We note that if $\vec{F}\cdot d\vec{l} = -dV$ then

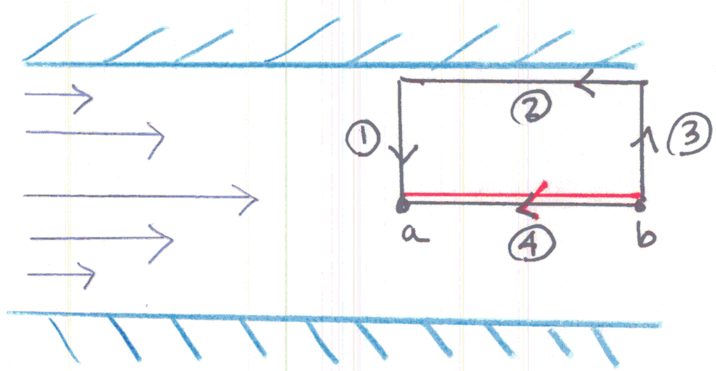
$$\int_{P_1}^{P_2} \vec{F}\cdot d\vec{l} = -\int_{P_1}^{P_2} dV = V(P_1)-V(P_2) \leftarrow \underline{\text{independent of the path}} \qquad (14)$$

We have shown on p.18.1 that $dV = \vec{\nabla} V\cdot d\vec{l}$, and hence having shown that $\vec{\nabla}\times\vec{F} = 0 \Rightarrow \vec{F}\cdot d\vec{l} = dV$ we have also shown that

$$\vec{F}\cdot d\vec{l} = -dV = -\vec{\nabla} V\cdot d\vec{l} \Rightarrow \boxed{\vec{F} = -\vec{\nabla} V} \qquad (15)$$

Hence altogether $\boxed{\vec{F} = -\vec{\nabla} V \Leftrightarrow \vec{\nabla}\times\vec{F} = 0}$ (16)

<u>Physical Picture:</u> Using the river example from p.18 we can see physically why a velocity field for which $\vec{\nabla}\times\vec{v} \neq 0$ is not conservative:




Clearly the work done in going from $b \to a$ along ④ is greater than along the path ③②①

# The Laplacian: $\nabla^2 \equiv \Delta$ [also $\boxed{4}$, ⑤, ...]

This is an important operator which arises in many interesting physics equations. The Laplacian of a scalar field is defined by

$$\nabla^2 u(\vec{r}) = \vec{\nabla}\cdot(\vec{\nabla} u(\vec{r})) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial u}{\partial x} + \hat{y}\frac{\partial u}{\partial y} + \hat{z}\frac{\partial u}{\partial z}\right) \tag{1}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Notes: a) We will later define the Laplacian of a vector field, $\nabla^2\vec{V}$.

b) In the upcoming discussion of tensor analysis we will show how to express $\nabla^2$ in an arbitrary coordinate system.

## Physical Interpretation of the Laplacian:

Let $u(\vec{r})$ have the value $u_0$ at the origin of a Cartesian coordinate system. Construct a cube of side a around the origin. The average value $\bar{u}$ of $u$ inside this cube is then given by

$$\bar{u} = \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz \; u(x,y,z) \tag{2}$$

Assuming that a is small, so that $u(x,y,z)$ varies slowly over the cube, Expand $u(x,y,z)$ in a Taylor series about the origin:

$$u(x,y,z) = u_0 + x\left(\frac{\partial u}{\partial x}\right)_0 + y\left(\frac{\partial u}{\partial y}\right)_0 + z\left(\frac{\partial u}{\partial z}\right)_0 \tag{3}$$

$$+ \frac{1}{2}x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + \frac{1}{2}y^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0 + \frac{1}{2}z^2\left(\frac{\partial^2 u}{\partial z^2}\right)_0 + xy\left(\frac{\partial^2 u}{\partial x\partial y}\right)_0 + xz\left(\frac{\partial^2 u}{\partial x\partial z}\right)_0 + yz\left(\frac{\partial^2 u}{\partial y\partial z}\right)_0 + \cdots$$

Inserting the expansion (3) into (2) we see that the terms linear in $x$ or $y$ or $z$ vanish since

$$\int_{-a/2}^{a/2} dx \cdot x \cdots = \frac{1}{2} x^2 \Big|_{-a/2}^{a/2} = 0 \quad \text{etc.} \tag{4}$$

The same holds for the terms proportional to $xy$, $xz$, and $yz$. Thus the only surviving contributions are of the form:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_0 \int_{-a/2}^{a/2} dx \left(\tfrac{1}{2}x^2\right) \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz = \left(\frac{\partial^2 u}{\partial x^2}\right)_0 \left[\frac{1}{2}\cdot\frac{1}{3}x^3\right]_{-a/2}^{a/2} \Big[y\Big]_{-a/2}^{a/2} \Big[z\Big]_{-a/2}^{a/2} = \frac{a^5}{24}\left(\frac{\partial^2 u}{\partial x^2}\right)_0 \tag{5}$$

Combining Eqs. (2) & (5) we have

$$\bar{u} = u_0 + \frac{1}{a^3}\cdot\frac{a^5}{24}\left[\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + \left(\frac{\partial^2 u}{\partial y^2}\right)_0 + \left(\frac{\partial^2 u}{\partial z^2}\right)_0\right] = u_0 + \frac{a^2}{24}\left(\nabla^2 u\right)_0 \tag{6}$$

Hence finally:

$$\boxed{\nabla^2 u(x=y=z=0) = \frac{24}{a^2}\left[\bar{u} - u_0\right]} \tag{7}$$

Thus the Laplacian measures the difference between the value of a function at a given point, $u_0$, and the average of the function in an infinitesimal neighborhood around the point. This gives a physical interpretation of the heat/diffusion equations:

$$\nabla^2 u(\vec{r},t) = K \frac{\partial u(\vec{r},t)}{\partial t}$$

$u$ = temperature; $K^{-1}$ = heat conductivity

$u$ = density; $K^{-1}$ = diffusion constant

---

In 4-dimensions:

$$\Box u(\vec{x},t) = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\vec{x},t) = 0$$

One of the most important relations involving $\nabla^2$ is:

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta^3(\vec{r})} \quad (1)$$

This has important experimental and theoretical consequences which we discuss below. To establish this result we first show that $\nabla^2(1/r) = 0$ when $r \neq 0$: Using $r = (x^2+y^2+z^2)^{1/2}$

$$\nabla^2(1/r) = \vec{\nabla}\cdot\vec{\nabla}(1/r) = \vec{\nabla}\cdot\left[\hat{x}\frac{\partial}{\partial x}(1/r) + \hat{y}\frac{\partial}{\partial y}(1/r) + \hat{z}\frac{\partial}{\partial z}(1/r)\right] \quad (2)$$

$$\frac{\partial}{\partial x}(1/r) = -x(x^2+y^2+z^2)^{-3/2} = \frac{-x}{r^3} \quad (3)$$

Hence $$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5} \quad (4)$$

Hence $$\nabla^2\left(\frac{1}{r}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{r}\right) \quad (5)$$

$$= \left(-\frac{1}{r^3}+\frac{3x^2}{r^5}\right) + \left(-\frac{1}{r^3}+\frac{3y^2}{r^5}\right) + \left(-\frac{1}{r^3}+\frac{3z^2}{r^5}\right) \quad (6)$$

$$\therefore\ \nabla^2\left(\frac{1}{r}\right) = -\frac{3}{r^3} + \frac{3}{r^5}(x^2+y^2+z^2) \equiv 0 \quad (r \neq 0) \quad (7)$$

When $r=0$ the various differentiations leading to (7) would no longer be valid, so we must be more careful: Consider

$$\int \nabla^2\left(\frac{1}{r}\right) d^3r \equiv \int \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) dV = \int \vec{\nabla}\left(\frac{1}{r}\right)\cdot d\vec{S} \quad (8)$$

Gauss' Theorem

$$\vec{\nabla}\left(\frac{1}{r}\right) = (\hat{\imath}\,\partial_x + \hat{\jmath}\,\partial_y + \hat{k}\,\partial_z)\left(\frac{1}{r}\right) = -(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)\frac{1}{r^3} = -\frac{\vec{r}}{r^3} \quad (9)$$

$$\therefore \int \nabla^2\left(\frac{1}{r}\right) dV = -\int \frac{\vec{r}\cdot d\vec{S}}{r^3} \equiv -\int d\Omega = -4\pi \quad (10)$$

(We note from (10) that since $\vec{F}\cdot d\vec{s}$ has the same sign at all points on the surface of a sphere that $\int \vec{r}\cdot d\vec{s}/r^3$ cannot be zero when integrated over the surface.)

It follows from (10) that $\nabla^2(1/r)$ cannot be zero everywhere, or else Eq. (10) could not hold. In fact we see that (7) & (10) can be made compatible if (1) holds since then

$$\int \nabla^2\left(\frac{1}{r}\right) dV = \int \nabla^2\left(\frac{1}{r}\right) dx\,dy\,dz = -4\pi \int \delta(x)\delta(y)\delta(z)\,dx\,dy\,dz$$

$$= -4\pi \int \delta(x)dx \int \delta(y)dy \int \delta(z)\,dz = -4\pi V \qquad (11)$$

Experimental Consequences:

For Newtonian gravity the potential energy is

$$V_{12}^{(N)}(r) = -\frac{G\,m_1 m_2}{r} \qquad (12)$$

And for the the Coulomb interaction $V_{12}^{(C)}(r) = +\dfrac{q_1 q_2}{r}$ (13)

In both cases $\nabla^2 V_{12}^{(N,C)} = 0 \quad (r \neq 0)$. (14)

However, suppose that in the gravitational case there was an additional contribution such that

$$V_{12}(r) = -\frac{G m_1 m_2}{r}\left(1 + \alpha e^{-r/\lambda}\right) \quad ; \ \alpha, \lambda = \text{constants} \qquad (15)$$

Then by an elementary calculation we find

$$\boxed{\nabla^2 V_{12}(r) = \frac{\alpha G m_1 m_2}{\lambda^2 r}\, e^{-r/\lambda} \neq 0} \qquad (16)$$

Measuring $\nabla^2 V_{12}(r) \neq 0$ is thus a means of testing for deviations from Newton's law of gravity, or from Coulomb's law.

# CARTESIAN TENSOR NOTATION FOR VECTOR IDENTITIES

## Definitions:

1) $\vec{u}\cdot\vec{V} = \sum_i u_i V_i \equiv u_i V_i$

2) $(\vec{u}\times\vec{V})_i = \sum_{j,k} \epsilon_{ijk} u_j V_k \equiv \epsilon_{ijk} u_j V_k$

3) $(\vec{\nabla}\phi)_i = \partial_i \phi$

4) $\vec{\nabla}\cdot\vec{V} = \partial_i V_i$

5) $(\vec{\nabla}\times\vec{V})_i = \epsilon_{ijk}\partial_j V_k$

6) $\nabla^2\phi = \vec{\nabla}\cdot(\vec{\nabla}\phi) = \partial_i(\partial_i\phi) = \partial_i\partial_i\phi$

7) $(\nabla^2\vec{V})_i = \underbrace{\partial_j\partial_j}_{\text{dummy indices}} \underbrace{V_i}_{\text{free index}}$

## Some Simple Theorems:

a) $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$ (8)

$\hookrightarrow [\vec{\nabla}\times(\vec{\nabla}\phi)]_i = \epsilon_{ijk}\partial_j(\partial_k\phi) = \epsilon_{ijk}\partial_j\partial_k\phi = 0$ (9)

b) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0$ ; see p.17 (10)

c) $\vec{\nabla}\cdot(\vec{u}\times\vec{V}) = \partial_i(\epsilon_{ijk}u_j V_k) = \epsilon_{ijk}(u_j\partial_i V_k + V_k\partial_i u_j)$ (11)

$= -\epsilon_{jik}u_j\partial_i V_k + \epsilon_{kij}V_k\partial_i u_j = -\vec{u}\cdot(\vec{\nabla}\times\vec{V}) + \vec{V}\cdot(\vec{\nabla}\times\vec{u})$

$$\therefore \boxed{\vec{\nabla}\cdot(\vec{u}\times\vec{V}) = \vec{V}\cdot(\vec{\nabla}\times\vec{u}) - \vec{u}\cdot(\vec{\nabla}\times\vec{V})} \qquad (12)$$

## d) Laplacian of a Vector Field

Consider $\vec{\nabla}\times(\vec{\nabla}\times\vec{V})$:

$$[\vec{\nabla}\times(\vec{\nabla}\times\vec{V})]_i = \epsilon_{ijk}\,\partial_j\,(\epsilon_{klm}\,\partial_l V_m) = \epsilon_{ijk}\,\epsilon_{klm}\,\partial_j\partial_l\,V_m \qquad (13)$$

$$= (\delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im})\,\partial_j\partial_l V_m = \partial_i\partial_j V_j - \partial_j\partial_j V_i = \partial_i(\vec{\nabla}\cdot\vec{V}) - \nabla^2 V_i \qquad (14)$$

$$\therefore\ [\vec{\nabla}\times(\vec{\nabla}\times\vec{V})]_i = -\nabla^2 V_i + \partial_i\,(\vec{\nabla}\cdot\vec{V}) \qquad (15)$$

In more familiar notation:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = -\nabla^2\vec{V} + \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) \qquad (16)$$

Hence finally: $$\boxed{\nabla^2\vec{V} = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{V})} \qquad (17)$$

Since all the quantities on the r.h.s. are well-defined this defines what we mean by $\nabla^2\vec{V}$.

$\vec{B}=\vec{\nabla}\times\vec{A} \Leftrightarrow \vec{\nabla}\cdot\vec{B}=0$ (2)

# Some Theorems in Potential Theory

We have already established some elementary results, such as

$$\vec{F} = -\vec{\nabla} V \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \tag{1}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \Leftrightarrow \vec{\nabla} \cdot \vec{B} = 0 \tag{2}$$

To establish some other results we require a discussion of the Dirac $\delta$- function.

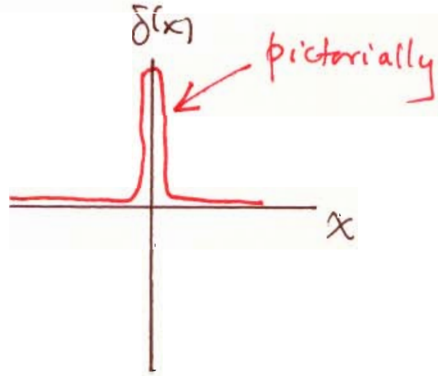
## Review of the Dirac $\delta$-Function

In 1-dimension $\delta(x)$ is defined by

$$\boxed{\int_{-\infty}^{\infty} dx \; \delta(x) f(x) = f(0)} \tag{3}$$

Equivalently, define $\delta(x)$ by

$$\left.\begin{aligned} &\delta(x) = 0 \quad x \neq 0 \\ &\int_{-\infty}^{\infty} dx \, \delta(x) = 1 \end{aligned}\right\} \tag{4}$$




<u>IMPORTANT!!</u> $\delta(x)$ is <u>not</u> an ordinary mathematical function such as $e^{-x}$. A relation such as (3) must be understood as holding when $\delta(x)$ is integrated along with a smooth convergent test function such as $e^{-x^2}$, which vanishes as $x \to \pm\infty$. With this understanding <u>the following relations hold when appearing under an integral with $f(x)$:</u>

## Useful Relations Involving $\delta(x)$:

(a) $\delta(x) = \delta(-x)$

(b) $\delta'(x) = -\delta'(-x)$

(c) $x\delta(x) = 0$

(d) $x\delta'(x) = -\delta(x)$ (5)

(e) $\delta(ax) = \frac{1}{|a|}\delta(x)$ ; $a$ = constant

(f) $\delta(x^2-a^2) = \frac{1}{|2a|}\left[\delta(x-a)+\delta(x+a)\right]$

(g) $\int dx\, \delta(x-a)\delta(x-b) = \delta(a-b)$

(h) $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$

(i) $\delta[g(x)] = \sum_i \frac{1}{|\partial g/\partial x|_{x_i}}\delta(x-x_i)$ ; $\boxed{g(x_i)=0\text{, so } x_i \text{ are the roots of } g(x)}$

You will be asked to establish these for homework; here we illustrate with 2 examples:

(5d): $$\int_{-\infty}^{\infty} dx\,[x\delta'(x)]f(x) \equiv \int_{-\infty}^{\infty} dx\, x\left[\frac{d}{dx}\delta(x)\right]f(x) = \int_{-\infty}^{\infty} dx\, \underbrace{xf(x)}_{u}\cdot\underbrace{\frac{d}{dx}\delta(x)}_{dV} \qquad (6)$$

$$= \cancel{xf(x)\delta(x)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\cdot\delta(x)\cdot\underbrace{\frac{d}{dx}\left[xf(x)\right]}_{f(x)+xf'(x)} = -\int_{-\infty}^{\infty} dx\,\delta(x)f(x) - \int_{-\infty}^{\infty} dx\cdot \underbrace{x\delta(x)}_{0} f'(x)$$

Hence: $$\int_{-\infty}^{\infty} dx\left[x\delta'(x)\right]f(x) = \int_{-\infty}^{\infty} dx\left[-\delta(x)\right]f(x) \Rightarrow \underbrace{x\delta'(x) \sim -\delta(x)}_{\text{under an integral!!}} \qquad (7)$$