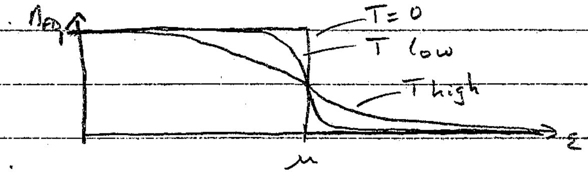
7.3 Degenerate Fermi Gas

Electrons in a metal, protons/neutrons inside nucleus, electrons in white dwarf star, neutrons in neutron star...

Example: Ideal gas where $\frac{V}{N} << v_Q$. Electrons at room temp $v_Q \simeq (4\,nm)^3$. But in metal there is 1 electron per $\sim (0.2\,nm)^3$

Examine n_{FD} vs T



Notice at $T=0$, n_{FD} is a step function $n_{FD} = 1$ for $\varepsilon < \mu$ and 0 for $\varepsilon > \mu$
Why? At $T=0$, the system is in its ground state. For Fermions, order the single particle states from low energy to high energy. There are N particles $\Rightarrow$ the first N states have occupancy 1, all others have occupancy 0.

The energy $\varepsilon_N = \mu(T=0) \equiv \varepsilon_F \rightarrow$ the Fermi Energy.
When a gas of fermions mostly occupies states with $\varepsilon < \varepsilon_F$
Degenerate Fermi Gas

Notice $\left(\frac{\partial U}{\partial N}\right)_{S,V} = \mu$ at $T=0$. $U(N+1) = U(N) + \varepsilon_F = U(N) + \mu$

Particles in Box

$E = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2) \Rightarrow \varepsilon_F = \frac{h^2 n_{max}^2}{8mL^2}$ (Remember $n_x = 1, 2, 3, \ldots, n_y = 1, 2, \ldots$)
But this doesn't help unless you know how n_{max} depends on N

Think of $n_x^2 + n_y^2 + n_z^2$ as a spherical radius
$N = \#$ of states with $n_x^2 + n_y^2 + n_z^2 < n_{max}^2$
This is like finding volume of $\frac{1}{8}$ of sphere
For spin ½ every n_x, n_y, n_z is 2 states (↑ and ↓)
$N = 2 \times \frac{1}{8}\frac{4\pi}{3} n_{max}^3 \Rightarrow n_{max} = (3N/\pi)^{1/3}$

$$\Rightarrow \varepsilon_F = \frac{h^2}{8mL^2}\left(\frac{3N}{\pi}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3N}{\pi L^3}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}$$

To find the total internal energy $U = 2 \sum_{n_x n_y n_z} \frac{h^2(n_x^2+n_y^2+n_z^2)}{8mL^2}$

$U \cong 2 \frac{h^2}{8mL^2} \frac{1}{8} \int_0^{n_{max}} n^2 \, 4\pi n^2 \, dn$

$= \frac{h^2}{8mL^2} \pi \, n_{max}^5/5 = \frac{h^2 n_{max}^2}{8mL^2} \frac{\pi}{5} n_{max}^3 = \varepsilon_F \cdot \frac{\pi}{5} \frac{3N}{\pi} = \frac{3}{5} N \varepsilon_F$

$\Rightarrow$ The average energy per particle is $\frac{3}{5}\varepsilon_F$

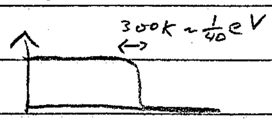
Estimate for metal $\frac{N}{V} \approx \frac{1}{(0.2\times10^{-9}\,m)^3} = 1.25\times10^{29} \frac{1}{m^3}$

$\varepsilon_F = \frac{(6.626\times10^{-34}\,Js)^2}{8\times9.11\times10^{-31}\,kg}\left(\frac{3}{\pi} 1.25\times10^{25} \frac{1}{m^3}\right)^{2/3} = 1.46\times10^{-18}\,J \approx 9\,eV \approx 10^5\,K\,k_B$

Room temperature $\sim 1/40$ eV ~ 300 K

$\varepsilon_F >> k_B T$

$\Rightarrow$ For electrons in metal



The degeneracy pressure $P = -\left(\frac{\partial U}{\partial V}\right)_{S,N} = -\frac{3}{5}N \frac{\partial \varepsilon_F}{\partial V} = -\frac{3}{5}N\left(-\frac{2}{3}\right)\frac{\varepsilon_F}{V} = \frac{2}{3}\frac{U}{V}$

This is mostly cancelled by the attractive electrostatic forces.

The Bulk Modulus $B = -V\left(\frac{\partial P}{\partial V}\right)_T = -V \frac{\partial}{\partial V}\left(\frac{C}{V^{5/3}}\right) = +\frac{5}{3}\frac{C}{V^{5/3}} = \frac{5}{3}\frac{2}{3}\frac{U}{V} = \frac{10}{9}\frac{U}{V}$

Prob 7.20 $\frac{N}{V} \cong 10^{32} \frac{1}{m^3}$ $T = 10^7$ K

$\varepsilon_F = \frac{(6.626\times10^{-34}\,Js)^2}{8\times9.11\times10^{-31}}\left(\frac{3}{\pi} 10^{32}\frac{1}{m^3}\right)^{2/3} = 1.26\times10^{-16}\,J \approx 800\,eV \sim 9\times10^6\,K\cdot k_B$

Since $\varepsilon_F \sim k_B T$, the electron gas is neither degenerate nor nearly classical

Prob 7.23 White Dwarf (show that electron degeneracy pressure can prevent star from collapsing)

a) Compute Gravitational PE. Imagine assembling from shell

$\Delta U = -\frac{Gm}{r}\,dm = -\frac{G}{r}\left(\frac{4}{3}\pi r^3 \rho\right) 4\pi r^2\,dr\,\rho$

$U_g = \int -\frac{G}{r}\left(\frac{4}{3}\pi r^3\right)\rho\, 4\pi r^2 dr\,\rho = -G\rho^2\left(\frac{4\pi}{3}\right)^2 \int_0^R r^4 dr = -G\rho\left(\frac{4\pi}{3}\right)^2 \frac{3}{5}R^5$

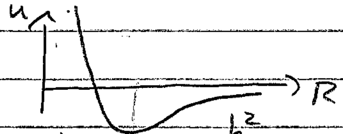
$= -\frac{3}{5}GM^2/R$

b) Guess that most of the KE is from nonrelativistic electrons

$U_{KE} = \frac{3}{5}N\varepsilon_F = \frac{3}{5}\frac{M}{(2m_p)}\frac{h^2}{8m_e}\left(\frac{3}{\pi}\frac{M}{2m_p \frac{4\pi}{3}R^3}\right)^{2/3}$

$= \left(\frac{3}{80}\left[\frac{9}{8\pi^2}\right]^{2/3}\right) h^2 (M/m_p)^{5/3}\frac{1}{m_e}\frac{1}{R^2} = (0.0088)\left(\frac{M}{m_p}\right)^{5/3}\frac{h^2}{m_e R^2}$

c) $U = U_g + U_{KE}$



$\frac{dU}{dR} \overset{?}{=} 0 = \frac{3}{5}\frac{GM^2}{R^2} - (0.0088)\left(\frac{M}{m_p}\right)^{5/3}\frac{2h^2}{m_e R^3}$

$\Rightarrow R = \frac{10}{3}(0.0088)\frac{1}{G}\frac{h^2}{M^{1/3}m_p^{5/3}m_e} = 0.029\frac{h^2}{Gm_e m_p^{5/3}}\frac{1}{M^{1/3}}$

Notice R decreases with $M^{1/3}$

d) $R = 0.029\frac{(6.626\times10^{-34}\,\mathrm{Js})^2}{6.673\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}\,(2\times10^{30}\,\mathrm{kg})^{1/3}(1.67\times10^{-27}\,\mathrm{kg})^{5/3}(9.11\times10^{-31}\,\mathrm{kg})} = 7.2\times10^6\,\mathrm{m} = 7200\,\mathrm{km}$!

$\rho = 2\times10^{30}\,\mathrm{kg}/\left(\frac{4\pi}{3}(7.2\times10^6\,\mathrm{m})^3\right) = 1.3\times10^9\,\mathrm{kg/m^3} \sim 1.3\times10^6\times$ denser than water

e) $\varepsilon_F = \frac{h^2}{8m_e}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3} = \frac{h^2}{8m_e}\left(\frac{3\rho}{\pi 2m_p}\right)^{2/3} = \frac{(6.626\times10^{-34}\,\mathrm{Js})^2}{8\times9.11\times10^{-31}\,\mathrm{kg}}\left(\frac{3}{\pi 2}\frac{1.3\times10^9\,\mathrm{kg/m^3}}{1.67\times10^{-27}\,\mathrm{kg}}\right)^{2/3} = 3.1\times10^{-14}\,\mathrm{J}$

$\Rightarrow T_F = \varepsilon_F/k_B = 2.3\times10^9\,K \gg$ temperature

f) From prob 7.22 $U_{KE} = \frac{3}{4}N\varepsilon_F = \frac{3}{4}N\frac{hc}{2}\left(\frac{3N}{\pi V}\right)^{1/3} = \frac{3}{8}hc\left(\frac{M}{2m_p}\right)^{4/3}\left(\frac{9}{4\pi^2R^3}\right)^{1/3}$

$= 0.091\,hc\,(M/m_p)^{4/3}\frac{1}{R}$

$\Rightarrow U = U_g + U_{KE} = \left[0.091\,hc\,(M/m_p)^{4/3} - \frac{3}{5}GM^2\right]/R$

If [ ] < 0 then the minimum is R = 0!

If [ ] > 0 then R increases until KE becomes nonrelativistic

g) The breakeven $0.091\,hc\left(\frac{M}{m_p}\right)^{4/3} = \frac{3}{5}GM^2$

$\Rightarrow M \gg \left[0.091\,hc\,\frac{5}{3G}\right]^{3/2}\frac{1}{m_p^2} = 3.4\times10^{30}\,\mathrm{kg}$

Some considerations relativistic $\varepsilon_F \sim mc^2 = 8.2\times10^{-14}\,\mathrm{J}$

The Fermi energy for sun is less than mc^2

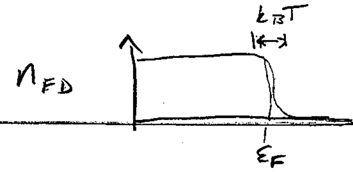
Best modern calculation ~ Stability mass ~ 1.4 solar masses

$\sim 3\times10^{30}\,\mathrm{kg}$

What happens? Star collapses until lower energy $p + e \to n + \nu_e$

Supernova → neutron star

Small T, but not T = 0



How does U increase with T?

Number of affected particles $\sim N \frac{k_B T}{\varepsilon_F}$

Energy acquired $\sim k_B T$

$\Rightarrow \quad U = \frac{3}{5} N \varepsilon_F + \text{Const}\; N \frac{k_B^2 T^2}{\varepsilon_F} = \frac{3}{5} N\varepsilon_F + \frac{\pi^2}{4} N \frac{k_B^2 T^2}{\varepsilon_F}$

$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2}{2} N k_B^2\, T/\varepsilon_F$

Prob 7.25 $\quad C_V/Nk_B = \frac{\pi^2}{2} \frac{k_B T}{\varepsilon_F} \simeq \frac{\pi^2}{2} \frac{1/40 \text{ eV}}{7 \text{ eV}} \simeq 0.017$

Vibrations contribut $C_V/Nk_B \simeq 3$

$\Rightarrow$ electrons contribute $\sim 0.5\%$ to heat capacity at room temp

Introduce the concept of density of states

Example Particles in Box: $U = 2 \times \frac{1}{8} 4\pi \int_0^\infty n_{FD}(\varepsilon(n))\, \varepsilon(n)\, n^2 dn$

with $\varepsilon(n) = \frac{h^2}{8mL^2} n^2$

Turn into integral over energy $\quad n = \sqrt{\frac{8mL^2}{h^2}} \sqrt{\varepsilon} \quad dn = \frac{1}{2}\sqrt{\frac{8mL^2}{h^2}} \frac{1}{\sqrt{\varepsilon}} d\varepsilon$

$\Rightarrow U = \int_0^\infty n_{FD}(\varepsilon)\, \varepsilon\, \pi \frac{8mL^2}{h^2} \varepsilon \frac{1}{2} \sqrt{\frac{8mL^2}{h^2}} \frac{1}{\sqrt{\varepsilon}} d\varepsilon \equiv \int_0^\infty n_{FD}(\varepsilon)\, \varepsilon\, g(\varepsilon) d\varepsilon$

$g(\varepsilon) = \frac{\pi}{2} \left(\frac{8mL^2}{h^2}\right)^{3/2} \sqrt{\varepsilon} = \frac{\pi}{2} \frac{(8m)^{3/2}}{h^3} V \sqrt{\varepsilon}$ is the density of states

units of g is $1/\text{energy}$

$g(\varepsilon) \cdot \Delta\varepsilon$ is # of states between ε and $\varepsilon + \Delta\varepsilon$

For $T=0 \quad N = \int_0^{\varepsilon_F} g(\varepsilon) d\varepsilon = \frac{\pi}{2} \left(\frac{(8m)^{3/2}}{h^3}\right) V \frac{2}{3} \varepsilon_F^{3/2}$

From Eq 7.39 $\quad \varepsilon_F = \frac{h^2}{8m} \left(\frac{3N}{\pi V}\right)^{2/3}$ ✓

For $T=0 \quad U = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon) d\varepsilon = \frac{\pi}{2} \frac{(8m)^{3/2}}{h^3} V \frac{2}{5} \varepsilon_F^{5/2} = \frac{3}{5} N \varepsilon_F$

Eq 7.42 $\quad U = \frac{3}{5} N \varepsilon_F$ ✓

How to deal with small T (not $T=0$)

$$N = \int_0^\infty g(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}+1}d\varepsilon \quad \text{and} \quad U = \int_0^\infty g(\varepsilon)\frac{\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}d\varepsilon$$

Use the left equation to determine μ; once you have μ, you can use the right equation to get

μ decreases with temperature because $g(\varepsilon)$ increases with ε (n_{FD} is smaller for $\varepsilon < \mu$ and larger for $\varepsilon > \mu$)
U increases with T because particles reach higher ε.

Sommerfeld gave first treatment of $\mu(T)$ and $U(T)$.

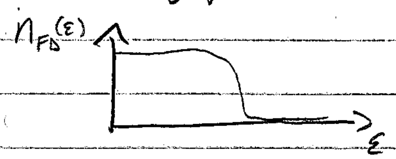
I will do general treatment for N and U

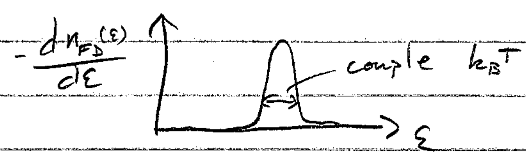
Need to do integral $A \equiv \int_0^\infty \frac{dF(\varepsilon)}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1}d\varepsilon$
where F is going to be related to $g(\varepsilon)$ or $\varepsilon g(\varepsilon)$

$$A = F(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\Big|_0^\infty + \int_0^\infty F(\varepsilon)\left(-\frac{d}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\right)d\varepsilon$$

For the functions we will use $F(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\Big|_0^\infty = 0$

The reason for integration by parts is that $-\frac{d}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1}$ is strongly peaked at $\varepsilon=\mu$





$$-\frac{d}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \beta\frac{e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)}+1)^2}$$

Define $x = \beta(\varepsilon-\mu) = \frac{\varepsilon-\mu}{k_BT} \Rightarrow \varepsilon = \mu + k_BTx$

$$\Rightarrow A = \int_{-\frac{\mu}{k_BT}}^{\infty} F(\mu+k_BTx)\,\cancel{\beta}\frac{e^x}{(e^x+1)^2}\,\cancel{k_BT}\,dx$$

Now we can examine some general features to make approximation

Small $T \Rightarrow \mu/k_BT >> 1 \Rightarrow e^{-\mu/k_BT} \cong 0$

Thus, the lower limit of integration can be moved to $-\infty$.

Also, since $k_BT << \mu$, make a Taylor series expansion

$$F(\mu + k_BTx) = F(\mu) + k_BTx\, F'(\mu) + \tfrac{1}{2}(k_BT)^2 x^2 F''(\mu) + \dots$$

Finally, $\frac{e^x}{(e^x+1)^2} = \frac{1}{(e^{x/2}+e^{-x/2})^2}$ is a symmetric function of x

Prob 7.30 Show A only depends on even powers of T.

$$A = \int_{-\infty}^{\infty} \left(F(\mu) + k_BT\,F'(\mu)\,x + \frac{(k_BT)^2}{2}F''(\mu)x^2 + \dots\right)\frac{1}{(e^{x/2}+e^{-x/2})^2}\,dx$$

all odd powers of x integrate to 0

$$\Rightarrow A = F(\mu)\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}\,dx + \frac{(k_BT)^2}{2}F''(\mu)\int_{-\infty}^{\infty}\frac{x^2e^x}{(e^x+1)^2}\,dx$$

$$\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}\,dx = -\frac{1}{e^x+1}\Big|_{-\infty}^{\infty} = -\frac{1}{e^{\infty}+1} - -\frac{1}{e^{-\infty}+1} = 0+1 = 1$$

$$\int_{-\infty}^{\infty}\frac{x^2e^x}{(e^x+1)^2}\,dx = 2\int_0^{\infty}\frac{x^2e^x}{(e^x+1)^2}\,dx = 2\int_0^{\infty}-\frac{d}{dx}\left[\frac{x^2}{e^x+1}\right] + \frac{2x}{e^x+1}\,dx$$

$$= 4\int_0^{\infty}\frac{x}{e^x+1}\,dx = 2\zeta(2) = \frac{\pi^2}{3} \quad \text{Eqs B.35 + B.45}$$

$$A = F(\mu) + \frac{\pi^2}{6}(k_BT)^2F''(\mu) + (\ )(k_BT)^4 \dots$$

Let's evaluate for $N = \int g(\varepsilon)\, n_{FD}(\varepsilon)\, d\varepsilon$

$$\frac{dF}{d\varepsilon} = g(\varepsilon) = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3}V\sqrt{\varepsilon} \Rightarrow F = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3}V\,\frac{2}{3}\varepsilon^{3/2}$$

$$\Rightarrow N = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3}V\,\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{6}(k_BT)^2\,\frac{\pi}{2}\frac{(8m)^{3/2}}{h^3}V\,\frac{1}{2\mu^{1/2}}$$

What a horrible expression!!

To get simplification note $\frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{3} = \frac{N}{\varepsilon_F^{3/2}}$

$$\Rightarrow N = N \cdot \left(\frac{\mu}{\varepsilon_F}\right)^{3/2} + \frac{\pi^2}{6}(k_B T)^2 \frac{3N}{4\mu^{1/2}\varepsilon_F^{3/2}}$$

$$1 = \left(\frac{\mu}{\varepsilon_F}\right)^{3/2} + \frac{\pi^2 (k_B T)^2}{8\mu^{1/2}\varepsilon_F^{3/2}} \Rightarrow \frac{\mu}{\varepsilon_F} = \left(1 - \frac{\pi^2}{8}\frac{(k_B T)^2}{\mu_{1/2}\varepsilon_F^{3/2}}\right)^{2/3}$$

$$\frac{\mu}{\varepsilon_F} \cong 1 - \frac{\pi^2}{8}\frac{(k_B T)^2}{\varepsilon_F^2}\frac{2}{3} = 1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2$$

For $U = \int \varepsilon g(\varepsilon) n_{FD}(\varepsilon) d\varepsilon$ $\quad \frac{dF}{d\varepsilon} = \varepsilon g(\varepsilon) = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \varepsilon^{3/2}$

$$\Rightarrow F = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{5}\varepsilon^{5/2}$$

$$U = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{5}\mu^{5/2} + \frac{\pi^2}{6}(k_B T)^2 \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{3}{2}\mu^{1/2}$$
$$= \frac{3}{5} N \frac{\mu^{5/2}}{\varepsilon_F^{3/2}} + \frac{\pi^2}{6}(k_B T)^2 \frac{9}{4} N \frac{\mu^{1/2}}{\varepsilon^{3/2}}$$

Plug in for $\mu = \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$

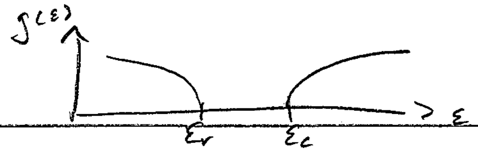
$$U = \frac{3}{5} N \varepsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]^{5/2} + \frac{3\pi^2}{8}(k_B T)^2 N \varepsilon_F^{-1}\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]^{1/2}$$

$$= \frac{3}{5} N \varepsilon_F \left[1 - \frac{5\pi^2}{24}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right] + \frac{3\pi^2}{8} N \varepsilon_F \left(\frac{k_B T}{\varepsilon_F}\right)^2$$

$$= \frac{3}{5} N \varepsilon_F + \frac{\pi^2}{4} N \varepsilon_F \left(\frac{k_B T}{\varepsilon_F}\right)^2$$

This matches the hand waving argument from start of section.

Prob 7.33



For $\varepsilon < \varepsilon_v$ $\quad g(\varepsilon) = g_0\,(\varepsilon_v - \varepsilon)^{1/2}$

$\varepsilon > \varepsilon_c$ $\quad g(\varepsilon) = g_0\,(\varepsilon - \varepsilon_c)^{1/2}$

a) $N = \int g(\varepsilon)\, n_{FD}(\varepsilon)\, d\varepsilon$ $\qquad g_0 = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V$

Remember $n_{FD}(\varepsilon)$ is symmetrical in the sense the decrease at an $\varepsilon < \mu$ is exactly the increase at the corresponding $\varepsilon > \mu$.

Since $g(\varepsilon)$ is symmetrical about $\varepsilon = \frac{\varepsilon_v + \varepsilon_c}{2}$,

$\mu = \frac{\varepsilon_v + \varepsilon_c}{2}$

b) The number in the conduction band

$$N_c = \int_{\varepsilon_c}^{\infty} g(\varepsilon)\, n_{FD}(\varepsilon)\, d\varepsilon = g_0 \int_{\varepsilon_c}^{\infty} \sqrt{\varepsilon - \varepsilon_c}\; n_{FD}(\varepsilon)\, d\varepsilon$$

$$= g_0 \int_0^{\infty} \sqrt{\varepsilon}\; n_{FD}(\varepsilon + \varepsilon_c)\, d\varepsilon$$

As before this integral can't be done analytically, but approximations will work when $\varepsilon_c - \varepsilon_v \gg k_BT$

$$n_{FD}(\varepsilon + \varepsilon_c) = \frac{1}{e^{\beta(\varepsilon - \mu + \varepsilon_c)} + 1} = \frac{1}{e^{\beta(\varepsilon_c - \mu)}\, e^{\beta\varepsilon} + 1} \cong e^{-\beta(\varepsilon_c - \mu)}\, e^{-\beta\varepsilon}$$

$$\Rightarrow N_c = g_0\, e^{-\beta(\varepsilon_c - \mu)} \int_0^{\infty} \sqrt{\varepsilon}\, e^{-\beta\varepsilon}\, d\varepsilon \qquad \varepsilon = x^2 \quad d\varepsilon = 2x\,dx$$

$$= g_0\, e^{-\beta(\varepsilon_c - \varepsilon_v)/2}\; 2\int_0^{\infty} x^2 e^{-\beta x^2} dx = g_0\, e^{-\frac{\varepsilon_c - \varepsilon_v}{2k_BT}}\, \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}}$$

$$= \frac{\pi}{2}\frac{(8mk_BT)^{3/2}}{h^3}\frac{\sqrt{\pi}}{2} V e^{-\frac{(\varepsilon_c - \varepsilon_v)}{2k_BT}} = \frac{2V}{v_Q}\, e^{-\frac{(\varepsilon_c - \varepsilon_v)}{2k_BT}}$$

where $v_Q = \left(\frac{h}{\sqrt{2\pi m k_BT}}\right)^3$ from Eq. 7.18

Our approximation in the integral will work when $e^{-(\varepsilon_c - \varepsilon_v)/2k_BT}$ is small

For Si $\quad \varepsilon_c - \varepsilon_v = 1.11\,eV \quad k_BT = \frac{1}{40}\,eV \Rightarrow e^{-22.2} = 2.3\times10^{-10}$ ✓

Prob 7.34 Asymmetric bands $\varepsilon < \varepsilon_v \quad g(\varepsilon) = g_{ov}(\varepsilon_v - \varepsilon)^{1/2}$

$\varepsilon > \varepsilon_c \quad g(\varepsilon) = g_{oc}(\varepsilon - \varepsilon_c)^{1/2}$

a) Suppose $g_{oc} > g_{ov}$ => if μ is in center, N will increase with T (because integrand in conduction band is >) to prevent This $\mu \to$ smaller with increasing T

b) $N_c = \int_{\varepsilon_c}^{\infty} g_{oc}(\varepsilon-\varepsilon_c)^{1/2} \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\, d\varepsilon \cong g_{oc}\int_0^\infty \sqrt{\varepsilon}\, e^{-\beta(\varepsilon_c-\mu)} e^{-\beta\varepsilon} d\varepsilon$

$= g_{oc} e^{-\beta(\varepsilon_c-\mu)} \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} = \frac{1}{2} g_{oc}\sqrt{\pi (k_B T)^3}\, e^{-\beta(\varepsilon_c-\mu)}$

c) $N_{holes} = \int_{-\infty}^{\varepsilon_v} g_{ov}(\varepsilon_v-\varepsilon)^{1/2}\left[1 - \frac{1}{e^{\beta(\varepsilon-\mu)}+1}\right] d\varepsilon$

$[\ ] = \frac{e^{\beta(\varepsilon-\mu)}}{e^{\beta(\varepsilon-\mu)}+1} = \frac{1}{e^{-\beta(\varepsilon-\mu)}+1}$

$N_{holes} = g_{ov}\int_{-\infty}^{0} (-\varepsilon)^{1/2} \frac{1}{e^{-\beta(\varepsilon_v-\mu)}e^{-\beta\varepsilon}+1}\, d\varepsilon = g_{ov} e^{\beta(\varepsilon_v-\mu)} \int_{-\infty}^{0} (-\varepsilon)^{1/2} e^{\beta\varepsilon} d\varepsilon$

$= g_{ov} e^{\beta(\varepsilon_v-\mu)} \frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} = \frac{1}{2} g_{ov}\sqrt{\pi(k_B T)^3}\, e^{\beta(\varepsilon_v-\mu)}$

d) $N_{holes} = N_c \Rightarrow g_{oc} e^{-\beta(\varepsilon_c-\mu)} = g_{ov} e^{\beta(\varepsilon_v-\mu)}$

$\Rightarrow e^{2\beta\left(\mu - \frac{\varepsilon_c+\varepsilon_v}{2}\right)} = \frac{g_{ov}}{g_{oc}} \Rightarrow \mu = \frac{\varepsilon_c+\varepsilon_v}{2} + \frac{1}{2}k_B T \ln\left(\frac{g_{ov}}{g_{oc}}\right)$

Notice at $T=0$ μ is still in the center

e) $\mu - \frac{\varepsilon_c+\varepsilon_v}{2} = \frac{1}{2}k_B T \ln\left(\frac{0.44}{1.09}\right) = -0.45\, k_B T$

Since the gap for Si is $\sim 1\,eV$ this shift ($\sim -0.01\,eV$) is not very important.