COMPLETENESS OF $\{f_n(x)\}$ & CONVERGENCE:

- For the ∞-dim case this is important for the same reasons as in the finite dim case
- Completeness has physical consequences:

  Example: To obtain a complete set of solutions to the DIRAC EQUATION one had to include negative frequency solutions. This lead to the discovery of ANTIMATTER (positrons)

- As noted above there are 2 ways to specify $f(x)$ in $[a,b]$
  - a) Specify $f(x_i)$ $a \le x_i \le b$ at an ∞ (c_0) number of points
  - b) Choose a basis in $[a,b]$ and expand $f(x) = \sum_{n=0}^{\infty} c_n f_n(x)$.

    This requires a denumerable ∞ $(\aleph_0)$ of points. This is sufficient for the continuous functions we are studying.

- In the ∞-dim case the completeness of $\{f_n(x)\}$ — because they have finite norms — ensures that any function formed from them will also have a finite norm i.e. it will belong to the same Hilbert space:

$$f(x) = \sum_n^{\infty} f_n(x) \Rightarrow \langle f|f\rangle < \infty \qquad (1)$$

This is important since there are many examples where a sequence of terms having some property converges to an expression which does not:

Ex: $$(1-x)^{1/2} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \dots \qquad (2)$$

For $x = 1/2$ l.h.s $= \sqrt{1/2} = \sqrt{2}/2 =$ irrational, but each term on the r.h.s is rational

UNDERSTANDING CONVERGENCE

For the finite dim case there is no ambiguity in what it means to add up ~~xxx~~ a sum of terms. However, in the infinite dim case one must deal with convergence, and the different types of convergence.

Q: What does it mean for a sequence of terms to converge to $f(x)$?

Consider: $$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots \qquad (1)$$

Define: $$f_n(x) \equiv \sum_{i=0}^{n} a_i x^i \qquad (2)$$

We expect that if the series in (1) converges then the difference between successive f_n's should get smaller:

$$\lim_{n\to\infty} |f_{n+1}(x) - f_n(x)| \to 0 \qquad (3)$$

Definition: $\{f_n(x)\}$ is called a SEQUENCE

Uniform Convergence: In $[a,b]$ $f_n(x)$ converges uniformly to $f(x)$ if for every x in $a \le x \le b$ and for every $\epsilon > 0$ $\exists$ $N(\epsilon)$ such that

$$\boxed{|f(x) - f_n(x)| < \epsilon \quad \text{for } n > N(\epsilon)} \qquad (4)$$

This can be replaced by the CAUCHY CRITERION which does not require knowing $f(x)$ in advance:

$$\boxed{|f_r(x) - f_s(x)| < \epsilon \quad \text{for } r,s > N} \qquad (5)$$

CAUCHY CRITERION FOR UNIFORM CONVERGENCE

(A) COMPLETENESS OF A SET OF FUNCTIONS – UNIFORM CONVERGENCE

Let $g(x)$ = piecewise continuous on $[a,b]$

$\{f_i(x)\} \stackrel{?}{=}$ "basis" set of functions on $[a,b]$. They will be a basis if they are complete.

Define: $g_n(x) = \sum_{i=1}^{n} a_i f_i(x)$

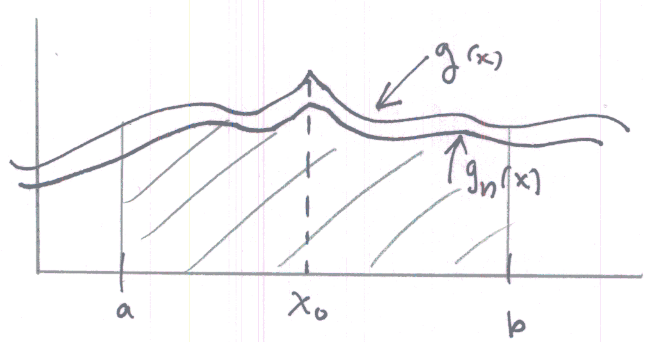
$\{f_i(x)\}$ will be complete in the sense of uniform convergence if

$$\boxed{|g(x) - g_n(x)| = \left|g(x) - \sum_{i=1}^{n} a_i f_i(x)\right| < \epsilon \quad \text{for } n > N(\epsilon)} \quad (1)$$

For a given set of $\{f_i(x)\}$ this comes down to asking whether a_i can be found satisfying (1).

(B) COMPLETENESS – CONVERGENCE IN THE MEAN

Uniform convergence may be too strong a criterion for some purposes: Suppose we are evaluating an integral over $[a,b]$; then it would not really be necessary for $g_n(x)$ to converge to $g(x)$ at every x:



It may be that the "smooth" approximation to $g(x)$ by $g_n(x)$ is sufficient for evaluating the area under the curve: This leads to the idea of convergence in the mean:

Definition:

A sequence of functions $h_n(x)$ converges in the mean ("on average") to $h(x)$ if

$$\lim_{n\to\infty} \int_a^b dx \, |h(x) - h_n(x)|^2 \to 0 \qquad (1)$$

or

$$\lim_{n\to\infty} \int_a^b dx \, \left|h(x) - \sum_{i=1}^{n} k_i(x)\right|^2 \to 0 \qquad (2)$$

COMPLETENESS IN THE SENSE OF CONVERGENCE IN THE MEAN:

Define $g_n(x) = \sum_{i=1}^{n} a_i f_i(x)$ (3)

$\{f_i(x)\}$ are complete in the sense of convergence in the mean if

$$\boxed{\lim_{n\to\infty} \int_a^b dx \, \left|g(x) - \sum_{i=1}^{n} a_i f_i(x)\right|^2 \to 0} \qquad (4)$$

NOTES:
- When talking of completeness, which form must be specified
- Uniform convergence $\Rightarrow$ convergence in the mean, but not vice versa.

Thus one must be more specific when writing

$$g(x) = \lim_{n\to\infty} \sum_{m=1}^{n} a_m f_m(x) = \sum_{m=1}^{\infty} a_m f_m(x) \qquad (5)$$

to specify which form of convergence is meant by the limiting process

F18/1/85

To Show that Uniform Convergence $\Rightarrow$ Convergence in the Mean:

Assume uniform convergence $\Rightarrow$ $\exists$ an integer N such that for $\forall x$ in $[a,b]$ and all $\epsilon > 0$ then

$$|h(x) - h_n(x)| < \epsilon \quad \text{for } n > N(\epsilon). \qquad (1)$$

Then consider $\displaystyle\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon^2 \int_a^b dx = \epsilon^2 (b-a)$ (2)

Hence uniform convergence $\Rightarrow$ $\displaystyle\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon^2 (b-a)$ for $n > N(\epsilon)$. (3)

What we now want to show is that $\displaystyle\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon'$, where ϵ' is some pre-assigned number. [for $n > N$]

Since ϵ' can be made arbitrarily small, this is what we mean by $\displaystyle\lim_{n\to\infty} \int_a^b |h(x) - h_n(x)|^2 = 0$. To achieve what we want choose

$$\epsilon^2 (b-a) = \epsilon' \Leftrightarrow \epsilon = \sqrt{\epsilon'/(b-a)}$$

Then if $|h(x) - h_n(x)| < \epsilon = \sqrt{\epsilon'/(b-a)}$ $\Rightarrow$

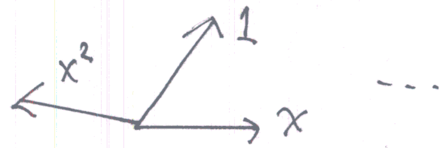
$$\int_a^b dx\, |h(x) - h_n(x)|^2 < \epsilon^2 (b-a) = \left[\frac{\epsilon'}{b-a}\right] \cdot (b-a) = \epsilon' \checkmark$$

<u>Corollary:</u> Completeness in the sense of uniform convergence $\Rightarrow$ Completeness in the sense of convergence in the mean

THE WEIERSTRASS APPROXIMATION THEOREM

Theorem: The polynomials $1, x, x^2, \ldots$ form a <u>complete set</u> in any closed interval $a \leq x \leq b$, in the (strong) sense of uniform convergence. [not proved in class]

<u>Schematically:</u>



However, these do not in general form a CON set: For example consider the basis $\{f_n(x)\} = \{x^n\}$ $n = 0, 1, \ldots$. Then

$$\langle x^2 | x^4 \rangle_{-1}^{1} \equiv \int_{-1}^{1} dx\, (x^2)^* x^4 = \tfrac{1}{7} x^7 \Big|_{-1}^{1} = \tfrac{2}{7} \neq 0 \leftarrow$$

Also $$\langle x^2 | x^2 \rangle = \int_{-1}^{1} dx\, (x^2)^* x^2 = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5} \neq 1 \leftarrow$$

<u>Solution:</u> Use <u>GRAM-SCHMIDT</u> METHOD to form CON Sets

$1, x, x^2, x^3, \ldots$

GRAM-SCHMIDT

YOUR CHOICE!

$[-1,1]$	$[-\infty,\infty]$	$[0,\infty]$	$[-\pi,\pi]$	$\ldots$
Legendre	Hermite	Laguerre	Fourier	
$P_0(x) = 1$	$H_0(x) = 1$	$L_0 = 1$	$\sin(nx), \cos(nx)$	
$P_1(x) = x$	$H_1(x) = 2x$	$L_1 = 1 - x$		
$P_2(x) = \frac{1}{2}(3x^2 - 1)$	$H_2(x) = -2 + 4x^2$	$L_2 = 2 - 4x + x^2$		
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$	$H_3(x) = -12x + 8x^3$	$L_3 = 6 - 18x + 9x^2 - x^3$		
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$	$H_4(x) = 12 - 48x^2 + 16x^4$			
	$\vdots$			

p.177 ← [Merzbacher, QM, p.56] → p.195 & Thornton/Estle/Lane QM p.55

Legendre Polynomials:

These form a CON set in $[-1,1]$. Note the GRAM–SCHMIDT method produces Legendre polynomials $\equiv \bar{P}_n(x)$ which are normalized as

$$\int_{-1}^{1} dx\, \bar{P}_n(x)\,\bar{P}_m(x) = \delta_{mn} \qquad (1)$$

The usual "textbook" functions are normalized as: $\int_{-1}^{1} dx\, P_n(x) P_m(x) = \left(\frac{2}{2n+1}\right)\delta_{mn}$ (2)

We return below to discuss these normalization issues. In using G–S we have:

$$\boxed{\bar{P}_0(x) = \text{Const}\cdot 1 = \frac{1}{\sqrt{2}}\cdot 1} \Rightarrow \int_{-1}^{1} dx\, \bar{P}_0(x)\,\bar{P}_0(x) = \frac{1}{2}\int_{-1}^{1} dx = 1 \checkmark$$

Next consider: $\bar{P}_1(x) = \dfrac{x - \langle \bar{P}_0(x)|x\rangle \bar{P}_0(x)}{|x - \langle \bar{P}_0(x)|x\rangle \bar{P}_0(x)|}$ $\Leftrightarrow$ $\overset{\text{Recall}}{y_2 = \dfrac{x_2 - \langle y_1|x_2\rangle y_1}{|x_2 - \langle y_1|x_2\rangle y_1|}}$ (3)

$$\langle \bar{P}_0(x)|x\rangle = \int_{-1}^{1} dx\, \left(\tfrac{1}{\sqrt{2}}\right)\cdot x = \frac{1}{\sqrt{2}}\,\frac{1}{2}x^2\Big|_{-1}^{1} = 0 \qquad (3a)$$

Hence: $\bar{P}_1(x) = \dfrac{x}{|x|} = \text{Const}\cdot x \;;\; \langle \bar{P}_1(x)|\bar{P}_1(x)\rangle = 1 \Rightarrow \text{Const} = \sqrt{\tfrac{3}{2}}$ (4)

Check $\langle \bar{P}_1(x)|\bar{P}_1(x)\rangle = \left(\sqrt{\tfrac{3}{2}}\right)^2 \int_{-1}^{1} dx\, x\cdot x = \frac{3}{2}\cdot\frac{x^3}{3}\Big|_{-1}^{1} = 1 \checkmark$ (5)

$$\therefore \boxed{\bar{P}_1(x) = \sqrt{\tfrac{3}{2}}\,x} \qquad (6)$$

This case was trivial since $\langle \bar{P}_0(x)|x\rangle = 0$ in Eq. (3a).

Next: $\bar{P}_2(x) = \dfrac{x^2 - \langle \bar{P}_0(x)|x^2\rangle \bar{P}_0(x) - \langle \bar{P}_1(x)|x^2\rangle \bar{P}_1(x)}{|x^2 - \ldots \quad \ldots |^2}$ (6)

$$\bar{P}_2(x) = \frac{x^2 - \left(\frac{1}{\sqrt{2}}\int_{-1}^{1} x'^2 dx'\right)\cdot\frac{1}{\sqrt{2}} - \overbrace{\left(\sqrt{\frac{3}{2}}\int_{-1}^{1} x'^3 dx'\right)}^{\equiv 0}\cdot\sqrt{\frac{3}{2}}\,x}{\left|\qquad\qquad\qquad\right|}$$

$$= \frac{x^2 - \frac{1}{2}\int_{-1}^{1} dx'\, x'^2}{\left|\qquad\right|} = \frac{x^2 - \frac{1}{3}}{|x^2 - 1/3|} \;;\; |x^2 - 1/3|^2 = \int_{-1}^{1} dx'\,(x'^2 - 1/3)(x'^2 - \tfrac{1}{3}) \qquad (7)$$

$$|x^2 - 1/3|^2 = \int_{-1}^{1} dx' \left\{ x'^4 - \frac{2}{3}x'^2 + \frac{1}{9} \right\} = \frac{8}{45} \Rightarrow \boxed{\bar{P}_2(x) = \frac{x^2 - 1/3}{\sqrt{8/45}} = \sqrt{\frac{5}{8}}\,(3x^2 - 1)} \qquad (8)$$

Notes: [1] We see already that $P_n(x)$ contains only even powers of x if n = even, and odd powers of x if n = odd. This could have been deduced at the outset from a PARITY ARGUMENT, which we give later. Here we simply note that this arises from the fact that integrals of odd powers vanish in $[-1,1]$ as in (7).

[2] The expression in (8) can be rewritten as:

$$\bar{P}_2(x) = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \equiv \sqrt{\frac{5}{2}}\,\underbrace{P_2(x)}_{\text{"textbook" expression}} \qquad (9)$$

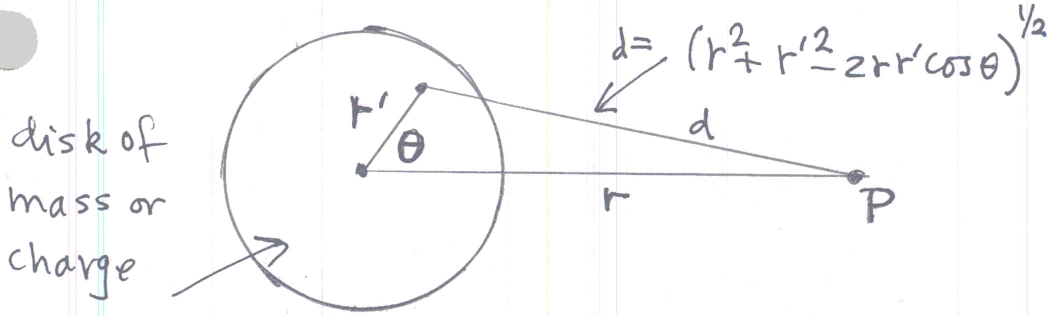
These two conventions are related via

$$\boxed{\bar{P}_n(x) = \sqrt{\frac{2n+1}{2}}\;P_n(x)} \qquad (10)$$

The "textbook" $P_n(x)$ are normalized differently for reasons we now discuss.

[3] Apart from an overall normalization, $\bar{P}_n(x)$ or $P_n(x)$ are unique in $[-1,1]$

Other Derivations of $P_n(x)$:



$$\frac{1}{d} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta) \qquad r' < r \qquad (1)$$

In (1) set $r = 1$, $r' \equiv t$ $\quad |t| < 1$, $\quad x = \cos\theta$; (1) $\Rightarrow$

$$(1 - 2tx + t^2)^{-1/2} = \sum_{\ell=0}^{\infty} t^{\ell} P_\ell(x) \qquad |t| < 1 \qquad (2a)$$

GENERATING FUNCTION FOR $P_\ell(x)$

or

$$(1 - 2tx + t^2)^{-1/2} = \sum_{\ell=0}^{\infty} t^{-\ell} P_\ell(x) \qquad |t| > 1 \qquad (2b)$$

To show that the generating function generates the usual $P_\ell(x)$:

$$(1 - 2tx + t^2)^{-1/2} = [1 - (2tx - t^2)]^{-1/2} \equiv [1 - z]^{-1/2}$$

First note that $|z| < 1$: For fixed x $\quad \frac{d}{dt} z(t,x) = 2x - 2t \Rightarrow t = x \qquad (3)$

Also $\frac{d^2}{dt^2} z(t,x) = -2 \Rightarrow$ $t = x$ is a maximum. At $t = x$ we have

$$z(t,x)_{x=t} = (2tx - t^2)_{x=t} = 2t^2 - t^2 = t^2 < 1 \text{ when } |t|^2 < 1 \qquad (4)$$

Hence altogether $|z| < 1 \Rightarrow$ $\boxed{(1-z)^{-1/2} = 1 + \frac{1}{2}z + \frac{3}{8}z^2 + \frac{15}{48}z^3 + \frac{35}{128}z^4 + \ldots}$ (5)

Then:

$$z^2 = 4t^2x^2 + t^4 - 4t^3x$$

$$z^3 = -t^6 + 10t^5x - 16t^4x^2 + 8t^3x^3 \qquad (6)$$

$$z^4 = t^8 - 12t^7x + 36t^6x^2 - 40t^5x^3 + 16t^4x^4$$

$\vdots$