FOURIER SERIES:

Fourier Series expansions can also be obtained from a generalized (2-dim) version of Weierstrass' theorem:

$$\boxed{f(x,y) = \sum_{n,m=0}^{\infty} C_{nm}\, x^n y^m} \quad (1)$$

~~Similarly~~ Changing to polar coordinates with $r=1$,

$$x^n = (r\cos\theta)^n \to \cos^n\theta \qquad (2)$$
$$y^m = (r\sin\theta)^m \to \sin^m\theta$$

$$\therefore\ f(x,y) \Rightarrow f(\theta) = \sum_{n,m=0}^{\infty} C_{nm} \cos^n\theta \sin^m\theta \qquad (3)$$

$$\cos^n\theta = \left[\tfrac{1}{2}(e^{i\theta}+e^{-i\theta})\right]^n \quad ; \quad \sin^m\theta = \left[\tfrac{1}{2i}(e^{i\theta}-e^{-i\theta})\right]^m \qquad (4)$$

Eqs. (3) & (4) $\Rightarrow$ that we can also expand $f(\theta)$ in terms of $e^{\pm in\theta}$

$$f(\theta) = \sum_{n=-\infty}^{\infty} A_n' e^{in\theta} \quad (5)$$ note from (4) that negative powers of n will appear

To connect to the usual Fourier expansions:

$$\text{Eq. (5)} \Rightarrow f(\theta) = \sum_{n=-\infty}^{\infty} A_n' (\cos n\theta + i\sin n\theta)$$

$$= A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + i\sum_{n=1}^{\infty} A_n' \sin n\theta \qquad (6)$$

$$+ \sum_{n=-\infty}^{-1} A_n' \cos n\theta + i\sum_{n=-\infty}^{-1} A_n' \sin n\theta$$

→ in this line rename $\boxed{m = -n}$

$$(6) \Rightarrow \sum_{-\infty}^{-1} A_n' \cos n\theta + i \sum_{-\infty}^{-1} A_n' \sin n\theta \tag{7}$$

$$= \sum_{m=1}^{\infty} A_{-m}' \underbrace{\cos(-m\theta)}_{\cos(m\theta)} + i \sum_{m=1}^{\infty} A_{-m}' \underbrace{\sin(-m\theta)}_{-\sin(m\theta)}$$

Since m is just a dummy summation variable rename $m \to n$:

$$(7) = \sum_{n=1}^{\infty} A_{-n}' \cos n\theta + \sum_{n=1}^{\infty} (-iA_{-n}') \sin n\theta \tag{8}$$

Hence: $$f(\theta) = \underbrace{A_0'}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \underbrace{(A_n' + A_{-n}')}_{a_n} \cos n\theta + \sum_{n=1}^{\infty} \underbrace{i(A_n' - A_{-n}')}_{b_n} \sin n\theta \tag{9}$$

Finally: $$\boxed{f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \quad (10)}$$ FOURIER EXPANSION

We note that since $\cos(n\pi) = \cos(-n\pi)$ and $\sin(n\pi) = 0$

Eq.(10) holds only for functions for which $f(\pi) = f(-\pi)$.

Summary to this Point:

(a) We have shown previously that $\left\{ f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \right\}$ are orthonormal in $[-\pi, \pi]$

(b) Now we have shown that $\frac{1}{\sqrt{2\pi}} e^{inx}$ are complete, via Weierstrass theorem.

If we combine these statements $\Rightarrow$ $\frac{1}{\sqrt{2\pi}}$ e^{inx} form a CON set on $[-\pi, \pi]$ in the sense of uniform convergence.

Determining the Fourier Coefficients

For any vector expansion

$$|f\rangle = \sum_n |f_n\rangle \underbrace{\langle f_n|}_{I} \overbrace{|f\rangle}^{c_n} \Leftrightarrow f(x) = \sum_n c_n f_n(x) \qquad (1)$$

$$\therefore c_n = \langle f_n|f\rangle = \int_{-\pi}^{\pi} dx\, f_n^*(x)\, f(x) \qquad f_n = \left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\} \qquad (2)$$

Since the f_n are complete $\Rightarrow$ Bessel's equality: $\sum_n |c_n|^2 = \sum_n |\langle f_n|f\rangle|^2 = |f|^2$ (3)

From (1) & (2): $$f(x) = \sum_{n=-\infty}^{\infty} A_n \frac{e^{inx}}{\sqrt{2\pi}} \equiv \sum_{n=-\infty}^{\infty} \bar{A}_n e^{inx} \qquad (4)$$

Then: $$\boxed{A_n = \langle f_n|f\rangle = \int_{-\pi}^{\pi} dx \left\{\frac{1}{\sqrt{2\pi}} e^{-inx}\right\} f(x)} \qquad (5)$$

If we wish to carry out the expansion in terms of $\cos(nx)$ & $\sin(nx)$

Then from the previous results:

$$\boxed{a_0 = 2\,(A_0/\sqrt{2\pi}) = \sqrt{\frac{2}{\pi}} \int_{-\pi}^{\pi} dx \cdot \left(\frac{1}{\sqrt{2\pi}} f(x)\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x)} \qquad (6)$$

$$a_n = \frac{1}{\sqrt{2\pi}}(A_n + A_{-n}) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\pi}^{\pi} dx \left\{e^{-inx} + e^{inx}\right\} f(x) \qquad (7)$$

$$\boxed{\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, \cos(nx)\, f(x)} \qquad (8)$$

$$b_n = \frac{i}{\sqrt{2\pi}}(A_n - A_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} dx\, \underbrace{(e^{+inx} + e^{-inx})}_{-2i\sin(nx)} f(x) \qquad (9)$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, \sin(nx)\, f(x)} \qquad (10)$$

COMPLEX VARIABLES

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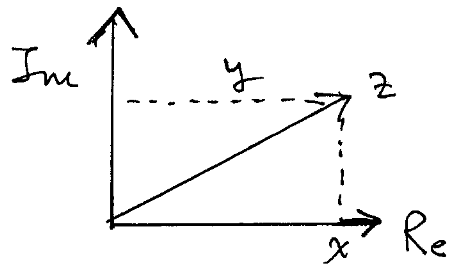
Let $i = \sqrt{-1}$; $z = x + iy$ (x, y are real) (1)

Pictorially:



The theory of complex numbers can be developed by viewing them as 2-dim vectors. From (1) & figure we develop the following simple rules:

addition : $z_1 = a + ib$ $\quad z_2 = c + id \Big\} \Rightarrow (z_1 + z_2) = (a+c) + i(b+d)$ (2)

multiplication : $z_1 z_2 = (a+ib)(c+id) = ac + i^2 bd + i(bc + ad)$

$$\boxed{\text{New Vector} = z_1 z_2 = \underbrace{(ac - bd)}_{x\text{-component}} + i \underbrace{(bc + ad)}_{y\text{-component}}} \quad (3)$$

Recall ; $i^2 = -1$; $i^3 = -i$; $i^4 = +1$ (4)

In a practical sense the presence of $i = \sqrt{-1}$ merely serves to define a prescription for multiplication, which can be summarized compactly as

$$\boxed{(a,b)(c,d) = (ac - bd,\ bc + ad)} \quad (5)$$

The theory of complex numbers can be developed using (5) directly instead of (3).

division: $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2}\left(\dfrac{x_2 - iy_2}{x_2 - iy_2}\right) = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\dfrac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$ (6)

Complex Conjugation : $\boxed{z = x + iy \Rightarrow z^* = \bar{z} \equiv x - iy}$ (7)

More generally: $i \to -i$

Rules for Complex Conjugation

1) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$
2) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$
3) $z+\bar{z} = 2x = 2\,\mathrm{Re}\, z$
4) $z-\bar{z} = 2iy = 2i\,\mathrm{Im}\, z$

Proofs are trivial: For example: $\overline{z_1 z_2} = (ac-bd) - i\,(bc+ad)$

Compare to $\bar{z}_1 \bar{z}_2 = (a-ib)(c-id) = (ac-bd) - i\,(bc+ad)$

Absolute Value = Modulus = Magnitude of a Complex Number :

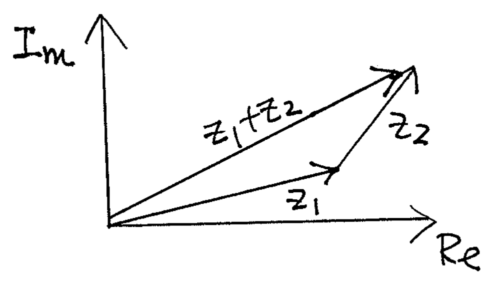
$$|z| \equiv |x+iy| \equiv \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

Evidently: (1) $|z|^2 = x^2+y^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$

(3) $|z| \geq \mathrm{Re}(z)$; $|z| \geq \mathrm{Im}(z)$

(3) $|z_1 z_2| = |z_1||z_2|$

(4) $|z_1+z_2| \leq |z_1| + |z_2|$ [triangle inequality]



Complex Numbers in Polar Coordinates:

Depending on the problem polar coordinates may be more useful than Cartesian coordinates:

$$x \to r\cos\theta \; ; \; y \to r\sin\theta \; ; \; z = x+iy \to r(\cos\theta + i\sin\theta)$$

$$= r\, e^{i\theta}$$

$$r \to \sqrt{x^2+y^2}$$

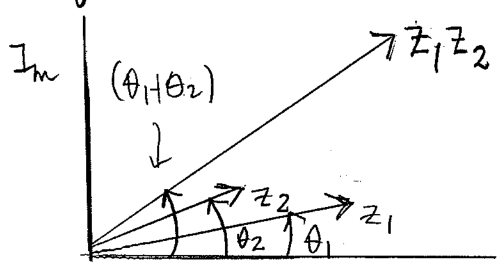
Hence: $\boxed{r = \sqrt{x^2+y^2} = |z| \; ; \; \theta = \tan^{-1}(y/x) \equiv \arg z}$

<u>Key Problem with Polar Coordinates:</u> θ is not unique
$\Rightarrow$ branch cuts, ... (more later!!)

Notes: ① $r = e^{i\theta} = e^{i(\theta+2\pi)} = \ldots = e^{i(\theta+2n\pi)}$ $\quad n = 0, 1, 2, \ldots$

② <u>multiplication:</u> $z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$ (1)

<u>Physical Interpretation:</u> Multiplication of one complex number by another changes the length of the number being multiplied and also rotates it:



$\boxed{\text{De Moivre's Theorem:}}$ $\quad z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$ (2)

$\therefore \boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$ (3)

n can be an integer or any rational number here

<u>Application:</u> Find the nth root(s) of a given complex number z:
(Find z_0 such that $z^{1/n} = z_0$) $\Rightarrow (z^{1/n})^n = z_0^n \Rightarrow z_0^n = z$

Write $z_0 = r_0 e^{in\theta_0}$; $z = re^{i\theta} \Rightarrow r_0^n e^{in\theta_0} = re^{i\theta}$ (4)

<u>Principle:</u> When equating two complex numbers in Cartesian space write

$$\left.\begin{array}{l} z_1 = x_1 + iy_1 \\ z_2 = x_2 + iy_2 \end{array}\right\} \Rightarrow \text{If } z_1 = z_2 \text{ then: } \begin{array}{l} x_1 = x_2 \\ y_1 = y_2 \end{array}$$

In polar coordinates: $\left.\begin{array}{l} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{array}\right\}$ If $z_1 = z_2$ then $r_1 = r_2$ (5)

but $\theta_1 = \theta_2 \pm 2n\pi$

$\hookrightarrow$ leads to multivalued functions

Application: $z^{1/n} = z_0 \Rightarrow r e^{i\theta} = r_0^n e^{in\theta_0}$

$\therefore \; r = r_0^n \Rightarrow \boxed{r_0 = r^{1/n}}$ (6) $\qquad \theta = n\theta_0 \pm \underbrace{2\pi k}_{e^{i2\pi k}=1}$ ← integer

$$\therefore \boxed{\theta_0 = \frac{\theta}{n} \pm \frac{2\pi k}{n}} \quad (7)$$

Hence the full solution z_0 is given by $z_0 = r_0 e^{i\theta_0} \Rightarrow$

$$z_0 = r^{1/n} e^{i\left(\frac{\theta}{n} \pm 2\pi\frac{k}{n}\right)} \quad (8)$$

$$\boxed{z_0 = r^{1/n} e^{i\theta/n} e^{2\pi i\left(\frac{n-k}{n}\right)}} \quad (9)$$

Q: Since k is an arbitrary integer how many distinct roots do we find

A: n roots

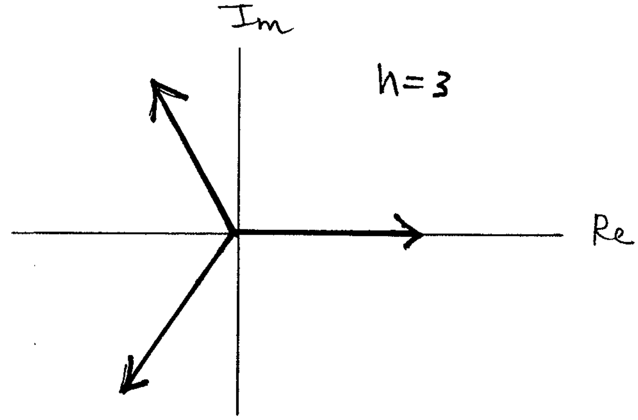
Example: $n = 3$: $e^{2\pi i\left(\frac{n-k}{n}\right)} = \underbrace{e^{2\pi i}}_{1} (k=0)$; $e^{2\pi i \cdot \frac{2}{3}} (k=1)$; $e^{2\pi i\left(\frac{1}{3}\right)} (k=2)$

$\underbrace{e^{2\pi i \cdot 0}}_{1} (k=3)$; $e^{2\pi i\left(-\frac{1}{3}\right)} (k=4) = \underbrace{e^{2\pi i} e^{-\frac{2\pi i}{3}}}_{e^{4\pi i/3}} (k=4)$

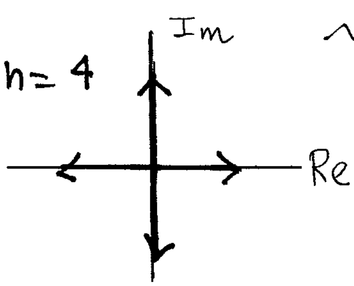
Hence after a while the roots repeat leaving only 3 independent roots:

$1, \quad e^{i\frac{4}{3}\pi}, \quad e^{i\frac{2}{3}\pi}$

$0^\circ \qquad 240^\circ \qquad 120^\circ$



$r = 1 \Rightarrow$ "nth roots of unity"



Analytic Functions: Cauchy-Riemann Conditions

Any function $w = f(z)$ can be written in the form

$$W(z) = u(x,y) + i\,v(x,y)$$

Ex: $$W = f(z) = z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + \underbrace{2ixy}_{i\,v(x,y)} \quad (1)$$

It is critical to identify those functions which have derivatives. Such functions are said to be analytic

$$\boxed{\text{analytic} \longleftrightarrow \text{differentiable (a unique derivative exists)}} \quad (2)$$

Some functions may be analytic everywhere in the complex plane except at isolated points ("poles") or lines ("branch cuts") ....

Consider ① $$w = f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y) \quad (3)$$

$$= \underbrace{e^x\cos y}_{u(x,y)} + i\underbrace{e^x\sin y}_{v(x,y)} \quad (4)$$

$\leftarrow$ this function is analytic everywhere

② $$W = f(z) = \bar{z} = x - iy \;;\quad u(x,y) = x \quad v(x,y) = -y \quad (5)$$

This function is not analytic

Cauchy-Riemann Conditions:

We will show shortly that there is a simple test for analyticity: $W(z) = u(x,y) + i\,v(x,y)$ is analytic iff

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;;\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \quad (6)$$