Then $\vec{\nabla} g(r) = [\hat{i}\partial_x + \hat{j}\partial_y + \hat{k}\partial_z]\, g(r) = \frac{c}{2}[\hat{i}(2)(x-x') + \cdots] = c\vec{r}$

Compare this to $\vec{\nabla}' g(r) = [\hat{i}\partial_{x'} + \hat{j}\partial_{y'} + \hat{k}\partial_{z'}]\, g(r) = \frac{c}{2}[\hat{i}(2)(x-x')(-1) + \cdots] = -c\vec{r}$ (22)

here is where the sign is coming from

This establishes the validity of the trick in (20).

Using (20) we then return to the expression for $\vec{D}$ in (17) and replace $\vec{\nabla} \to -\vec{\nabla}'$. Since we do this twice, there is no sign change:

$$\vec{D} = \frac{1}{4\pi}\int d^3x' \left[\vec{C}(\vec{x}')\cdot\vec{\nabla}'\right]\left[\vec{\nabla}'\left(\frac{1}{r(\vec{x},\vec{x}')}\right)\right] \tag{23}$$

Consider one of the components of $\vec{D}$, $D_\alpha$ ($\alpha = 1, 2$, or $3$). We can integrate by parts using the identity

this acts on everything to the right

$$\int d^3x'\, \vec{\nabla}'\cdot\left\{\vec{C}(\vec{x}')\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right)\right\} = \int d^3x' \left(\vec{\nabla}'\cdot\vec{C}\right)\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) + \int d^3x' \left[\vec{C}\cdot\vec{\nabla}'\right]\frac{\partial}{\partial x'_\alpha}\left(\frac{1}{r}\right) \tag{24}$$

Comparing (23) & (24) we see that

$$4\pi\vec{D} = \underbrace{\int d^3x'\, \vec{\nabla}'\cdot\left\{\vec{C}(\vec{x}')\vec{\nabla}'\left(\frac{1}{r}\right)\right\}}_{A} - \underbrace{\int d^3x' \left[\vec{\nabla}'\cdot\vec{C}(\vec{x}')\right]\vec{\nabla}'\left(\frac{1}{r}\right)}_{B} \tag{25}$$

$\{\ \} \equiv \vec{F}(\vec{x}')$

Keep in mind that we are trying to show that $\vec{D} = 0$, so we begin by showing that (A) $= 0$. This is a general and very widely used argument! Write

$$(A) \equiv \int_V d^3x'\, \vec{\nabla}'\cdot\vec{F}(\vec{x}') \overset{\text{Gauss}}{=} \int_S d\vec{S}'\cdot\vec{F}(\vec{x}') \tag{26}$$

Here we make the standard argument that if $\vec{F}(\vec{x}\,')$ depends on a source function $\vec{c}(\vec{x}\,')$ which is localized in space. Then a Gaussian surface $S$ can be found (taking $S$ large enough!) so that no flux from $\vec{c}(\vec{x}\,')$ crosses $S$, and hence Ⓐ $\equiv 0$.

[A similar argument is often used for the 4-dimensional version of Gauss' theorem, but care must be used there, since sources are not always localized in time!!]

Since Ⓐ $\equiv 0$, the combination of Eqs. (11), (13), (14), & (25) give

$$\vec{\nabla}\times\vec{V}(\vec{x}) = \vec{c}(\vec{x}) - \frac{1}{4\pi}\int d^3x' \left[\vec{\nabla}'\cdot\vec{c}(\vec{x}\,')\right]\vec{\nabla}'\left(\frac{1}{r}\right) \qquad (27)$$

We are trying to show that $\vec{\nabla}\times\vec{V}(\vec{x}) = \vec{c}(\vec{x})$; this follows by noting that Eq. (27) would hold if we replace $\vec{c}(\vec{x}\,')$ by $\vec{\nabla}'\times\vec{V}(\vec{x}\,')$. This is a self-consistency argument; We conclude from (27) that in fact $\vec{\nabla}\times\vec{V}(x) = \vec{c}(\vec{x})$, where $\vec{V}(\vec{x})$ is given by (2).

//

Uniqueness of Solutions:

Question: Having shown that $\vec{\nabla}\cdot\vec{V} = S$ and $\vec{\nabla}\times\vec{V} = \vec{C}$, where $\vec{V}$ is given by (2), we now ask whether there can be 2 solutions $\vec{V}_1$ and $\vec{V}_2$ which work? Specifically can we find $\vec{V}_{1,2}$ such that

$$\boxed{\vec{\nabla}\cdot\vec{V}_{1,2}(\vec{x}) = S(\vec{x}) \quad \text{and} \quad \vec{\nabla}\times\vec{V}_{1,2}(\vec{x}) = \vec{c}(\vec{x})} \qquad (28)$$

Consider $\vec{W}(\vec{x}) = \vec{V}_1(\vec{x}) - \vec{V}_2(\vec{x})$. We want to show that $\vec{W}(\vec{x}) \equiv 0$.

From (28)

$$\vec{\nabla}\cdot\vec{W}(\vec{x}) = \vec{\nabla}\cdot[\vec{V}_1(\vec{x}) - \vec{V}_2(\vec{x})] = S(\vec{x}) - S(\vec{x}) = 0 \qquad (29)$$
$$\vec{\nabla}\times\vec{W}(\vec{x}) = \vec{\nabla}\times[\vec{V}_1(x) - \vec{V}_2(\vec{x})] = \vec{c}(\vec{x}) - \vec{c}(\vec{x}) = 0$$

Since $\vec{\nabla}\times\vec{W}=0$ it follows from 18.3 (16) that $\vec{W}$ can be expressed as

$$\vec{W}=-\vec{\nabla}\psi \quad \leftarrow \text{scalar field} \qquad (30)$$

Then $\vec{\nabla}\cdot\vec{W}=0 \Rightarrow$ $\boxed{\nabla^2\psi(x)=0 \text{ everywhere}}$ (31)

To proceed using (31) we apply Gauss' theorem to the vector $\psi\vec{\nabla}\psi$:

$$\int \psi\vec{\nabla}\psi\cdot d\vec{S} = \int \vec{\nabla}\cdot(\psi\vec{\nabla}\psi)\,d\mathcal{V} \equiv \int \partial_i(\psi\partial_i\psi)\,d\mathcal{V} \qquad (31)$$

$\leftarrow$ volume element

$$= \int[(\partial_i\psi)(\partial_i\psi)+\psi\partial_i\partial_i\psi]\,d\mathcal{V} = \int(\vec{\nabla}\psi)^2\,dV+\int\psi\nabla^2\psi\,dV$$

Hence $\boxed{\int \psi\vec{\nabla}\psi\cdot d\vec{S} = \int[(\vec{\nabla}\psi)^2+\underbrace{\psi\nabla^2\psi}_{=0\ (31)}]\,d\mathcal{V}}$ (32)

GREEN'S IDENTITY

We will return to show that if there are no sources at $\infty$ then the l.h.s. of (32) vanishes. Accepting this for the moment [See below ☆]

We then have from (32)

$$\int \underbrace{(\vec{\nabla}\psi)^2}_{\text{positive definite (non-negative)}}\,d\mathcal{V}=0 \quad \Rightarrow \quad \boxed{\vec{\nabla}\psi\equiv 0} \quad (33)$$

(34)

But from (30) $\vec{\nabla}\psi\equiv 0 \Rightarrow \vec{W}(x)=\vec{V}_1(x)-V_2(x)\equiv 0 \Rightarrow \boxed{\vec{V}_1(x)=\vec{V}_2(x)}$

In other words, the only way that (28) can hold is if $\vec{V}_1=\vec{V}_2$ so that in the end there is a unique solution.

☆ To complete the proof it remains to show that the l.h.s. of (32) $\to 0$.

Since $\psi(\vec{x})$ is a solution of $\nabla^2\psi(\vec{x})=0$ we expand $\psi(\vec{x})$ in the form:

$$\psi(\vec{x})\cong R(r)\,Y(\theta,\phi) \qquad (35)$$

Then $\nabla^2\psi(\vec{x})=0 \Rightarrow \frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=0 \Rightarrow$

$$R(r)=\sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-n-1}\right) \qquad (36)$$

$A_n$ and $B_n$ are constants chosen to satisfy the boundary conditions appropriate to a given problem. Since we are assuming that there are no sources at $\infty$ it follows that $A_n \equiv 0$ for all $n$. Since only the $B_n$ survive the leading term is $B_0/r$ so that [up to a constant]

$$\psi \sim B_0/r \Rightarrow \vec{\nabla}\psi = -B_0\frac{\hat{r}}{r^2}$$

Hence $\int \psi\vec{\nabla}\psi\cdot d\vec{S} = \int\left(\frac{B_0}{r}\right)\left(-\frac{B_0\hat{r}}{r^2}\right)\cdot d\vec{S} = -B_0^2\int\frac{1}{r}\underbrace{\left(\frac{\hat{r}\cdot d\vec{S}}{r^2}\right)}_{d\Omega} \qquad (37)$

$$\boxed{\therefore \int \psi\vec{\nabla}\psi\cdot d\vec{S} \sim -B_0^2\int\frac{1}{r}d\Omega = -\frac{4\pi}{r} \xrightarrow[r\to\infty]{} 0} \qquad (38)$$

Simply stated, since we assume on physical grounds that there are no sources at $\infty$, we can find a Gaussian surface for sufficiently large $r$ such that there is no flux of $\psi\vec{\nabla}\psi$ through $d\vec{S}$, and hence the l.h.s of (38) and (32) vanishes. This then completes the proof of uniqeness.

#

Side Comment: Returning to (29) and this proof of uniqeness we see that if we have a field $\vec{E}(\vec{x})$ for which

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{E}(\vec{x}) &= 0\\ \vec{\nabla}\times\vec{E}(\vec{x}) &= 0\end{aligned}\right\} \text{ at all points in space} \qquad (39)$$

and if there are no sources at $\infty$, then $\vec{E}(\vec{x}) \equiv 0 \qquad (40)$

# TENSOR ANALYSIS

## $g_{\mu\nu}$ & all that!!

# TENSORS

$ds$ = distance from $\vec{x}$ to $\vec{x}+d\vec{x}$

## 3 dimensions

$$ds^2 = dx^2 + dy^2 + dz^2 \equiv g_{ij}\, dx^i dx^j$$

a) CARTESIAN

$dx^1 = dx$
$dx^2 = dy$
$dx^3 = dz$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij} \quad ; \; i,j = 1,2,3$$

$$g \equiv \det g_{ij} = +1$$

b) SPHERICAL

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$
$$\equiv g_{ij}\, dx^i dx^j$$

$dx^1 = dr$
$dx^2 = d\theta$
$dx^3 = d\phi$

$$; \; g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$g_{rr} = 1$
$g_{\theta\theta} = r^2$
$g_{\phi\phi} = r^2 \sin^2\theta$

$$g \equiv \det g_{ij} = r^4 \sin^2\theta$$

## 3-dim volume element

$$\int d\text{Volume} = \int \sqrt{g}\, dx^1 dx^2 dx^3 \equiv \int \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3$$

$$= \int 1\, dx^1 dx^2 dx^3 = \int dx\, dy\, dz \qquad \text{CARTESIAN}$$

$$= \int (r^2 \sin\theta)\, dr\, d\theta\, d\phi \qquad \text{SPHERICAL}$$

$$= \int (r^2 dr)(\sin\theta\, d\theta)\, d\phi$$

## 3+1 dimensions (relativity)

time

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \quad (\text{Sum on } \mu, \nu = 1, 2, 3, 0)$$

**Minkowski** ("flat" space) $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

can't travel to past!!

**Coordinates**

$$dx^1 = dx \,;\; dx^2 = dy \,;\; dx^3 = dz \,;\; dx^0 = c\,dt$$

43.3

## TRANSFORMATION OF VECTORS & TENSORS

For an arbitrary transformation $x^\mu \to x'^\mu$

we define the following objects:

a) CONTRAVARIANT VECTOR $V'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$

example: $dx'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} dx^\nu$

b) COVARIANT VECTOR $U'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} U_\nu(x)$

example (the gradient)

$$\frac{\partial\phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{d\phi}{dx^\nu}$$

c) MIXED TENSOR

$$T^{\mu\lambda}{}_{\nu}(x) \sim V^{\mu}(x)\, W^{\lambda}(x)\, U_{\nu}(x)$$

3rd RANK TENSOR (3 indices)

Can always view a mixed tensor this way. This helps keep track of indices.

Example: $T'^{\mu\lambda}{}_{\nu}(x') = \left(\dfrac{\partial x'^{\mu}}{\partial x^{\alpha}}\right)\left(\dfrac{\partial x'^{\lambda}}{\partial x^{\beta}}\right)\left(\dfrac{\partial x^{\gamma}}{\partial x'^{\nu}}\right) T^{\alpha\beta}{}_{\gamma}(x)$

d) MOST IMPORTANT TENSOR = metric tensor

( defines the coordinate system we are in )

$$\boxed{g_{\mu\nu}(x) \equiv \left(\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\right)\left(\frac{\partial \xi^{\beta}}{\partial x^{\nu}}\right) \eta_{\alpha\beta}(\xi)}$$

flat space (Minkowski coods)

any other coordinates

To verify that $g_{\mu\nu}$ is a tensor note that:

$$g'_{\mu\nu}(x') \equiv \frac{\partial \xi^{\alpha}}{\partial x'^{\mu}} \frac{\partial \xi^{\beta}}{\partial x'^{\nu}} \eta_{\alpha\beta} = \left(\frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\left(\frac{\partial \xi^{\beta}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) \eta_{\alpha\beta}(\xi)$$

$$= \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) \underbrace{\left(\frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \frac{\partial \xi^{\beta}}{\partial x^{\sigma}} \eta_{\alpha\beta}(\xi)\right)}_{g_{\rho\sigma}(x)} = \left(\frac{\partial x^{\rho}}{\partial x'^{\mu}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) g_{\rho\sigma}(x)$$

✓