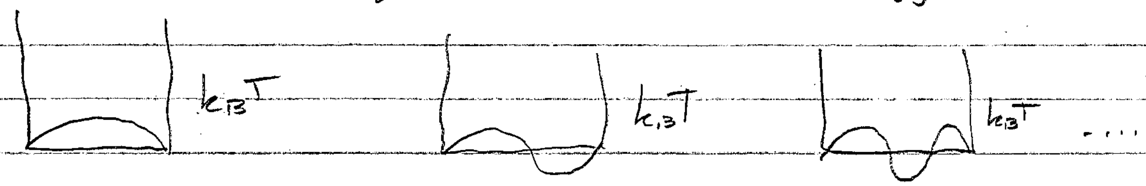
7.4 Blackbody Radiation

Thermodynamics of electromagnetic field

Classical E+M plus thermo $\rightarrow$ ultraviolet catastrophe
Every classical mode has $\frac{1}{2}k_BT + \frac{1}{2}k_BT$ energy.
There are an ∞ of modes $\Rightarrow$ ∞ E+M energy in a cavity



From Quantum, if $hf \ll k_BT$, then should get classical result. If $hf > k_BT$, result is wildly different.

Classically, each mode behaves like a harmonic oscillator

QM $\rightarrow$ energies are quantized $E_n = 0hf, 1hf, 2hf, \ldots$

Can find $\bar{E}$ in mode using $\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$

$$Z = 1 + e^{-\beta hf} + e^{-2\beta hf} + \ldots = 1 + (e^{-\beta hf})^1 + (e^{-\beta hf})^2 + (e^{-\beta hf})^3 + \ldots$$

$$= \frac{1}{1-e^{-\beta hf}}$$

$$\Rightarrow \quad \bar{E} = -(1-e^{-\beta hf})\left(- - - \frac{hf\, e^{-\beta hf}}{(1-e^{-\beta hf})^2}\right) = \frac{hf\, e^{-\beta hf}}{1-e^{-\beta hf}}$$

$$\bar{E} = hf/(e^{\beta hf}-1) = hf/(e^{hf/k_BT}-1)$$

Suppose $\frac{hf}{k_BT} \ll 1$, $e^{hf/k_BT} - 1 = \frac{hf}{k_BT}$ $\Rightarrow \bar{E} = k_BT$ ✓

Suppose $hf/k_BT \gg 1$ $e^{hf/k_BT} - 1 = e^{\frac{hf}{k_BT}}$ $\Rightarrow \bar{E} = hf\, e^{-hf/k_BT}$

Note $\bar{E} = hf\,\bar{n}_{pl}$ $\Rightarrow$ $\bar{n}_{pl} = \frac{1}{e^{hf/k_BT}-1} = \frac{1}{e^{\beta\varepsilon}-1}$

$\Rightarrow$ Interpretation is $E_n = 0hf, 1hf, 2hf, \ldots nhf \ldots$
n is the number of photons, the photons are bosons with chemical potential $\mu = 0$!

Now we need to figure out how to count the modes.

Suppose you have a cube, the modes can be indexed by j_x, j_y, j_z (book uses n_x, n_y, n_z)

$$\sin\left(\frac{\pi j_x}{L} x\right) \sin\left(\frac{\pi j_y}{L} y\right) \sin\left(\frac{\pi j_z}{L} z\right) \qquad j_x = 1, 2, \ldots;\ j_y = 1, 2, \ldots; \ldots$$

Remember from E+M $\quad k = \sqrt{k_x^2 + k_y^2 + k_z^2} \qquad k_x = \frac{\pi j_x}{L},\ k_y = \frac{\pi j_y}{L} \ldots$

and $\omega = kc \Rightarrow f = kc/(2\pi)$

$$\varepsilon = hf = \frac{hkc}{2\pi} = \frac{hc}{2L}\sqrt{j_x^2 + j_y^2 + j_z^2} = \frac{hcj}{2L}$$

where $j \equiv \sqrt{j_x^2 + j_y^2 + j_z^2}$

Just like the ideal gas we can get U by converting sums to integrals

$$U = \overset{\text{polarization}}{2} \sum_{j_x j_y j_z} \frac{hc}{2L}\sqrt{j_x^2+j_y^2+j_z^2}\ \bar{n}_{Pl}\left(\frac{hc}{2L}\sqrt{j_x^2+j_y^2+j_z^2}\right)$$

$$= \frac{hc}{L}\,\frac{1}{8}\,4\pi \int_0^\infty j\,\frac{1}{e^{\frac{hc}{2Lk_BT}j}-1}\,j^2\,dj$$

Convert the j integral to ε integral: $j = \frac{2L}{hc}\varepsilon$

$$U = \frac{hc}{2L}\,\pi\left(\frac{2L}{hc}\right)^4 \int_0^\infty \frac{\varepsilon^3}{e^{\varepsilon/k_BT}-1}\,d\varepsilon$$

$$\Rightarrow \frac{U}{V} = \frac{8\pi}{(hc)^3}\int_0^\infty \frac{\varepsilon^3}{e^{\varepsilon/k_BT}-1}\,d\varepsilon$$

One interpretation: $u(\varepsilon) = \frac{8\pi}{(hc)^3}\frac{\varepsilon^3}{e^{\frac{\varepsilon}{k_BT}}-1}$ is the energy density per unit photon energy

To evaluate $\frac{U}{V}$ define $\varepsilon = k_BT \cdot x$

$$\Rightarrow \frac{U}{V} = \frac{8\pi}{(hc)^3}(k_BT)^4 \int_0^\infty \frac{x^3}{e^x-1}\,dx$$

Important point is that $\frac{U}{V} \propto T^4$

Appendix A: $\int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{\pi^4}{15} \Rightarrow \boxed{\frac{U}{V} = \frac{8\pi^5}{15(hc)^3}(k_BT)^4}$

Prob 7.37 maximum of $\frac{x^3}{e^x - 1}$

$\frac{d}{dx}() = \frac{3x^2}{e^x - 1} - \frac{x^3 e^x}{(e^x-1)^2} = 0 \Rightarrow \frac{x e^x}{e^x - 1} = 3$

$\Rightarrow \frac{x}{3} = 1 - e^{-x} \Rightarrow e^{-x} = 1 - x/3$

$e^{-2.82} = 0.0596 \qquad 1 - \frac{2.82}{3} = 0.0600$

7.38 How does $u(\varepsilon)$ change with T? Suppose T doubles?

ε of maximum $\varepsilon = 2.82\, k_B T$

height? $u(\varepsilon) = \frac{8\pi}{(hc)^3} (k_B T)^3 \frac{(\varepsilon/k_B T)^3}{e^{\varepsilon/k_B T} - 1}$

the height $\propto T^3$

The total energy in cavity from photons compared to 1 mole monatomic gas? $V = 1\,m^3$, $T = 300k$

$U = \frac{3}{2} N k_B T = \frac{3}{2} RT = 3.7\,kJ$

$U = 1m^3 \frac{8\pi^5}{(6.626 \times 10^{-34} Js \; 3 \times 10^8 m/s)^3} (1.38 \times 10^{-23} \frac{J}{k} 300k) = 9.2 \times 10^{-5} J$

much less!

What is heat capacity $C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{32\pi^5}{15}\left(\frac{k_B T}{hc}\right)^3 V k_B$

The entropy $S(T) = \int_0^T \frac{C_V(T')}{T'} dT' = \frac{32\pi^5}{45} V k_B \left(\frac{k_B T}{hc}\right)^3$

$= \frac{4}{3} U/T$

Entropy $1m^3$, $T = 300k$ $\quad S = \frac{4}{3} \frac{9.2 \times 10^{-5} J}{300k} = 4.1 \times 10^{-7} \frac{J}{K}$

very small!

The microwave background of the universe looks like blackbody radiation at 2.735K!

Look at fit in Fig 7.20

Prob 7.45 What pressure is exerted on walls from photons?

$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$ Problem! $S(T) = \frac{32\pi^5}{45} V k_B \left(\frac{k_B T}{hc}\right)^3$

As you change V, you must also change T to keep S constant.

So I don't need to write long constants

$S \equiv a \cdot V \cdot T^3$ $\quad U \equiv b \cdot V \cdot T^4$

$\Rightarrow T = (S/aV)^{1/3}$ $\quad U = b\left(\frac{S}{a}\right)^{1/3} V^{-1/3}$

$P = -\frac{\partial U}{\partial V} = \frac{1}{3} b \left(\frac{S}{a}\right)^{1/3} V^{-4/3} = \frac{1}{3}\frac{U}{V} = \frac{8\pi^5}{45}\left(\frac{k_B T}{hc}\right)^3 k_B T$

At 1500 K $P = \frac{8\pi^5}{45}\left(\frac{1.38\times10^{-23}\frac{J}{K} 1500\,K}{6.626\times10^{-34}\,Js\; 3\times10^8\,m/s}\right)^3 1.38\times10^{-23}\frac{J}{K} 1500\,K$

$= 0.0013\ Pa$

Compare to 1 atm from air (factor of $\sim 10^8$

At the center of the sun $T = 1.5\times10^7\,K$

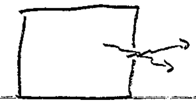
$P = 0.0013\ Pa\ (10^4)^4 = 1.3\times10^{13}\ Pa$

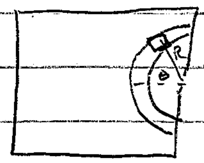
From e^- + protons $P = 2\times10^5\frac{kg}{m^3}\ 1.38\times10^{-23}\frac{J}{K}\ 1.5\times10^7 K\ \frac{1}{1.67\times10^{-27}kg}$

$= 2.5\times10^{16}\ Pa$

Still ~ 1000x more than photons

An important idea is the power escaping through a hole of area A. (Important $A^{1/2}$ is much greater than average wave length.





Energy in little volume

$$E = \frac{U}{V} \cdot R^2 \sin\theta \, d\theta \, d\phi \, dR$$

$$= \frac{U}{V} R^2 \sin\theta \, d\theta \, d\phi \, c \, \delta t$$

Probability energy in volume will go directly through hole $\frac{A \cos\theta}{4\pi R^2}$

$$\Rightarrow \frac{\text{Power}}{A} = \frac{U}{V} c \cancel{R^2} \sin\theta \, d\theta \, d\phi \frac{\cos\theta}{4\pi \cancel{R^2}}$$

$$= \frac{U}{4\pi V} c \cos\theta \sin\theta \, d\theta \, d\phi$$

Integrate over ϕ and θ

$$\text{Power}/A = \frac{U}{4\pi V} c \int_0^{2\pi} \left[\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \right] d\phi$$

$$= \frac{U}{4\pi V} c \; 2\pi \frac{1}{2} \sin^2\theta \Big|_0^{\pi/2} = \frac{U}{V} \frac{c}{4}$$

This should make sense $\frac{\text{Power}}{A} \propto$ energy density x speed

$$\frac{\text{Power}}{A} = \frac{2\pi^5}{15} \frac{(k_B T)^4}{h^3 c^2} = \sigma T^4$$

σ = Stefan Boltzman const. $= 5.67 \times 10^{-8} \frac{W}{m^2 K^4}$

Radiation from object

$$\text{Power} = \sigma \cdot e \cdot A \cdot T^4$$

emissivity (e); surface area of object (A)

$0 < e < 1$ $\quad$ e = fraction of photons absorbed/emitted

e = 1 for blackbody

e = 0 for perfectly reflecting body

The solar radiation at earth orbit is $1370 \frac{W}{m^2}$.
What is T of sun?
$P_{sun} = P_{earth} \frac{4\pi R_{s-e}^2}{4\pi R_s^2} = 1370 \frac{W}{m^2} \left(\frac{1.5\times10^{11} m}{7.0\times10^{8} m}\right)^2 = 6.3\times10^7 \frac{W}{m^2}$
$\Rightarrow T = (6.3\times10^7 W/m^2 / 5.67\times10^{-8} \frac{W}{m^2K^4})^{1/4} = 5800\ K$

What is T of earth if earth is blackbody?

$P_{hitting\ earth} = P_{radiated} \Rightarrow 1370 \frac{W}{m^2} \pi R_{earth}^2 = \sigma T^4 \cdot 4\pi R_{earth}^2$
$\Rightarrow T = [1370\ W/m^2 / 4\times 5.67\times10^{-8} \frac{W}{m^2K^4}]^{1/4} = 280\ K$

Actual T of earth ~ 288 K

Book says ~ 30% sunlight reflected
$\Rightarrow P_{hitting\ earth} \times 0.3 \Rightarrow T = 255 K$
I think this isn't correct because $P_{radiated}$
will also be multiplied by 0.3.

Greenhouse gases leaves $P_{hitting\ earth}$ ~ unchanged
but $P_{radiated} = e\ \sigma T^4\ 4\pi R_{earth}^2$
decreases because of changed wavelength
Wavelength from sun ~ 20x smaller than
radiated from earth