1.2 Ideal Gas

For low density gas (over modest Temp range)

Ideal Gas "Law"

$$PV = nRT$$

P: pressure; V: volume; n: # of moles; R: $8.31 \frac{J}{mol \cdot K}$; T: Temp in $\underline{\underline{K}}$

SI units $P = \frac{N}{m^2} = Pa$ (pressure = $\frac{\text{Force}}{\text{area}}$)
$V = m^3$
$T = K$

Be careful of Chemist units 1 atm, 1 liter

1 mole of molecules/atoms $= 6.02 \times 10^{23} = N_A$ ← Avogadro's number

In terms of the number of atoms/molecules $N = n N_A$

$$\Rightarrow \quad PV = N \frac{R}{N_A} T = N k_B T$$

k_B: Boltzman's const $= 1.381 \times 10^{-23} \frac{J}{K}$
??? convert K to J ???

Of course Ideal Gas "Law" is an <u>approximation</u> for actual materials

Note $\boxed{N k_B = n R}$

Prob 1.9 What is volume of 1 mole of air, at room temp and 1 atm pressure?

$P = 1.01 \times 10^5 \, Pa \qquad T = 300 \, K$

$V = 1 \, mol \; 8.31 \frac{J}{mol \cdot K} \cdot 300 K / 1.01 \times 10^5 Pa = 0.025 \, m^3 \approx \frac{1}{40} m^3$

units $\cancel{mol} \frac{J}{\cancel{mol} \cdot \cancel{K}} \cancel{K} \frac{1}{N/m^2} = \frac{N \, m}{N/m^2} = m^3$ ✓

What kind of gas???

Prob 1.10 Number of air molecules in this room

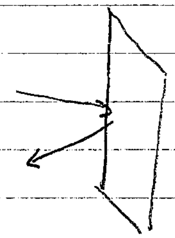
$V = \underset{\text{height}}{3m} \cdot \underset{\text{depth}}{4m} \cdot \underset{\text{long}}{6m} = 72 m^3 \qquad n = \frac{72 m^3}{\frac{1}{40} m^3} = 2900 \, mol = 2 \times 10^{27}$

Note another way to express: $P = \left(\frac{N}{V}\right) k_B T$

$\frac{N}{V} = \frac{\#}{\text{volume}}$ = number density

Derivation of Ideal Gas Law

Assumptions: gas is 1 type of atom*, distance between atom-atom collisions large*, atom has specular reflection off wall*,



Before $\vec{V}_b = (V_x, V_y, V_z)$

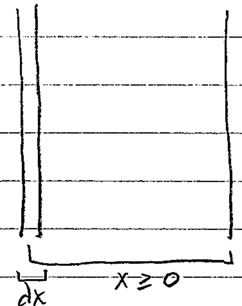
After $\vec{V}_a = (-V_x, V_y, V_z)$

momentum impulse to wall $= 2mV_x = I$

Random collisions of atoms with wall gives pressure $= F/A$

$$F = \frac{\text{total } I \text{ during time } \delta t}{\delta t}$$

Examine a small slice of gas a distance x from wall



atoms that hit wall have $V_x \geq \frac{x}{\delta t}$

Define $\mathcal{P}(V_x)dV_x$ to be probability atom has velocity between V_x and $V_x + dV_x$

$$I = \underbrace{\frac{N}{V} A \cdot dx}_{\text{number of atoms in slice}} \underbrace{\int_{x/\delta t}^{\infty} 2m V_x \mathcal{P}(V_x) dV_x}_{\text{impulse from atoms that reach wall}}$$

$$I_{TOT} = 2m\frac{N}{V}A \int_0^{\infty} \left[\int_{x/\delta t}^{\infty} V_x \mathcal{P}(V_x) dV_x \right] dx$$

Note $\int_{x/\delta t}^{\infty} V_x P(V_x)\, dV_x = \frac{d}{dx}\left[x \int_{x/\delta t}^{\infty} V_x \mathcal{P}(V_x) dV_x \right] + \frac{x^2}{\delta t^2} \mathcal{P}(x/\delta t)$

math -> $\int_0^{\infty} \frac{d}{dx}\left[x \int_{x/\delta t}^{\infty} V_x \mathcal{P}(V_x) dV_x \right] dx = x \int_{x/\delta t}^{\infty} V_x \mathcal{P}(V_x) dV_x \Big|_{x=0}^{x=\infty} = 0$ $\quad \mathcal{P}(V_x) \to 0$ faster than $1/V_x^3$ as $V_x \to \infty$

$$I_{TOT} = 2m\frac{N}{V}A \int_0^{\infty} \frac{x^2}{\delta t^2} \mathcal{P}(x/\delta t)\, dx \qquad x/\delta t \equiv V_x \qquad dx = \delta t\, dV_x$$

$$= 2m\frac{N}{V}A\delta t \int_0^{\infty} V_x^2 \mathcal{P}(V_x) dV_x = M\frac{N}{V}A\delta t \int_{-\infty}^{\infty} V_x^2 \mathcal{P}(V_x) dV_x$$

$$= M\frac{N}{V}A\delta t \underbrace{\overline{V_x^2}}_{\text{average of } V_x^2 \text{ not square of } \overline{V}_x}$$

Note assumed $\mathcal{P}(V_x) = \mathcal{P}(-V_x)$

Put it all together

$F = \frac{N}{V} A \, m \overline{v_x^2} \Rightarrow P = \frac{F}{A} = \frac{N}{V} m \overline{v_x^2}$

It must be true $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} \Rightarrow \overline{KE} = \frac{1}{2} m (\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2})$
$= \frac{3}{2} PV/N$

$P \cdot V = N k_B T = N \frac{2}{3} \overline{KE}$

$\Rightarrow$ Average $KE = \frac{3}{2} k_B T$

$\frac{3}{2} k_B$ is the conversion factor between T and $\overline{KE}$!!!

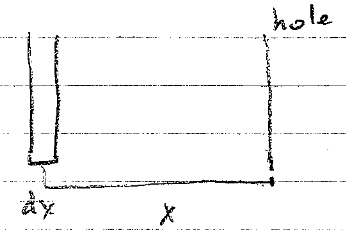
Note $v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}} = \sqrt{\frac{3 k_B T}{m}}$

Prob 1.18 Air is mostly N_2 $m = 4.65 \times 10^{-26}$ kg $v_{rms} = 517$ m/s at 300K

Prob 1.19 H_2 and O_2 are in thermal equil. What is ratio of v_{rms}? 4
2 amu 32 amu

slight diff from book

Prob 1.22 Calculate effusion through small hole



that go through hole during time δt have $v_x \geq x/\delta t$

Number of atoms in slice that go through hole

$$\# = \frac{N}{V} A\, dx \int_{x/\delta t}^{\infty} P(v_x)\, dv_x$$

$$\text{Total \# through hole} = \frac{N}{V} A \int_0^{\infty} \left[\int_{x/\delta t}^{\infty} P(v_x)\, dv_x \right] dx$$

Note $\int_{x/\delta t}^{\infty} P(v_x)\, dv_x = \frac{d}{dx}\left[x \int_{x/\delta t}^{\infty} P(v_x)\, dv_x \right] + \frac{x}{\delta t} P(v_x)$

$$\Rightarrow \text{Total \#} = \frac{N}{V} A \int_0^{\infty} \frac{x}{\delta t} P\left(\frac{x}{\delta t}\right) dx = \frac{N}{V} A\, \delta t\, \bar{v}_x$$

where $\frac{N}{V} = \frac{P}{m \overline{v_x^2}}$ and $\bar{v}_x = \int_0^{\infty} v_x P(v_x)\, dv_x$

$$\Rightarrow \frac{dN}{dt} = -A P \frac{\bar{v}_x}{m \overline{v_x^2}} = -\frac{A}{V} \frac{\bar{v}_x}{m \overline{v_x^2}} k_B T\, N$$

As a guesstimate use $\bar{v}_x = \sqrt{\overline{v_x^2}} = \sqrt{\frac{k_B T}{m}}$

$$\Rightarrow \frac{dN}{dt} = -\frac{A}{V} \sqrt{\frac{k_B T}{m}}\, N$$ (this is a factor of 2 diff from book)

If T is constant $N(t) = N(0)\, e^{-t/\tau}$ with $\tau = \frac{V}{A}\sqrt{\frac{m}{k_B T}}$

For $A = 1\,\text{mm}^2 = 10^{-6}\,\text{m}^2$ $\quad V = 1\,\text{liter} = 10^{-3}\,\text{m}^3$

$T = 300\,\text{K}$ $\quad m = 4.65 \times 10^{-26}\,\text{kg}$ $\quad \tau = 3.4\,\text{s}$

To make $\tau = 1$ hour A must be ~1000X smaller or about 30X smaller radius