Evidently, if $n > m$ then

$$D^n P_m = 0 \Rightarrow D^n = 0$$

Hence $\exists$ nonzero l.t.s which have the property that even though $D \neq 0$ $D^n = 0$. [We will later see that typically such operators do not have inverses.]

Linear Transformations obey usual algebraic relations:

a) $A0 = 0A = 0$

b) $AI = IA = A$

c) $A(B+C) = AB + AC$

d) $A(BC) = (AB)C$

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INVERSE OF A LINEAR TRANSFORMATION

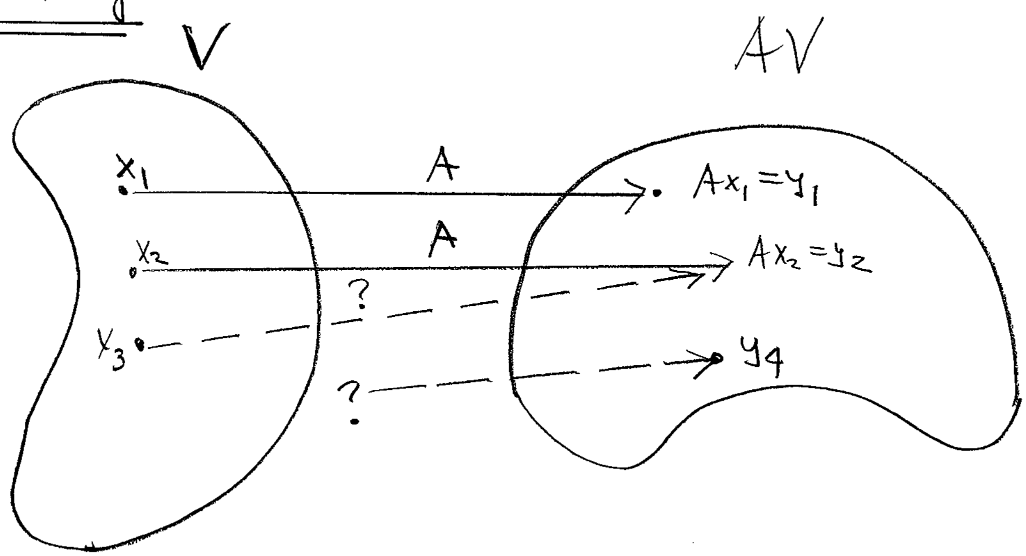
This subject is important because the solution of many algebraic problems requires finding the inverse, or at least knowing when an inverse exists.

A l.t. A has an inverse A^{-1} if

1) $x_1 \neq x_2 \Rightarrow Ax_1 \neq Ax_2$
   or
   $Ax_1 = Ax_2 \Rightarrow x_1 = x_2$ } "uniqueness"

2) For $\forall\, y \in V$ there exists at least one $x \in V \ni$ $Ax = y$. "parentage"

Pictorially



If the situations described by the - - - - lines hold then no inverse of A exists.

The picture explains intuitively why both "uniqueness" and "parentage" are necessary for A^{-1} to exist.

Definition of A^{-1}

If y_0 is any vector in AV then condition 2) $\Rightarrow$ $\exists$ an x_0 in V such that $y_0 = Ax_0$. x_0 is unique by condition 1). Then A^{-1} is defined by $\boxed{A^{-1}y_0 = x_0}$ (1)

If A^{-1} exists then $AA^{-1} = A^{-1}A = I$ (2)

Theorem: A, B, C are l.t.s, such that

$$AB = CA = I \tag{3}$$

then A^{-1} exists and $A^{-1} = B = C$

Proof: To prove that A^{-1} exists we have to show that conditions 1) & 2) on p. F106 hold.

1) if $Ax_1 = Ax_2$ then $CAx_1 = x_1$ and $CAx_2 = x_2$ (4)

$\therefore$ $Ax_1 = Ax_2 \Rightarrow CAx_1 = CAx_2 \Rightarrow x_1 = x_2$ ✓

2) Let y be any vector in AV, and define $x = By$.
Then $Ax = ABy = Iy = y$. This assigns to every y a vector Ax in V $\Rightarrow$ every y has a "parent" ✓

This establishes that A^{-1} exists. To find A^{-1},

$$AB = I \Rightarrow \underbrace{A^{-1}A}_{I}B = A^{-1}I \Rightarrow \boxed{A^{-1}I = A^{-1} = B} \tag{5}$$

$$CA = I \Rightarrow C\underbrace{AA^{-1}}_{I} = IA^{-1} = A \Rightarrow \boxed{C = IA^{-1} = A^{-1}} \quad \text{QED} \tag{6}$$

For a finite dimensional V either condition $AB = I$ <u>or</u> $CA = I$ is sufficient to prove that A^{-1} exists.

However, <u>for an infinite dimensional V both conditions are needed.</u>

<u>Examples:</u> Let V = infinite-dim vector space of polynomials $P(x)$

Then define
$$DP(x) = \frac{d}{dx}P(x)$$
$$SP(x) = \int_0^x P(t)\,dt$$

Even though $DSP(x) = \frac{d}{dx}\int_0^x P(t)\,dt = P(x)$ [fundamental thm of calculus]

$$\therefore DS = I$$

Nonetheless neither D nor S is invertible:

① D violates condition 1), since $D(x^2+3) = D(x^2+17)$ etc.

② S violates condition 2), since for $y = x^2+1$ there is no $x(t)$ such that $x^2+1 = \int_0^x x(t)\,dt$ [Hint: Try $x(t) = at+b$]

<u>FUNDAMENTAL THEOREM ON INVERSES A^{-1}:</u>

<u>Thm:</u> If $Ax = 0 \Rightarrow x = 0$, then a l.t. on a finite dimensional V is invertible. [Also if A^{-1} exists then $Ax = 0 \Rightarrow x = 0$]

<u>Proof:</u> If $Ax_1 = Ax_2 \Rightarrow Ax_1 - Ax_2 = 0 = A\underbrace{(x_1 - x_2)}_{x} = 0 \Rightarrow Ax = 0$

By assumption this implies $x = x_1 - x_2 = 0$ or $x_1 = x_2$

Hence defining $x = x_1 - x_2$ it follows that $Ax = 0 \Rightarrow x = 0$ satisfies condition 1) on p. F106

To prove condition 2) ["parentage"] from these assumptions let $\{x_1 \dots x_n\}$ be a finite basis in V. If it can be shown that $\{Ax_1 \dots Ax_n\}$ is also a basis then any y can be written as

$$y = \sum_i \alpha_i \underbrace{(Ax_i)}_{\text{in } AV} = A \sum_i \alpha_i \underbrace{x_i}_{\text{in } V} = Ax \quad = \text{Condition 2}$$

Q: is $\{Ax_1, \dots Ax_n\}$ a basis?

A: yes since there are n of them (n-dim space) – provided they are linearly independent. Form

$$\sum_i \alpha_i Ax_i = 0 \quad = A \underbrace{\sum_i \alpha_i x_i}_{x_i \text{ are l.i}} = 0 \Rightarrow \alpha_i = 0$$

$$\therefore \sum_i \alpha_i Ax_i = 0 \Rightarrow \alpha_i = 0 \Rightarrow Ax_i \text{ lin. indep.}$$

This completes the proof QED.

Theorem: If A & B have inverses then AB has an inverse and $(AB)^{-1} = B^{-1}A^{-1}$. Also $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and $(A^{-1})^{-1} = A$.

Proof: It is sufficient to prove that [see earlier theorem p. 107]

$$\left.\begin{aligned} (AB)\, B^{-1}A^{-1} &= A \underbrace{BB^{-1}}_{I} A^{-1} = AA^{-1} = I \\ B^{-1}A^{-1}(AB) &= B^{-1}A^{-1}AB = B^{-1}B = I \end{aligned}\right\} \text{trivial}$$

Rest is obvious.

MATRICES AS LINEAR TRANSFORMATIONS

By choosing an appropriate basis in a finite dim vector space, a linear transformation A can be expressed in terms of a matrix. (The same txfunction will have a different representation as a matrix w.r.to another basis)

Let $\{x_i\}$ be a basis for an n-dim V. Thus any vector Ax_j is some (other) vector in V and can be expressed in terms of $\{x_i\}$

$$Ax_j = \sum_i a_{ij}\, x_i \qquad (1)$$

For example: Let $j=7$: $Ax_7 = \sum_{i=1}^{N} a_{i7} x_i = a_{17} x_1 + a_{27} x_2 + \ldots a_{N7} x_N$ (2)

If we write the coefficients a_{ij} as a matrix, then

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{17} & a_{18} \\ a_{21} & & & & a_{27} & a_{28} \\ a_{31} & & & & a_{37} & a_{38} \\ \vdots & & & & \vdots & \vdots \end{pmatrix}$$

Hence the coefficients of x_i in expressing Ax_7 in terms of $\{x_i\}$ form the 7th column in this convention. So Ax_j for $j=7$ is another vector, represented by a column vector, as x_7 itself would be.

NOTE!! Different authors use different conventions!!
BE CONSISTENT!!!

Side Comment on $Ax=0$:

If $Ax=0 \Rightarrow x=0$ then A^{-1} exists. Why is this important?

(I) Eigenvalue Problem:

$$H\underset{\uparrow\ \text{eigenvalue}}{\lambda}\overset{\text{eigenvector}\ \downarrow}{x} \quad Hx = \lambda x = \lambda I x \qquad I = \text{unit matrix} \qquad (1)$$

Then $(1) \Rightarrow \quad (H-\lambda I)x = 0 \equiv Ax \begin{cases} \to x=0 \Leftrightarrow A^{-1} \text{ exists} \\ \to x\neq 0 \Leftrightarrow A^{-1} \text{ does not exist} \end{cases}$

For the eigenvalue problem $x=0$ is trivial, and is not the solution we want. Hence $x \neq 0 \Rightarrow A^{-1}$ does not exist. We show later that

$$A^{-1} = \frac{\text{Adj}\,A \leftarrow \text{a matrix}}{\det A \leftarrow \text{a number}} \qquad (3)$$

When A^{-1} does not exist it is because $\det A = 0$, which becomes the condition which leads to a solution for the eigenvalues λ:

$$\boxed{\underbrace{\det(H-\lambda I)=0}_{\text{characteristic equation}} = \text{polynomial in } \lambda}$$

(II) Linear Independence of the Solutions of a Differential Equation:

A 2ND order diff. eqn. has 2 lin. indep. solutions.

How do we know whether the solutions we have found are lin. indep.?

Consider more generally a set of functions $f_i(x)$: The condition for lin. indep. is

$$\boxed{\sum_{i=1}^{N} \alpha_i f_i(x) = 0 \Rightarrow \alpha_i \equiv 0 \quad \forall i}$$

Q: Given a set of functions $f_i(x)$ how do we know whether nonzero α_i can be found?

A: Rather than focus on α_i, focus on solutions

Then
$$\sum_i \alpha_i f_i(x) = 0 \quad \Rightarrow \qquad (1)$$
$$\sum_i \alpha_i f_i'(x) = 0 \qquad (2)$$
$$\sum_i \alpha_i f_i'' = 0 \qquad (3)$$
$$\vdots$$
$$\sum_i \alpha_i f_i^{(N-1)} = 0 \qquad (4)$$

Writing these explicitly:

$$f_1\alpha_1 + f_2\alpha_2 + \dots + f_N\alpha_N = 0 = m_{11}\alpha_1 + m_{12}\alpha_2 + \dots m_{1N}\alpha_N \qquad (5)$$
$$f_1'\alpha_1 + f_2'\alpha_2 + \dots + f_N'\alpha_N = 0 = m_{21}\alpha_1 + m_{22}\alpha_2 + \dots + m_{2N}\alpha_N \qquad (6)$$
$$\vdots \qquad\qquad \vdots$$
$$f_1^{(n-1)}\alpha_1 + f_2^{(n-1)}\alpha_2 + \dots + f_N^{(N-1)}\alpha_N = 0 = m_{N1}\alpha_1 + m_{N2}\alpha_2 + \dots + m_{NN}\alpha_N \qquad (7)$$

Define
$$M = \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \cdots & m_{NN} \end{pmatrix} \qquad \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} \qquad (8)$$

$$(5)-(7) \Rightarrow \quad \boxed{M\alpha = 0} \quad \Leftrightarrow \quad Ax = 0 \qquad (9)$$

From our discussion of A^{-1} we again note:

$$M\alpha = 0 \begin{cases} \rightarrow \alpha = 0 \Leftrightarrow M^{-1} \text{ exists} \\ \rightarrow \alpha \neq 0 \Leftrightarrow M^{-1} \text{ does not exist} \end{cases} \qquad (10)$$

The condition for lin. indep. of $f_i(x)$ is that $\alpha_i = 0 \Rightarrow \alpha = 0 \Rightarrow M^{-1}$ exists.

As before
$$M^{-1} = \text{Adj}\, M/\det M \quad \Rightarrow \text{lin. indep.} \Rightarrow \det M \neq 0 \qquad (11)$$

Define $W(x) = \det M(x) =$ WRONSKIAN

$$\boxed{\therefore\ W(x) \neq 0 \Rightarrow f_i(x) \text{ are linearly independent}} \qquad (12)$$

Applications:

[1] $y''(x) + w^2 y(x) = 0$ has 2 lin. indep. solutions

$$y_1(x) = \sin wx \qquad (13)$$
$$y_2(x) = \cos wx$$

To show that these are infact linearly independent:

$$M(x) = \begin{pmatrix} \sin \omega x & \cos \omega x \\ \omega \cos \omega x & -\omega \sin \omega x \end{pmatrix} \qquad (14)$$

$$W(x) = \det M(x) = -\omega \sin^2 wx - \omega \cos^2 wx = -\omega \neq 0 \checkmark$$

[2] For any 2 solutions $y_1(x)$ & $y_2(x)$

$$M(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \Rightarrow W(x) = y_1 y_2' - y_2 y_1' \qquad (15)$$

We show next semester that one can determine $W(x)$ without completely knowing the 2 solutions $y_1(x)$ & $y_2(x)$.

It can then be shown that

$$\boxed{\left.\begin{matrix} y_1(x) \\ W(x) \end{matrix}\right\} \Rightarrow y_2(x)} \qquad (16)$$

Hence, knowing $W(x)$ and one solution $y_1(x)$ we can find a second solution $y_2(x)$.

PROPERTIES OF MATRICES

[1] Matrices of same form can be added.
[2] Matrices can be multiplied by scalars } trivial

[3] Matrix multiplication:

Theorem: $A = (\alpha_{ij}) \quad B = (\beta_{ij}) \Rightarrow C = AB \equiv \gamma_{ij}$ where

basis vector

$$\boxed{\gamma_{ij} = \sum_k \alpha_{ik} \beta_{kj}}$$

Proof: $Cx_j = A(Bx_j) = A \sum_k \beta_{kj} x_k = \sum_k \beta_{kj} (Ax_k)$

$$= \sum_k \beta_{kj} \sum_i \alpha_{ik} x_i = \sum_i \left(\sum_k \alpha_{ik} \beta_{kj} \right) x_i$$

But by definition $Cx_j \equiv \sum_i \gamma_{ij} x_i$ $\Bigg\} \Rightarrow \sum_i \left(\sum_k \alpha_{ik} \beta_{kj} \right) x_i = \sum_i \gamma_{ij} x_i$

Hence $\sum_i \left(\gamma_{ij} - \sum_k \alpha_{ik} \beta_{kj} \right) x_i = 0$

Since $x_i \in \{x_i\} =$ basis, x_i are lin. indep. $\Rightarrow (\ldots) = 0$

Hence $\gamma_{ij} = \sum_k \alpha_{ik} \beta_{kj}$ Q.E.D

Comment: $Ax_j = \sum_i \alpha_{ij} x_i$ is an isomorphism

$$A \Leftrightarrow (\alpha_{ij})$$

SPECIAL MATRICES

[1] PAULI MATRICES:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Along with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for a basis for any 2x2 matrix M:

$$\boxed{M = aI + \vec{b}\cdot\vec{\sigma}} \quad a, \vec{b} = \text{complex (in general)}$$

a) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

b) $[\sigma_x, \sigma_y] = 2i\,\sigma_z$; More generally $[\sigma_a, \sigma_b] = 2i\,\epsilon_{abc}\,\sigma_c$

c) $\{\sigma_x, \sigma_y\} = \sigma_x\sigma_y + \sigma_y\sigma_x = 0$, etc.

d) $\sigma_i^{-1} = \sigma_i^{\dagger}$

[2] Diagonal Matrices : $D = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{33} \end{pmatrix}$

a) Any 2 nxn diag. matrices commute, and their product is also diagonal

b) The eigenvalues of the matrix are its eigenvalues.
[This is why we speak about "diagonalizing the Hamiltonian" in quantum mechanics]