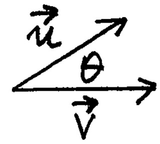
BRIEF REVIEW OF VECTORS

Scalar product: $\vec{u}, \vec{v} \Rightarrow \vec{u}\cdot\vec{V} = |\vec{u}||\vec{V}|\cos\theta$ (1)



Also: We can write

$$\vec{V} = V_1\hat{x} + V_2\hat{y} + V_3\hat{z}$$
$$\vec{u} = u_1\hat{x} + u_2\hat{y} + u_3\hat{z} \qquad (2)$$

Then:

$$\vec{u}\cdot\vec{V} = u_1V_1 + u_2V_2 + u_3V_3 = \sum_i u_iV_i \equiv u_iV_i \qquad (3)$$

Einstein Summation convention

$$(\vec{u}\cdot\vec{V})(\vec{S}\cdot\vec{T}) = u_iV_iS_jT_j \quad \text{etc.}$$ ← more later

Example of Scalar Product: Energy Conservation

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} \Rightarrow \vec{F}\cdot\vec{V} = m\frac{d\vec{V}}{dt}\cdot\vec{v} = m\frac{dV_i}{dt}V_i \qquad (4)$$

$$\therefore \vec{F}\cdot\vec{V} = m\frac{d}{dt}\underbrace{\left(\tfrac{1}{2}V_iV_i\right)}_{\frac{1}{2}V^2} = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) \qquad (5)$$

$$\int_a^b \vec{F}\cdot\vec{V}\,dt = \int_a^b dt\left[\frac{d}{dt}\underbrace{\left(\tfrac{1}{2}mv^2\right)}_{KE}\right] = KE(b) - KE(a) \qquad (6)$$

$$\int_a^b F\cdot\frac{d\vec{r}}{dt}\,dt = \int_a^b \vec{F}\cdot d\vec{r} = -\left(PE(b) - PE(a)\right) \qquad (7)$$

Hence finally: $\boxed{KE(a) + PE(a) = KE(b) + PE(b)}$ (8)

ENERGY CONSERVATION

Change of Basis & Orthogonal Transformations:

Although we describe a vector by its components relative to some specific coordinate system, <u>the choice of any particular system is arbitrary</u>. This implies that we must understand how vectors appear when described in different coordinate systems.

Consider 2 Cartesian coordinate systems K & K′ with the same origin. Then

$$\vec{V} = V_1\hat{x} + V_2\hat{y} + V_3\hat{z} \qquad \text{in } K$$
$$\vec{V} = V_1'\hat{x}' + V_2'\hat{y}' + V_3'\hat{z}' \qquad \text{in } K' \tag{9}$$

Note: $$\vec{V}\cdot\hat{x} = V_1 \underbrace{\hat{x}\cdot\hat{x}}_{=1} + V_2 \underbrace{\hat{y}\cdot\hat{x}}_{=0} + V_3 \underbrace{\hat{z}\cdot\hat{x}}_{=0} = V_1 \tag{10}$$

Hence we can also write: $$\vec{V} = (\vec{V}\cdot\hat{x})\hat{x} + (\vec{V}\cdot\hat{y})\hat{y} + (\vec{V}\cdot\hat{z})\hat{z}$$
$$\vec{V} = (\vec{V}\cdot\hat{x}')\hat{x}' + (\vec{V}\cdot\hat{y}')\hat{y}' + (\vec{V}\cdot\hat{z}')\hat{z}' \tag{11}$$

[Later we will introduce <u>DIRAC</u> <u>NOTATION</u> which allows us to write

$$|V\rangle = \sum_n |n\rangle\langle n|V\rangle = \sum_n |n\rangle V_n$$]

Returning to (11), let $\vec{V} = \hat{x}', \hat{y}',$ or $\hat{z}'$. This allows us to relate K & K′:

$$\hat{x}' = (\hat{x}'\cdot\hat{x})\hat{x} + (\hat{x}'\cdot\hat{y})\hat{y} + (\hat{x}'\cdot\hat{z})\hat{z}$$
$$\hat{y}' = (\hat{y}'\cdot\hat{x})\hat{x} + (\hat{y}'\cdot\hat{y})\hat{y} + (\hat{y}'\cdot\hat{z})\hat{z} \tag{12}$$
$$\hat{z}' = (\hat{z}'\cdot\hat{x})\hat{x} + (\hat{z}'\cdot\hat{y})\hat{y} + (\hat{z}'\cdot\hat{z})\hat{z}$$

The 9 quantities in (...) are the <u>direction</u> <u>cosines</u> between the various 6 axes. We will show shortly that not all of these 9 quantities are independent. Eq. (11) can be written in matrix notation:

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} \hat{x}'\cdot\hat{x} & \hat{x}'\cdot\hat{y} & \hat{x}'\cdot\hat{z} \\ \hat{y}'\cdot\hat{x} & \hat{y}'\cdot\hat{y} & \hat{y}'\cdot\hat{z} \\ \hat{z}'\cdot\hat{x} & \hat{z}'\cdot\hat{y} & \hat{z}'\cdot\hat{z} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad (13)$$

$$\hookrightarrow R \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad (14)$$

Briefly:

$$x' = R\,x \quad ; \quad R \equiv (a_{ij})$$

To show that the a_{ij} are not all linearly independent consider

$$\hat{x}'\cdot\hat{x}' = 1 = \underbrace{(\hat{x}'\cdot\hat{x})^2}_{a_{11}^2}\underbrace{\hat{x}\cdot\hat{x}}_{1} + \underbrace{(\hat{x}'\cdot\hat{y})^2}_{a_{12}^2}\underbrace{\hat{y}\cdot\hat{y}}_{1} + \underbrace{(\hat{x}'\cdot\hat{z})^2}_{a_{13}^2}\underbrace{\hat{z}\cdot\hat{z}}_{1} \qquad (15)$$

Hence: $\hat{x}'\cdot\hat{x}' = 1 = a_{11}^2 + a_{12}^2 + a_{13}^2$ (all other terms vanish) (16)

This can be written as: $\sum_{n=1}^{3} a_{1n}^2 = 1$ (17)

More generally (including the other relations of this form)

$$\boxed{\sum_{n=1}^{3} a_{in}^2 = 1 \qquad i = 1, 2, 3} \qquad (18)$$

Similarly, Consider the off-diagonal matrix elements:

$$\hat{x}' \cdot \hat{y}' = 0 = a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = \sum_{n=1}^{3} a_{1n}a_{2n} \qquad (19)$$

There are 3 relations of this form.

All of the relations in (18) & (19) can be subsumed into one equation:

$$\boxed{\sum_{n=1} a_{in}a_{jn} = \delta_{ij} \equiv a_{in}a_{jn}} \qquad (20)$$

Einstein summation convention

$\delta_{ij} = 0 \quad i \neq j$
$\delta_{ij} = 1 \quad i = j$

Eq. (20) comprises 6 equations among the 9 a_{ij}, which leaves only 3 of the a_{ij} as independent.

One can further constrain the transformation matrix $R = (a_{ij})$ by noting that (20) is also satisfied by an inversion described by the matrix

$$P = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \Rightarrow a_{ij} = -\delta_{ij} \qquad (21)$$

$\hookrightarrow a_{11} = a_{22} = a_{33} = -1 \;;\; a_{ij} = 0 \quad i \neq j$

Note that (21) $\Rightarrow \sum_n a_{in}a_{jn} = \sum_n (-\delta_{in})(-\delta_{jn}) = \delta_{ij}$ ✓ (22)

If we want to exclude inversion then we can note that with the additional condition $\det R = +1$ we get what we want:

Hence finally:

$$\boxed{\begin{array}{l} R = (a_{ij}) \text{ where} \\ \delta_{ij} = a_{in}a_{jn} \\ \det R = +1 \end{array}} \qquad (23)$$

Transformation of Arbitrary Vectors Under Rotations

Once the a_{ij} are determined, which fix a specific rotation, we can then determine how any other vector transforms:

$$\vec{V} = V_1'(\hat{x}') + V_2'(\hat{y}') + V_3'(\hat{z}') \tag{24}$$

$$= V_1'(a_{11}\hat{x} + a_{12}\hat{y} + a_{13}\hat{z}) + V_2'(a_{21}\hat{x} + a_{22}\hat{y} + a_{23}\hat{z}) + V_3'(a_{31}\hat{x} + a_{32}\hat{y} + a_{33}\hat{z})$$

$$= \underbrace{(a_{11}V_1' + a_{21}V_2' + a_{31}V_3')}_{\equiv V_1}\hat{x} + \underbrace{(a_{12}V_1' + a_{22}V_2' + a_{32}V_3')}_{\equiv V_2}\hat{y} + \underbrace{(a_{13}V_1' + a_{23}V_2' + a_{33}V_3')}_{\equiv V_3}\hat{z} \tag{25}$$

Hence: $V_1 = a_{11}V_1' + a_{21}V_2' + a_{31}V_3'$; etc.

More generally: $$\boxed{V_i = a_{ni} V_n'} \tag{26}$$

This relation can be inverted by writing

$$\sum_i a_{mi} V_i = \sum_i \sum_n \underbrace{a_{mi}\, a_{ni}}_{\delta_{mn}} V_n' = V_m' \tag{27}$$

→ See comment next page

Hence finally, $$\boxed{V_m' = a_{mi} V_i} \tag{28}$$

Using (28) we can define a (3-dimensional) vector V_i $(i=1,2,3)$ as an ordered triplet of numbers which transform as in (28) under a rotation defined by (23) above.

Comment on the Orthogonality Relation in (27):

In (23) we write

$$a_{in}a_{jn} \equiv \sum_n a_{in}a_{jn} = \delta_{ij} \qquad (23)$$

If we rename the indices so that $i \to m$ $j \to n$ $n \to i$ then Eq. (23) becomes:

$$a_{mi}\,a_{ni} = \delta_{mn} \qquad (29)$$

which is the expression used in (27) above

Using the orthogonality relation (29) in this form we can prove that the scalar product of 2 vectors $\vec{u}$ and $\vec{v}$ is invariant under rotations:

$$\vec{u}\cdot\vec{v} = u_i v_i \overset{(26)}{=} (a_{mi}u_m')(a_{ni}v_n') = \underbrace{(a_{mi}a_{ni})}_{\delta_{mn}}u_m'v_n' = u_n'v_n' \equiv \vec{u}'\cdot\vec{v}' \checkmark \qquad (30)$$

This is a simple example of the application of a general principle that the laws of physics must be expressed solely in terms of quantities which have well-defined behavior under transformations such as rotations. These include scalars, vectors, and tensors T_{ij} which transform as

$$T'_{mn} = a_{mi}\,a_{nj}\,T_{ij}$$

More later when we discuss tensor analysis.

Orthogonality Relations in 2-Dimensions

We illustrate the previous results by explicitly exhibiting the details of the 2-dimensional case: We have

$$R = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad (1)$$

Orthogonality $\Rightarrow \quad a_{in} a_{jn} = \sum_{n=1}^{2} a_{in} a_{jn} = \delta_{ij} \Rightarrow \qquad (2)$

$i=1 \quad j=1 \qquad a_{11} a_{11} + a_{12} a_{12} = a_{11}^2 + a_{12}^2 = \delta_{11} = 1 \qquad (3)$

$i=1 \quad j=2 \qquad a_{11} a_{21} + a_{12} a_{22} = \delta_{12} = 0 \qquad (4)$

$i=2 \quad j=1 \qquad a_{21} a_{11} + a_{22} a_{12} = \delta_{21} = 0 \qquad (5)$

$i=2 \quad j=2 \qquad a_{21} a_{21} + a_{22} a_{22} = \delta_{22} = 1 \qquad (6)$

$$a_{21}^2 + a_{22}^2 = 1$$

We see immediately that Eqs. (3)-(6) are solved by

$$a_{11} = a_{22} = \cos\theta \qquad a_{12} = -a_{21} = \sin\theta \qquad (7)$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow \det R = \cos^2\theta + \sin^2\theta = +1 \qquad (8)$$

If we did not wish to impose the condition $\det R = +1$, then Eqs. (3)-(6) could also have been solved by

$$R' = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad (9)$$

Vector Product ($\equiv$ cross product = outer product)

Given $\vec{u}, \vec{v}$ define $\vec{w} = \vec{u} \times \vec{v}$ where

a) $|\vec{w}| = |\vec{u}||\vec{v}| \sin(\vec{u}, v)$ b) $\hat{w} = \perp$ to $\vec{u}, \vec{v}$ with sense given by r.h. rule (1)

examples: $\hat{x} \times \hat{y} = \hat{z}$ $\vec{L} = \vec{r} \times \vec{p}$ (2)

Equivalently: $\vec{u} \times \vec{v} = \hat{x}(u_2 v_3 - u_3 v_2) + \hat{y}(u_3 v_1 - u_1 v_3) + \hat{z}(u_1 v_2 - u_2 v_1)$ (3)

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \qquad u_1 = u_x, \text{ etc.}$$

this relation will be useful later in studying properties of determinants.

To prove various identities involving vector products it is convenient to introduce the symbol (= permutation symbol)

$$\boxed{\epsilon_{ijk}} \quad (i, j, k = 1, 2, 3)$$

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1 \qquad (4)$$

$$\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$$

all others = 0 (e.g. $\epsilon_{113} = 0$)

Then $(\vec{w})_i \equiv w_i = \epsilon_{ijk} u_j v_k$ (summation understood) (5)

Check: $w_x \equiv w_1 = \epsilon_{1jk} u_j v_k = \underbrace{\epsilon_{123}}_{+1} u_2 v_3 + \underbrace{\epsilon_{132}}_{-1} u_3 v_2 = u_2 v_3 - u_3 v_2$ ✓ (6)

As we will discuss later, ϵ_{ijk} is actually an antisymmetric 3rd rank tensor. We now show that it satisfies the following important identity:

$$\boxed{\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im}} \quad \text{"}\delta_{il} - \delta_{im}\text{"} \qquad (7)$$

dummy index (can be replaced by any other index not in use)

free index

This identity can be established by choosing, in turn, all possible values for i, j, l, m. However, it can be established more simply by noting that since the only possible values of ϵ_{ijk} are $\pm 1, 0$ the right-hand side (r.h.s.) of (7) must be expressible in terms of δ_{il} ... etc. whose components are also $+1, 0$. Since $\epsilon_{ijk} = -\epsilon_{jik}$ and $\epsilon_{klm} = -\epsilon_{kml}$ the only combination of Kronecker δ-functions which can enter is

$$\epsilon_{ijk}\,\epsilon_{klm} = \underset{\uparrow \text{ constant}}{C}\left(\delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im}\right) \qquad (8)$$

Since $|\epsilon_{ijk}| = |\epsilon_{klm}| = 1, 0 \Rightarrow |C| = 1 = C = \pm 1$ (ϵ_{ijk} = real)

Finally the sign of C is fixed by taking any one example:

$$\epsilon_{12k}\,\epsilon_{k12} = \sum_k \epsilon_{12k}\,\epsilon_{k12} = \underset{0}{\underset{\shortparallel}{\epsilon_{121}\,\epsilon_{112}}} + \underset{0}{\underset{\shortparallel}{\epsilon_{122}\,\epsilon_{212}}} + \underset{+1}{\epsilon_{123}}\,\underset{+1}{\epsilon_{312}} = +1 \qquad (9)$$

$$= C(\delta_{11}\delta_{22} - \delta_{21}\delta_{12}) = C \Rightarrow C = +1 \checkmark \qquad (10)$$