2.4 Large Systems

Small numbers: You can write them out with hardly any trouble (for example 57683, 1024, ...)
Usual math rules apply: addition, subtraction, multiplication, division

Large numbers: are much, much larger than small numbers
Make by exponentiation (for example 10^{23}, 4.75×10^{36} etc)
Adding a small number to a large number doesn't change it in an appreciable way

$$36 + 10^{20} \cong 10^{20}$$ (Don't use this approx, if later subtracting the same large number.)

Very large numbers: are much, much larger than large numbers, Will arise when exponentiating large numbers (for example $10^{10^{23}}$)
Multiplying a very large number by a large number hardly changes it

$$10^{10}\,10^{10^{23}} = 10^{(10+10^{23})} \simeq 10^{10^{23}}$$

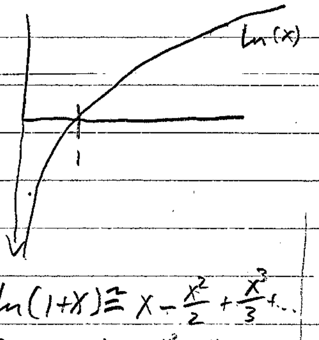
(Doesn't work if you are going to divide by same very large #

One useful trick $\ln(\text{large number}) \simeq$ small number
$\ln(\text{very large number}) \simeq$ large number

Remind: $e^a e^b = e^{(a+b)}$ and $(e^a)^b = e^{ab}$

Prob 2.12 pg 61 $e^{\ln x} = x$ $\quad \ln(e^x) = x \Rightarrow \ln(1) = 0$

a)



c) Prove $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$
use $\frac{d}{dx} e^{\ln x} = \frac{d\ln x}{dx} e^{\ln x} = x\frac{d\ln x}{dx} = \frac{dx}{dx} = 1$

b) Prove $\ln(ab) = \ln a + \ln b$
$ab = e^{\ln(ab)} = e^{\ln a + \ln b} = e^{\ln a} e^{\ln b} = ab$

Prove $\ln(a^b) = b\ln a$
$e^{\ln(a^b)} = e^{b\ln a} = (e^{\ln a})^b = a^b$

** Show $\ln(1+x) \cong x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$
$F(1+x) = F(1) + xF'(1) + \frac{x^2}{2}F''(1) + \ldots$

Stirling's Approximation $N! \simeq \sqrt{2\pi N}\, N^N e^{-N}$

(a better approximation is $N! \simeq \sqrt{2\pi(N+1/6)}\, N^N e^{-N}$

See Appendix B for a derivation of Stirling's Approx.

Gaussian

define

First $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$

To see this $\int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-ax^2} d(\sqrt{a}\, x) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-x^2} dx$

Try $\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$

$= \int_0^{\infty}\int_0^{2\pi} e^{-s^2}\, s\, ds\, d\phi = 2\pi \int_0^{\infty} e^{-s^2} s\, ds = 2\pi\left(-\frac{1}{2}e^{-s^2}\right)\Big|_0^{\infty}$

$= \pi$

$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \Rightarrow \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$

To find Stirling's approx use the integral form of the factorial

$$n! = \int_0^{\infty} x^n e^{-x} dx \qquad (\text{This is why } 0! = 1)$$

To see this relation $\int_0^{\infty} x^n e^{-x} dx = \int_0^{\infty} -\frac{d}{dx}\left[x^n e^{-x}\right] + n x^{n-1} e^{-x}\, dx$

$= n \int_0^{\infty} x^{n-1} e^{-x} dx$

$= n(n-1) \int_0^{\infty} x^{n-2} e^{-x} dx \ldots$

$$n! = \int_0^{\infty} x^n e^{-x} dx = \int_0^{\infty} e^{-x+n\ln x} dx$$

change variables $x = y+n \Rightarrow n! = \int_{-n}^{\infty} e^{-y-n+n\ln(y+n)} dy$

$-y-n+n\ln(y+n) = -y-n+n\ln\left[n\left(1+y/n\right)\right]$

$= -y-n+n\ln n + n\ln(1+y/n)$

$= -y-n+n\ln n + n\left(y/n - y^2/2n^2 + \ldots\right)$

$\simeq -n + n\ln n - y^2/2n$

$$\Rightarrow n! \simeq e^{-n+n\ln n} \int_{-n}^{\infty} e^{-y^2/2n} dy \simeq \sqrt{2\pi n}\, n^n e^{-n}$$

side note The gamma function $\Gamma(z+1) \equiv \int_0^{\infty} x^z e^{-x} dx$ defined for non-integer z

N	$N!$	$\sqrt{2\pi N}\, N^N e^{-N}$	$\sqrt{2\pi(N+\frac{1}{6})}\, N^N e^{-N}$
1	1	0.922	0.996
5	120	118	119.97
10	3,628,800	3,598,696	3,628,561

Stirling's approx is too small by factor $\sqrt{1+\frac{1}{6N}} \simeq 1+\frac{1}{12N}$

Einstein Solid

Use Stirling's approx to compute the multiplicity of an Einstein solid N, q large

$$\Omega(N,q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!\,(N-1)!} = \frac{N}{q+N}\,\frac{(q+N)!}{q!\,N!}$$

Because N, q are large numbers, $\Omega(N,q)$ will be a very large number

Prob 2.18 pg 64

$$\Omega(N,q) = \frac{N}{q+N}\,\frac{\sqrt{2\pi}\,\sqrt{q+N}\,[(q+N)/e]^{q+N}}{\sqrt{2\pi}\,\sqrt{q}\,(q/e)^q\,\sqrt{N}\,(N/e)^N} = \frac{1}{\sqrt{2\pi\,(q+N)q/N}}\left(\frac{q+N}{q}\right)^q\left(\frac{q+N}{N}\right)^N$$

In the high temperature limit $q \gg N$. What is the multiplicity in this limit?

$$\left(\frac{q+N}{q}\right)^q = \left(1+N/q\right)^q \simeq e^N$$

$$\left(\frac{q+N}{N}\right)^N = \left(\frac{q}{N}\right)^N\left(1+\frac{N}{q}\right)^N$$

N is a large #

$$\Omega(N,q) \simeq \frac{1}{\sqrt{2\pi}}\,\frac{N^{1/2}}{q}\left(\frac{qe}{N}\right)^N\left(1+\frac{N}{q}\right)^{N-1/2} \sim \left(\frac{qe}{N}\right)^N$$ because $\left(\frac{qe}{N}\right)^N$ is very large

We can use this form of $\Omega(N,q)$ to examine the sharpness of the multiplicity

Suppose system a has N_a and system b has N_b and $q = q_a + q_b$

$$\Omega_{TOT} = \left(\frac{e}{N_a}\right)^{N_a}\left(\frac{e}{N_b}\right)^{N_b} q_a^{N_a}\,(q-q_a)^{N_b}$$

We should expect Ω is peaked near $\bar{q}_a = \frac{N_a}{N_a+N_b} q = \frac{N_a}{N} q$ and $\bar{q}_b = \frac{N_b}{N_a+N_b} q = \frac{N_b}{N} q$

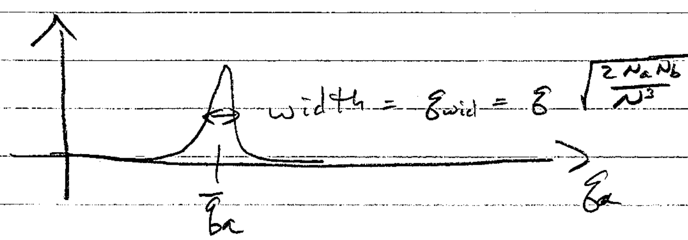
Write $q_a = \frac{N_a}{N_a+N_b} q + \delta_q \qquad q_b = \frac{N_b}{N_a+N_b} q - \delta_q$

$$\Rightarrow \Omega_{TOT} = \left(\frac{e}{N_a}\right)^{N_a} \left(\frac{e}{N_b}\right)^{N_b} (\bar{q}_a + \delta_q)^{N_a} (\bar{q}_b - \delta_q)^{N_b}$$

$$= \left(\frac{e}{N_a}\right)^{N_a} \left(\frac{e}{N_b}\right) e^{F(\delta_q)}$$

$$F(\delta_q) = N_a \ln(\bar{q}_a + \delta_q) + N_b \ln(\bar{q}_b - \delta_q)$$
$$= N_a \ln \bar{q}_a + N_a \ln\left(1 + \frac{\delta_q}{\bar{q}_a}\right) + N_b \ln \bar{q}_b + N_b \ln\left(1 - \frac{\delta_q}{\bar{q}_b}\right)$$
$$= N_q \ln \bar{q}_a + N_b \ln \bar{q}_b + N_a\left(\frac{\delta_q}{\bar{q}_a} - \frac{\delta_q^2}{2\bar{q}_a^2}\right) + N_b\left(-\frac{\delta_q}{\bar{q}_b} - \frac{\delta_q^2}{2\bar{q}_b^2}\right)$$
$$= N_q \ln \bar{q}_a + N_b \ln \bar{q}_b - \frac{N^2 \delta_q^2}{2 N_a q^2} - \frac{N^2 \delta_q^2}{2 N_b q^2}$$

$$\Rightarrow \Omega_{TOT} = \left(\frac{e\bar{q}_a}{N_a}\right)\left(\frac{e\bar{q}_b}{N_b}\right) e^{-\delta_q^2/q_{wid}^2} \qquad \frac{1}{q_{wid}^2} = \frac{N^2}{2q^2}\left(\frac{1}{N_a} + \frac{1}{N_b}\right) = \frac{N^3}{2N_aN_bq^2}$$



The book does the case $N_a = N_b \Rightarrow N = 2N_a \Rightarrow q_{wid} = q/2\sqrt{N_a}$
(Be careful: what the book calls N is my N_a)!

Let's examine a case $N_a = N_b = 10^{22}$ and $q = 10^{24}$
$\Rightarrow q_{wid} = 10^{24}/(2\times 10^{11}) = \frac{1}{2} 10^{13}$ huge number!
However, $\bar{q}_a = \frac{1}{2} q = \frac{1}{2} 10^{24} \Rightarrow q_{wid}/q_a = 10^{-11}$ <u>small number</u>!

At $\delta_q = \pm 10\, q_{wid}$ Ω_{TOT} has decreased by factor $e^{-100} \approx 4\times 10^{-44}$

$\Rightarrow$ You will always measure $\bar{q}_a$ to 1 part in 10^{10}

How would this be modified if the frequency of the oscillators were different? That is $\omega_a \neq \omega_b$

When systems are so large that fluctuations away from average values are unmeasurable are called <u>thermodynamic limit.</u> $\Rightarrow$ <u>Tiny fraction of macrostates occupied.</u>

Prob 2.25 You make N steps of length $+\Delta x$ or $-\Delta x$.

pg 67 a) at $x=0$

b) This is the same as the 2-state system $\Omega = N!/(N_R!\,(N-N_R)!)$ where N_R is the number of right steps

Use Stirling's approx. $\Omega = \dfrac{\sqrt{2\pi N}\, N^N e^{-N}}{\sqrt{2\pi N_R}\, N_R^{N_R} e^{-N_R} \sqrt{2\pi(N-N_R)}\,(N-N_R)^{N-N_R} e^{-(N-N_R)}}$

Change variables $N_R = \frac{N}{2} + x$

$$\Omega = \frac{\sqrt{2\pi N}\; N^N}{\sqrt{2\pi(\frac{N}{2}+x)}\,(\frac{N}{2}+x)^{\frac{N}{2}+x}\sqrt{2\pi(\frac{N}{2}-x)}\,(\frac{N}{2}-x)^{\frac{N}{2}-x}}$$

$$= \frac{N^N}{\sqrt{2\pi}} \frac{1}{\left[(\frac{N}{2})^2 - x^2\right]^{\frac{N}{2}+\frac{1}{2}}} \left(\frac{\frac{N}{2}-x}{\frac{N}{2}+x}\right)^x$$

$$= \frac{N^N}{\sqrt{2\pi}}\, e^{x\left[\ln(1-\frac{2x}{N}) - \ln(1+\frac{2x}{N})\right] - \frac{N+1}{2}\ln\left(1-\frac{x^2 4}{N^2}\right)} \frac{1}{(\frac{N}{2})^{N+1}}$$

$$= \frac{2^{N+1}}{\sqrt{2\pi}\,N}\, e^{x\left[-\frac{4x}{N}\right] + \frac{2x^2}{N}} = \frac{2^{N+1}}{\sqrt{2\pi}\,N}\, e^{-2x^2/N}$$

$x = \sqrt{N/2} = \sqrt{5000} = 71$

The displacement from the origin $2x = 142$