$ and $v(x,y)$ are harmonic conjugates of each other. This theorem then says that when f is analytic then $f = f(z)$ only.

Proof!
$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} \qquad (11)$$

$$(8) \Rightarrow \frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \;;\; \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$$