7.2 Bosons and Fermions

See book for 2 particles in 5 states.

When the number of single particle states is much larger than number of particles, $Z_1 >> N$, no worries because chance that two particles are in same state is small.

For ideal gas $\frac{V}{N} >> v_Q = \left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}$ means no worry

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a) Distinguishable 5,0,0,0,0 1 state
Boson 5,0,0,0,0 1 state
Fermion 1,1,1,1,1 1 state

b) Distinguishable 4,1,0,0,0 5 states
Boson 4,1,0,0,0 1 state
Fermion 1,1,1,1,0,1 1 state

c) Distinguishable 4,0,1,0,0 5 states 3,2,0,0,0 10 states 15 total
Boson 4,0,1,0,0 1 state 3,2,0,0,0 1 state 2 total
Fermion 1,1,1,1,0,0,1 1 state 1,1,1,0,1,1 1 state 2 total

and

Distinguishable 2,3,0,... 10 states; 3,1,1,0,... 20 states, 4,0,0,1,0... 5 states 35 total
Boson " 1 state " 1 state, " 1 state 3 total
Fermion 1,1,1,1,0,0,0,1 1 state; 1,1,1,0,1,0,1,0; 1 state 1,1,0,1,1,1 1 state 3 total

d) The states are same for distinguishable than for boson and the e^{-E/k_BT} factors are the same. <u>But</u> the multiplicity for Boson <u>excited</u> states is much less than for disting. particles.

$\Rightarrow$ Ground state much more likely for Bosons than you might expect from Boltzmann statistics

Distribution Functions

Concentrate on one single particle state of a system (for example, the $n=53$ state of harmonic oscillator).

The probability n particles are in this state is

$$P(n) = \frac{1}{Z} e^{-\beta[E - n\mu]} = \frac{1}{Z} e^{-n\beta(\varepsilon-\mu)}$$ where ε = energy for 1

Fermions $n = 0$ or $1 \Rightarrow Z = e^0 + e^{-\beta(\varepsilon-\mu)} = 1 + e^{-\beta(\varepsilon-\mu)}$

The average number of particles in this state is

$$\bar{n} = 0\,P(0) + 1\,P(1) = \frac{1}{Z}\left[0\,e^0 + 1 e^{-\beta(\varepsilon-\mu)}\right] = \frac{e^{-\beta(\varepsilon-\mu)}}{1+e^{-\beta(\varepsilon-\mu)}}$$

Fermi-Dirac statistics

$$\bar{n} = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \boxed{\frac{1}{e^{(\varepsilon-\mu)/k_BT}+1} = \bar{n}_{FD}}$$

Bosons $n = 0, 1, 2, \ldots \Rightarrow Z = e^0 + e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} + \ldots = 1+x+x^2+x^3+\ldots$

$x = e^{-\beta(\varepsilon-\mu)}$

The series $1+x+x^2+x^3+\ldots = \frac{1}{1-x}$ if $|x|<1 \Rightarrow \varepsilon > \mu$

$$\Rightarrow Z = \frac{1}{1-e^{-\beta(\varepsilon-\mu)}}$$

The average number of particles in this state is

$$\bar{n} = 0\,P(0) + 1\,P(1) + 2\,P(2) + \ldots = \frac{1}{Z}\left(0\cdot 1 + 1\cdot x + 2\cdot x^2 + 3x^3 + \ldots\right)$$

The series $0\cdot 1 + 1\cdot x + 2\cdot x^2 + 3\cdot x^3 + \ldots = x\frac{d}{dx}\left(1+x+x^2+x^3+\ldots\right)$

$$= x\frac{d}{dx}\left(\frac{1}{1-x}\right) = x/(1-x)^2$$

$$\Rightarrow \bar{n} = \frac{e^{-\beta(\varepsilon-\mu)}}{(1-e^{-\beta(\varepsilon-\mu)})^2}\frac{1}{\frac{1}{[1-e^{-\beta(\varepsilon-\mu)}]}} = \frac{e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}} = \boxed{\frac{1}{e^{\beta(\varepsilon-\mu)}-1} = \bar{n}_{BE}}$$

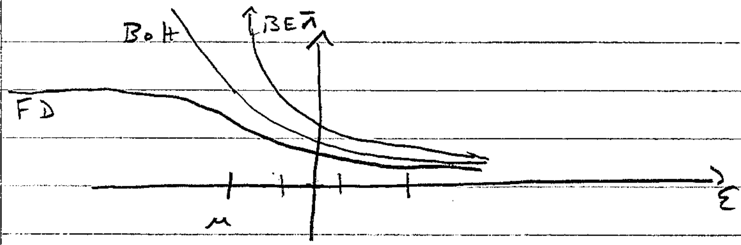
Distinguishable $P = \frac{1}{Z_1} e^{-\varepsilon/k_BT}$ $\quad \bar{n}_{Boltz} = \frac{N}{Z_1} e^{-\frac{\varepsilon}{k_BT}} = e^{-\beta(\varepsilon-\mu)}$ where use result from Prob 6.44 $\mu = -k_BT\ln(Z_1/N)$

General features of distributions

$$\bar{n}_{FD} = \frac{1}{e^{+\beta(\varepsilon-\mu)}+1}, \quad \bar{n}_{Bolt} = \frac{1}{e^{+\beta(\varepsilon-\mu)}}, \quad \bar{n}_{BE} = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$

$$\bar{n}_{FD} < \bar{n}_{Bolt} < \bar{n}_{BE}$$

When $\beta(\varepsilon-\mu) \gg 1$ $\quad \bar{n}_{FD} \cong \bar{n}_{Bolt} \cong \bar{n}_{BE}$



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$$\bar{n}_{BE}/\bar{n}_{FD} < 1.01 \Rightarrow \frac{\bar{n}_{BE}}{\bar{n}_{FD}} = \frac{e^{\beta(\varepsilon-\mu)}+1}{e^{\beta(\varepsilon-\mu)}-1} = \frac{1+e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}} \cong 1+2e^{-\beta(\varepsilon-\mu)}$$

$$\Rightarrow 0.01 = 2e^{-\beta(\varepsilon-\mu)} \Rightarrow \beta(\varepsilon-\mu) = \ln(200) = 5.3$$

Since $\varepsilon > 0$, this is always satisfied when $-\beta\mu > \ln 200$

$\Rightarrow \frac{V}{N}\frac{Z_{int}}{v_Q} > 200$ or $\frac{k_B T}{P v_Q} Z_{int} > 200$

But $\frac{k_B T}{P v_Q} \sim 10^8$

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$$N = \sum_s \bar{n}_{Bolt}(s) = \sum_s e^{-\beta\varepsilon(s)} e^{\beta\mu} = Z_1 e^{\beta\mu}$$

$$\Rightarrow \mu = \frac{1}{\beta}\ln(N/Z_1) = k_B T \ln\left(\frac{N}{Z_1}\right) = -k_B T \ln\left(\frac{Z_1}{N}\right)$$

Prob 7.12

Show Prob $\varepsilon = \mu + x$ is occupied equals prob. that stat $\varepsilon = \mu - x$ is unoccupied

$$\frac{1}{e^{\beta x}+1} \stackrel{?}{=} 1 - \frac{1}{e^{-\beta x}+1} = \frac{e^{-\beta x}+1-1}{e^{-\beta x}+1} = \frac{e^{-\beta x}}{e^{-\beta x}+1} = \frac{1}{e^{\beta x}+1} \checkmark$$