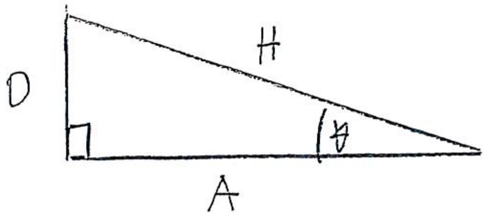
TRIG FUNCTIONS & THEIR INVERSES



Pythagoras: $A^2 + O^2 = H^2$ (1)

This can be realized by defining functions $\sin\theta$ & $\cos\theta$ so that:

$$O/H \equiv \sin\theta \Rightarrow O = H \cdot \sin\theta \; ; \quad A/H = \cos\theta \Rightarrow A = H \cdot \cos\theta \qquad (2)$$

Then Pythagoras $\Rightarrow$ $H^2 \sin^2\theta + H^2\cos^2\theta = H^2 \Rightarrow \boxed{\sin^2\theta + \cos^2\theta = 1}$ (3)

Having previously defined $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$ we can then define

$$\underline{\cos\theta \equiv \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)} \Rightarrow \cos^2\theta = \frac{1}{4}\left(\cancel{e^{2i\theta}} + \cancel{e^{-2i\theta}} + 2\right)$$

$$\underline{\sin\theta \equiv \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)} \Rightarrow \sin^2\theta = \frac{-1}{4}\left(\cancel{e^{2i\theta}} + \cancel{e^{-2i\theta}} - 2\right)$$

add these (4)

$$(4) \Rightarrow \sin^2\theta + \cos^2\theta = 1 \quad \checkmark$$

Note that this formula must hold for any system of units. However, when we define $\cos\theta$ & $\sin\theta$ in this way, θ is in radians. These can be checked by noting from (4) that

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \underbrace{e^{i\pi} = -1}_{\text{EULER}} = \underbrace{\cos(\pi)}_{-1} + i\,\underbrace{\sin(\pi)}_{0} \quad \checkmark \qquad (5)$$

$\leftarrow 180° \rightarrow$

The series expansions for $\cos\theta$ & $\sin\theta$ then follow from (4):

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} \qquad (6)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \qquad (7)$$

COMMENT: The technique of defining functions in terms of exponentials can be extended so that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ can be applied and used when

x = complex, matrix, quantum mechanical operator, ...

In the latter case, care must be taken to define what an ∞ series of matrices or operators means. More on this question later.

DERIVATIVES: Given the series expansions, we can now use those results to find the derivatives of $\sin x$ & $\cos x$.

(Henceforth the arguments of sin & cos will be assumed to be in radians, unless otherwise stated)

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\lim(\Delta x \to 0)\{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x\}}{\Delta x} \quad (8)$$

From the previous series solutions in (6) & (7) we see that when Δx is very small $\cos(\Delta x) \cong 1 - \frac{1}{2}(\Delta x)^2 \cong 1$, $\sin\Delta x \cong \Delta x$

(recall our previous result that $\frac{\sin\theta}{\theta} \xrightarrow{\theta \to 0} 1$)

Hence (8) $\Rightarrow$

$$\therefore \frac{d}{dx}\sin x \cong \lim_{\Delta x \to 0} \frac{\cancel{\sin x}\cdot 1 + \cos x \cdot \Delta x - \cancel{\sin x}}{\Delta x} = \cos x \quad (9)$$

$$\text{Similarly, } \frac{d}{dx}\cos x = \lim_{\Delta x \to 0}\left\{\frac{\cos(x+\Delta x) - \cos x}{\Delta x}\right\} = \lim_{\Delta x \to 0}\left\{\frac{\cos x \cos\Delta x - \sin x \cdot \sin\Delta x - \cos x}{\Delta x}\right\}$$

$$\therefore \frac{d}{dx}\cos x \cong \lim_{\Delta x \to 0}\left\{\frac{\cos x \cdot (\cancel{1} - \frac{\Delta x^2}{2}) - \sin x \cdot \Delta x - \cos x}{\Delta x}\right\} = -\sin x \quad (10)$$

INVERSE TRIG FUNCTIONS:

Define $y(x) = \sin^{-1}(x)$ (1)

"y is the angle whose sine is x" $\Rightarrow \underbrace{\sin y = \sin(\sin^{-1}(x)) = x}$

(2)

To find $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$ we write

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} \quad ; \quad x = \sin(y) \Rightarrow \frac{dx}{dy} = \cos(y)$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \quad (3)$$

Hence: $\boxed{y(x) = \sin^{-1}(x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} ;}$ (4)

Similarly: $\left.\begin{array}{l} y(x) = \cos^{-1}(x) \\ x = \cos y \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$ (5)

Hence $\boxed{y(x) = \cos^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$ (6)

Having introduced e^x and $\ln x$ we can derive a useful explicit expression for $\sin^{-1}(x)$, which holds for an arbitrary complex argument z. Start with

$$\underline{z = \sin\theta} = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \quad < \quad e^{i\theta} \equiv y \quad (7)$$

$$e^{i\theta} \cdot 2iz = e^{2i\theta} - 1 \Rightarrow e^{2i\theta} - 2ize^{i\theta} - 1 = 0 \quad (8)$$

$$y^2 - 2izy - 1 = 0$$

$$\therefore \; y = e^{i\theta} = \frac{2iz \pm \sqrt{-4z^2+4}}{2} = iz \pm \sqrt{1-z^2} \quad (9)$$

Choosing + [redacted] root $\Rightarrow$

Taking $\ln(\ldots)$ of both sides of (9) $\Rightarrow$

$$\ln(e^{i\theta}) = i\theta = \ln\left(iz + \sqrt{1-z^2}\right) \quad (10)$$

But $z = \sin\theta \Rightarrow \theta = \sin^{-1}(z) \Rightarrow$ $\boxed{\sin^{-1}(z) = -i\ln\left(iz + \sqrt{1-z^2}\right)}$ (11)

Using this formula we can directly check the previous result for the derivative of $\sin^{-1}(z)$. Let $u(z) = \left(iz + \sqrt{1-z^2}\right)$

Then
$$\frac{d}{dz}\sin^{-1}(z) = -i\,\frac{d\ln(u)}{du}\cdot\frac{du}{dz} = -i\cdot\frac{1}{\left(iz+\sqrt{1-z^2}\right)}\,\frac{d(\ldots)}{dz} \quad (12)$$

$$\frac{d(\ldots)}{dz} = i + \frac{1}{2}\,\frac{1}{\sqrt{1-z^2}}(-2z) = \frac{i\sqrt{1-z^2} - z}{\sqrt{1-z^2}} \quad (13)$$

Combining (12) & (13) we find

$$\frac{d}{dz}\sin^{-1}(z) = \frac{-i}{\left(iz+\sqrt{1-z^2}\right)} \otimes \frac{\left(i\sqrt{1-z^2} - z\right)}{\sqrt{1-z^2}} = \frac{1}{\sqrt{1-z^2}} \quad (14)$$

Which is the same result found previously.

HYPERBOLIC FUNCTIONS

$$\cosh x \equiv \frac{1}{2}\left(e^{x}+e^{-x}\right) ; \quad \sinh x = \frac{1}{2}\left(e^{x}-e^{-x}\right) \qquad (1)$$

$$\frac{d}{dx}\cosh x = \frac{1}{2}\left(e^{x}-e^{-x}\right) = \sinh x \qquad (2)$$

$$\frac{d}{dx}\sinh x = \frac{1}{2}\left(e^{x}-(-)e^{-x}\right) = \cosh x \qquad (3)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad ; \quad \sinh(x+y) = \sinh x \cdot \cosh y + \cosh x \cdot \sinh y \qquad (4)$$

Inverse Hyperbolic Functions

In analogy to $\sin x$ & $\cos x$ we write

$$y = \sinh^{-1} x \iff x = \sinh y$$

Using the "inverse function" trick we have:

$$\frac{dx}{dy} = \frac{d}{dy}\sinh y = \frac{d}{dy}\frac{1}{2}\left(e^{y}-e^{-y}\right) = \cosh y \qquad (5)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} \xrightarrow{(4)} \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}} \qquad (6)$$

Hence $\boxed{y = \sinh^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}}$ (7)

Similarly: $y = \cosh^{-1} x \Rightarrow x = \cosh y \Rightarrow \frac{dx}{dy} = \sinh y$ (8)

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}} \qquad (9)$$

In a similar manner we can define the hyperbolic tangent function $\equiv \tanh x$ as

$$\boxed{\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}} \qquad (10)$$

This function is interesting in MACHINE LEARNING because of its limits:

$$\tanh(-\infty) = \frac{\overset{\to 0}{e^{-\infty}} - e^{+\infty}}{e^{-\infty} + e^{+\infty}} \cong \frac{-e^{+\infty}}{e^{+\infty}} = -1 \qquad (11)$$

$$\tanh(+\infty) = \frac{e^{\infty} - e^{-\infty}}{e^{\infty} + e^{-\infty}} \cong \frac{e^{\infty}}{e^{\infty}} = +1$$

$$\tanh(0) = 0$$

Hence this function maps the whole real line (from $-\infty$ to $+\infty$) into the narrow range $[-1, 1]$, for any parameter of interest. This allows one to study different parameters that might describe the performance of a car engine (for example) on a common footing. For example, we might want to study the fuel efficiency of an engine as a function of bore, stroke, and compression ratio.

CONNECTION BETWEEN TRIG (CIRCULAR) FUNCTIONS AND HYPERBOLIC FUNCTIONS

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \Rightarrow \sin(ix) = \frac{1}{2i}\left(e^{i\cdot ix} - e^{-i(ix)}\right) \quad (1)$$

$$= \frac{1}{2i}\left(\underbrace{e^{-x} - e^{x}}_{-2\sinh x}\right) \Rightarrow \boxed{\sinh x = -i\sin(ix)} \quad (2)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \Rightarrow \boxed{\cos(ix) = \frac{1}{2}\left(e^{-x} + e^{x}\right) = \cosh x} \quad (3)$$

$$\tan x = \frac{\sin x}{\cos x} \Rightarrow \boxed{\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i\sinh x}{\cosh x} = i\tanh x} \quad (4)$$

We can similarly show that

$$\boxed{\begin{aligned} \coth x &= i\cot(ix) \\ \operatorname{sech} x &= \sec(ix) \\ \operatorname{csch} x &= i\csc(ix) \end{aligned}}$$

−25.2

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$$ ✓

$$\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}$$ ✓

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$ ✓

$$\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\csc^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cosh x)=\sinh x$$ ✓

$$\frac{d}{dx}(\sinh x)=\cosh x$$ ✓

$$\frac{d}{dx}(\tanh x)=\operatorname{sech}^2 x$$ ✓

$$\frac{d}{dx}(\coth x)=-\operatorname{csc}h^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x)=-\operatorname{sech} x\cdot\tanh x$$

$$\frac{d}{dx}(\operatorname{csch} x)=-\operatorname{csch} x\cdot\coth x$$

$$\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(\cosh^{-1}x\right)=\frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tanh^{-1}x\right)=\frac{1}{1-x^2}$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1}x\right)=\frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\operatorname{csch}^{-1}x\right)=\frac{-1}{|x|\sqrt{1+x^2}}$$

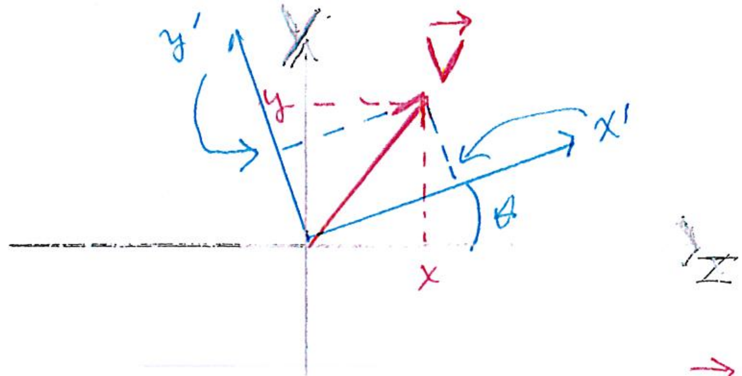
$$\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{1}{1-x^2}$$

$$\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$$

$$\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$$

$$\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

WHY IMAGINARY ANGLES?



It is easy to show that a vector $\vec{V}$ which has components $\vec{V} = (x, y)$ in the original (x,y) coordinate system, will have the components $\vec{V} = (x', y')$ in the coordinate system rotated by an angle θ, where x', y' are given by

$$x' = x\cos\theta + y\sin\theta \quad ; \quad y' = -x\sin\theta + y\cos\theta$$

These relations can be expressed in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

ROTATION MATRIX

In a 3-dimensional world, the same rotation in the x-y plane would be described by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

z axis is unaffected

SPECIAL THEORY OF RELATIVITY

Through the work of MINKOWSKI the familiar LORENTZ TRANSFORMATION which relate space & time can also be viewed as rotations – but in a 4-dimensional world and through an imaginary angle:

DEFINE: $x \to x_1$; $y \to x_2$; $z \to x_3$; $ict \to x_4$

For two observers with relative velocity v in the x-direction the usual LORENTZ TRANSFORMATIONS are

$$x' = \gamma(x - vt) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \; ; \; \boxed{\beta = v/c \equiv \tanh u}$$

$$t' = \gamma(t - v x/c^2)$$

Then:

$$\boxed{\begin{aligned} x_1' &= \gamma(x_1 + i\beta x_4) = x_1 \cosh u + x_4 \, i \sinh u \\ x_2' &= x_2 \qquad x_3' = x_3 \\ x_4' &= \gamma(x_4 - i\beta x_1) = -x_1 \sinh u + x_4 \cosh u \end{aligned}}$$

Since $\cosh u = \cos(iu)$ and $\sinh(u) = -i \sin(iu)$ it follows that LORENTZ TRANSFORMATIONS can be viewed as rotations through an imaginary angle iu in 4-dimensional space-time.

GRAPHING FUNCTIONS

To have a physical feeling for a mathematical function it is optimally helpful to graph it. Although this can always be done numerically, insight can usually be obtained by considering various limiting cases of some variable ($x=0, x\to\pm\infty$... and also by examining its derivatives.

[EXAMPLE 1] The previously discussed function $\underbrace{\tanh(x)}_{\equiv y(x)}$ is an example. We have already shown that $\tanh(0)=0$ and $\tanh(\pm\infty)=\pm 1 \equiv y(\pm\infty)$

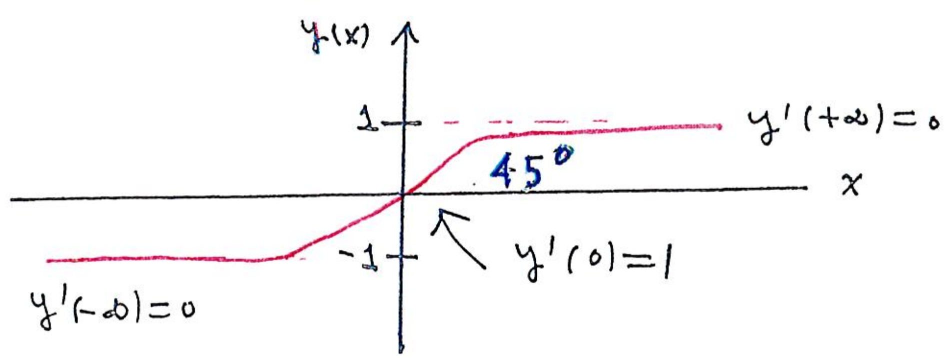
Further insight can be had by looking at its derivative:

$$\frac{d}{dx}\underbrace{\tanh(x)}_{y(x)} = \frac{d}{dx}\left(\frac{\sinh(x)}{\cosh(x)}\right) = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{(\cosh x)^2}$$

$$= 1/(\cosh x)^2 = \frac{1}{\left[\frac{1}{2}(e^{x}+e^{-x})\right]^2} \qquad (1)$$

$$\Rightarrow \frac{d}{dx}\underbrace{\tanh(x)}_{y(x)} = \frac{4}{(e^{x}+e^{-x})^2} \equiv y'(x) \Rightarrow \begin{array}{l} y'(+\infty) = y'(-\infty) = 0 \\ y'(0) = 1 \end{array} \qquad (2)$$

Combining these results with the previous results, $y(\pm\infty)=\pm 1$ we can sketch $y(x)=\tanh(x)$ as follows



[EXAMPLE 2] $y = f(x) = x^n e^{-x}$ $\quad (x \geq 0,\ n \geq 1)$

We see immediately that: $y(0) = 0$; $y(\infty) = 0$

$$y'(x) = x^n \underbrace{\left\{\frac{d}{dx} e^{-x}\right\}}_{-e^{-x}} + e^{-x} \underbrace{\left\{\frac{d}{dx} x^n\right\}}_{nx^{n-1}} = e^{-x}\left\{-x^n + nx^{n-1}\right\} \qquad (1)$$

$$\Rightarrow y'(x) = e^{-x} \cdot x^{n-1}\{-x+n\} \Rightarrow y'(x) = 0 \text{ when } x = n \qquad (2)$$

Recall that $y'(x) = 0$ can signify either a maximum or a minimum. To determine which it is we evaluate the 2ND derivative $y''(x)$ at $x = n$: From (1)

$$y''(x) = -e^{-x}\left\{-x^n + nx^{n-1}\right\} + e^{-x}\left\{-nx^{n-1} + n(n-1)x^{n-2}\right\}_{x=n}$$

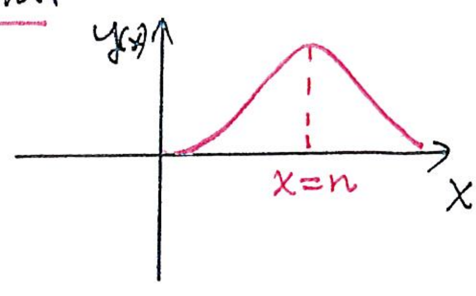
$$= e^{-x}\left\{x^n - nx^{n-1} - nx^{n-1} + n(n-1)x^{n-2}\right\}_{x=n}$$

$$= \underbrace{e^{-x}(x^{n-2})}_{\text{positive}}\left\{x^2 - 2nx + n(n-1)\right\}$$

when $x = n \Rightarrow \{\ldots\} = n^2 - 2n^2 + n(n-1)$
$= -n < 0$

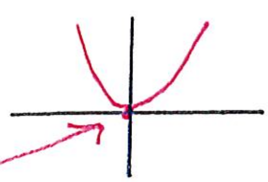
Since $y''(x=n) < 0 \Rightarrow$ maximum.*

Hence $y = x^n e^{-x}$ looks like



* USEFUL MNEMONIC:

Consider $y = x^2$



$y'(x) = 2x = 0$

$\Rightarrow x = 0$

$y'' = 2 \Rightarrow y''(0) = 2 = \text{positive} \Rightarrow$ $y'' = \text{positive} \Rightarrow \text{minimum}$

$y'' = \text{negative} \Rightarrow \text{maximum}$

[Example 3] From the text p. 23

$$y = f(x) = \frac{(x^2-5x+6)}{x-1} \cdot e^{-x/5} \qquad (1)$$

Step [1]: Examine $x \to +\infty$; We immediately note that $e^{-\infty/5} \to 0$.

Since $e^{-x/5}$ contains all powers of x (recall $e^y \equiv \sum_{n=0}^{\infty} y^n/n!$)

the behavior of $f(x)$ as $x \to \infty$ is dominated by $e^{-x/5}$

$\Rightarrow f(x) \to 0$ as $x \to \infty$

Step [2]: Examine the behavior of $f(x)$ as $x \to -\infty$. In this case

$$e^{-x/5} \longrightarrow e^{|x|/5} \longrightarrow \infty \qquad (2)$$

At the same time the rational function multiplying $e^{-x/5}$ is dominated by x^2 in the numerator, and by x in the denominator

Hence $$f(x) \xrightarrow{x \to -\infty} \frac{x^2}{x} e^{|x|/5} = x e^{|x|/5} \qquad (3)$$

Step [3]: Examine $f(x)$ for other finite values of x.

Here we note that (by inspection) $f(x)$ can be rewritten in the form

$$f(x) = \frac{(x-2)(x-3)}{x-1} e^{-x/5}$$

Hence $f(x=2)=0$ and $f(x=3)=0$, but $f(x=1) \longrightarrow \infty$

For homework you will be asked to compute $f'(x)$, and then use that result to reproduce Figure 1.6 of the text.

[EXAMPLE 4] Study the function $\boxed{y = f(x) = x^n \ln x}$

$n \geq 1$ is an integer

Here the behavior as $x \to +\infty$ is obvious: $f(x) \to +\infty$

The interesting question is what happens as $x \to 0$. Bear in mind that $\ln(0) \to -\infty$.

A convenient way to analyze y is to substitute $z = 1/x$ so that $x \to 0$ becomes $z \to \infty$. Then

$$f \to \frac{1}{z^n} \ln\left(\frac{1}{z}\right) = \frac{1}{z^n}(\ln 1 - \ln z) = -\frac{\ln z}{z^n} \xrightarrow{z \to \infty} \frac{\infty}{\infty} \qquad (1)$$

Applying l'Hopital's Rule: $f \to \dfrac{\frac{d}{dz}(-\ln z)}{\frac{d}{dz} z^n} = \dfrac{(-1/z)}{n z^{n-1}} = \dfrac{-1}{n}\dfrac{1}{z^n}$ (2)

Hence as $x \to 0 \leftrightarrow z \to \infty$ $f \to 0$.

$$\boxed{\text{So } y = f(x) = x^n \ln x \xrightarrow{x \to 0} 0} \qquad (3)$$

Hence the growth of $\ln x$ as $x \to \infty$ is weaker (slower) than the growth of any polynomial x^n.

Note that in this case we could have obtained the same result by applying l'Hopital's Rule directly in the form

$$y = f(x) = x^n \ln x \xrightarrow{x \to 0} \left(\frac{d}{dx} x^n\right)\left(\frac{d}{dx} \ln x\right) = (n x^{n-1}) \frac{1}{x} = n x^{n-2} \xrightarrow{x \to 0} 0$$

But you should be careful doing this elsewhere!!

SHANKAR

RECALL:

$\frac{d}{dx} e^x = e^x$

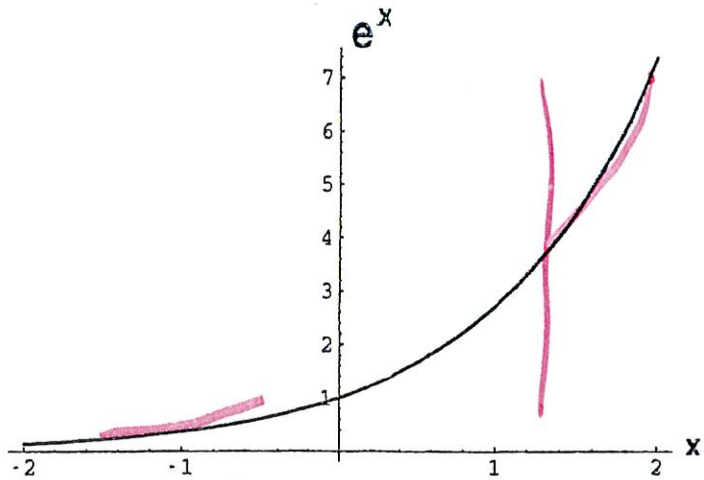


Figure 1.2. Plot of the function e^x. Notice its growth is proportional to the value of the function itself.

(Recall that every derivative of the function is also e^x which equals 1 at $x = 0$.) The ratio test tells us that in this case, since $a_n = 1/n!$,

$$R = \lim_{n \to \infty} \frac{(n+1)!}{n!} \to \infty, \tag{1.3.22}$$

i.e., that the series converges for all finite x. *Thus we have defined the function for all finite x based on what we knew at $x = 0$.*[2]

We are now ready to find e: simply set $x = 1$ in Eqn. (1.3.21). As we keep adding more terms to the sum we see it converges quickly to a value around 2.7183. We can now raise e to any power. For example, to find $e^{95/112}$ we just set $x = 95/112$ in the sum and compute as many terms as we want to get any desired accuracy. There is no trial and error involved. We may choose x to be any real number, say π or even e!

Figure 1.2 shows a plot of the exponential function for $-2 < x < 2$.

There is a second way to define the exponential function. Consider the following function defined by two integers M and N:

$$e^x_{N,M} = (1 + \frac{x}{N})^M \tag{1.3.23}$$

If we fix M and let $N \to \infty$, the result is clearly 1. On the other hand if we fix N and let $M \to \infty$, the result will either be 0 or ∞ depending on whether x

[2]When we study Taylor series later, we will see that this situation is quite unusual. Take for example, the function $1/(1-x)$. Suppose we only knew its Taylor series about the origin: $1 + x + x^2 + x^3 + \ldots$. The ratio test tells us the series converges only for $|x| < 1$. One then has to worry about how to reconstruct the function beyond that interval. This point will be discussed in Chapter 6. For the present let us thank our luck and go on.

SHANKAR

RECALL: $\frac{d}{dx} \ln x = \frac{1}{x}$

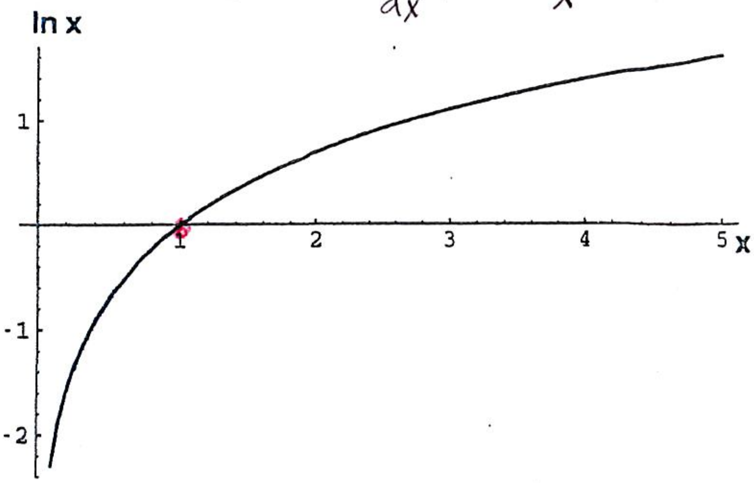


Figure 1.4. Plot of the function $\ln(x)$. Notice that $\ln 1 = 0$, $\ln e = 1$, $\ln x \to -\infty$ as $x \to 0$ and $\ln x \to \infty$ as $x \to \infty$.

How are we to get the ln of a number ≥ 2? There are general tricks which will be discussed in Chapter 6, but we will deal with this by deriving and using a property of the ln that you must know.

From the very definition, (Eqn. (1.3.39), for two numbers a and b,

$$a = e^{\ln a} \tag{1.3.50}$$
$$b = e^{\ln b} \tag{1.3.51}$$
$$ab = e^{\ln a + \ln b} \tag{1.3.52}$$
$$= e^{\ln ab} \quad \text{so that} \tag{1.3.53}$$
$$\ln(ab) = \ln a + \ln b. \tag{1.3.54}$$

Using the property

$$\ln(ab) = \ln a + \ln b \tag{1.3.55}$$

we can obtain the ln of a big number ab starting with the ln of smaller numbers a and b which in turn could be dealt with in the same way until we get to the stage where we need only the ln's of numbers less than 2. For example, knowing $\ln 1.6$ and $\ln 1.8$ we can get $\ln 2.88$ as the sum of the two logarithms. Fig. 1.4 depicts the ln function obtained by this or any other way.

We now know how to calculate a^x as follows:

$$a^x = (e^{\ln a})^x \tag{1.3.56}$$
$$= e^{x \ln a}. \tag{1.3.57}$$

Note that everything above is well defined: for any given a we can find $\ln a$ using the Taylor series, we can then exponentiate the result using the series for the exponential function.

SHANKAR

Catenary

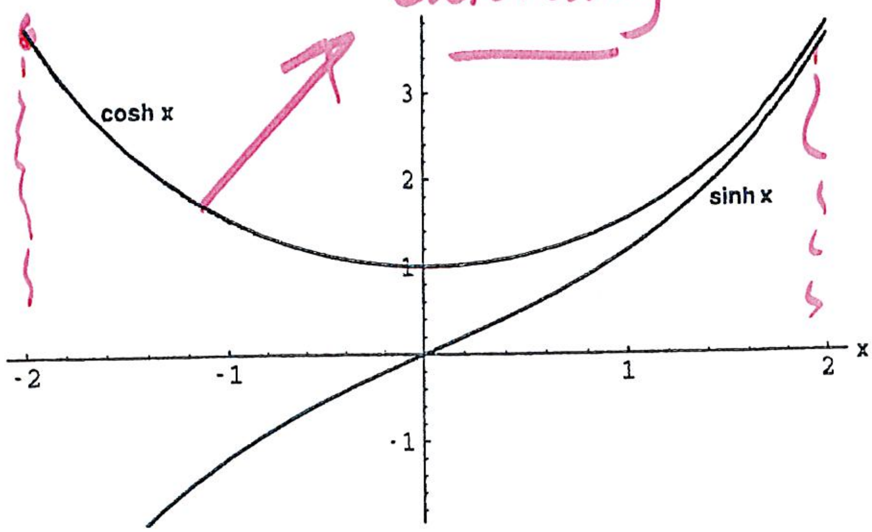


Figure 1.3. Plot of the hyperbolic sinh and cosh functions. Note that they are odd and even respectively and approach $e^x/2$ as $x \to \infty$.

From the function e^x we can generate the following *hyperbolic functions:*

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{1.3.28}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{1.3.29}$$

These functions are often called sh x and ch x, where the h stands for "hyperbolic". They are pronounced *sinch* and *cosh* respectively. Figure 1.3 is a graph of these functions.

They obey many identities such as

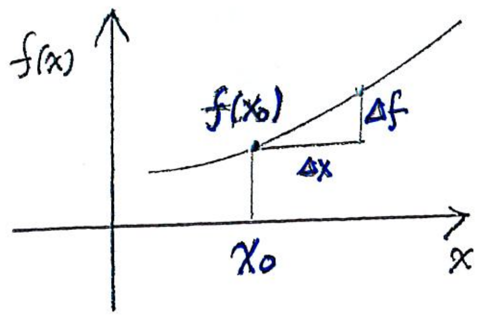
$$\cosh^2 x - \sinh^2 x = 1 \tag{1.3.30}$$

$$\sinh(x + y) = \sinh x \ \cosh y + \cosh x \ \sinh y \tag{1.3.31}$$

(which can be proved, starting from the defining Eqs.(1.3.28-1.3.29)) and numerous others which we cannot discuss in this chapter devoted to calculus. For the present note that these relations look a lot like those obeyed by trigonometric functions. The intimate relation between the two will be taken up later in this book. Readers wishing to bone up on this subject should work through the exercises in this chapter. Note that $\cosh x$ and $\sinh x$ are even and odd, respectively, under $x \to -x$.

Problem 1.3.1. *Verify that* $\sinh x$ *and* $\cosh x$ *are derivatives of each other. Verify Eqns. (1.3.30-1.3.31).*

DIFFERENTIALS



SEE Fig. 1.8 of text

From the figure, the change Δf in $f(x)$ from x_0 to $x_0+\Delta x$ is given by

$$\Delta f = \left.\frac{df}{dx}\right|_{x_0} (\Delta x) \;+\; \text{terms of order } (\Delta x)^2 \text{ and higher denoted by} \ldots \qquad (1)$$

In the limit as $\Delta x \longrightarrow dx$, $\Delta f \to df$ and we can write

$$\boxed{df(x_0) = \left.\frac{df(x)}{dx}\right|_{x_0} \cdot dx} \qquad (2)$$

As a practical matter we often use the approximation in (1) to estimate the small change Δf that will result from changing $x_0 \to x_0+\Delta x$.

1.8. Summary

Of the numerous ideas discussed in this chapter, the following are the key ones and should be at your fingertips.

- Definition of the derivative, derivative of a product of functions

$$D(fg) = gDf + fDg$$

a quotient of two functions

lo.dhi – hi.dlo
lo lo

$$D\left(\frac{f}{g}\right) = \frac{[gDf - fDg]}{g^2}$$

chain rule for a function of a function

$$\frac{df(u(x))}{dx} = \frac{df(u)}{\cancel{du}} \cdot \frac{du\cancel{(x)}}{dx}.$$

- The notion of the Taylor series

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2}f^{(2)}(0) + \ldots$$

about the origin or about the point a

$$f(a+x) = f(a) + xf^{(1)}(a) + \frac{x^2}{2}f^{(2)}(a) + \ldots$$

- The following series to the order shown

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$$

$$e^x = \lim_{N\to\infty}\left[1 + \frac{x}{N}\right]^N.$$

$$\cos x = 1 - \frac{x^2}{2!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \ldots$$

$$(1+x)^p = 1 + px + \frac{(p)(p-1)}{2}x^2 + \ldots$$

1.8. Summary

Of the numerous ideas discussed in this chapter, the following are the key ones and should be at your fingertips.

- Definition of the derivative, derivative of a product of functions

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- The notion of the Taylor series

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$$\sin x = x - \frac{x^3}{3!} + \ldots$$

$$(1+x)^p = 1 + px + \frac{(p)(p-1)}{2} x^2 + \ldots$$

- Definition of the hyperbolic function, in particular

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

their symmetry under $x \to -x$ and functional identities, especially

$$\cosh^2 x - \sinh^2 x = 1.$$

If you need a formula for $\cosh 2x$ or any other identity, you can get it from the definition of the hyperbolic functions in terms of exponentials. The power series for these functions can also be obtained from that of the exponential function.

- The $\ln x$ function, the identity

$$x = e^{\ln x}$$

and the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

and its derivative

$$D \ln x = \frac{1}{x}.$$

$$x^a = e^{a \ \ln x}.$$

- Trigonometric functions, identities and derivatives, radian measure for angles. These will not be listed here since you must have already learned them by heart as a child.

- The definition of the differential, that:

$$df = f' dx$$

is an exact relation which defines df in terms of the derivative at the point x, and that as $dx \to 0$, $df \to \Delta f$, the actual change in f.