PHYS550, Test 2, Fall 2018

You must show work to get credit. You should read the important math info at the bottom of the page before working any problem; it might save you steps.

(1) (5 pts) T. Li, et al, Phys. Rev. Lett. **109**, 163001 (2012) proposed an experiment that would consist of a circular ring of 100 identical $^9Be^+$ ions. Treat the ions as being equally spaced in a ring of radius 50 nm allowed to move around the circle but otherwise constrained. See diagram on blackboard. Find the lowest 3 energy levels in Joules and Kelvins.

(2) (5 pts) An operator $\hat{M} = \exp(i\hat{p}a/\hbar)$ where $i, a, \hbar$ are constants and $\hat{p}$ is the momentum operator. (a) What are the units of a? (b) Find $g(x)$ where $g(x) = \hat{M}f(x)$.

(3) (5 pts) The two states $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ are the solution of the same time dependent Schrödinger equation with a time dependent $\hat{H}(t)$. Derive the equation for the time derivative of $\langle\psi_1(t)|\psi_2(t)\rangle$. Solve this equation to obtain the time dependent projection.

(4) (5 pts) The lowest order relativistic correction to the kinetic energy gives $KE \simeq p^2/(2m) - p^4/(8m^3c^2)$. (a) Write down the representation of the time independent 3D Schrödinger in terms of spatial derivatives that includes this correction. (b) For the case $V(\vec{r}) = V(r)$, *show* whether or not the eigenstates are still separable as $R_{n\ell}(r)Y_{\ell m}(\theta, \varphi)$?

(5) (10 pts) (a) Determine the matrix elements $H_{0,n'}$ and H_{n0} of the relativistically corrected Hamiltonian (see previous problem) for a 1D harmonic oscillator. (b) *Show* whether or not the eigenstates of the *corrected* Hamiltonian still have the property $\psi_n(x) = (\pm 1)\psi_n(-x)$. (c) Is the ground state energy of the corrected Hamiltonian larger or smaller than that of the non-relativistic Hamiltonian? You need to prove your answer.

(6) (10 pts) In 2D, an eigenstate has the wave function $\psi(\rho, \varphi) = A\rho e^{i\varphi}e^{-\beta\rho^2}$. (a) Determine the constant A. (b) Convert the wave function into a function of x, y. (c) Compute the probability current. (d) Compute the integral of the probability current over all space. (e) Compute the integral of $M\vec{\rho} \times \vec{j}$ over all space.

(7) (10 pts) Two identical quarks with mass M interact with each other through a radial potential $V(r) = Fr$ with $F > 0$. Ignoring Fermi/Bose statistics, find the $\ell = 0$ eigenstates and eigenenergies using properties of the Airy functions described below.

<u>Math info:</u>

The solution of $y''(x) - xy(x) = 0$ are the Airy functions: $Ai(x)$ and $Bi(x)$. The $Bi(x) \to \infty$ as $x \to \infty$ and $Ai(x) \to 0$ as $x \to \infty$. The zeros of both the $Ai(x)$ and $Bi(x)$ are at $x < 0$. The n-th zero (n starts at 1) are denoted $Ai(\alpha_n) = 0$ and $Bi(\beta_n) = 0$.

Possibly useful integral $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{b^2/[4a]}$

Another possibly useful integral $\int_0^\infty x^n e^{-x}dx = n!$

In cylindrical coordinates the differential element is $\rho d\rho d\phi$

In spherical coordinates $-(\hbar^2/2\mu)d^2u_{n\ell}(r)/dr^2 + V_{eff}(r)u_{n\ell}(r) = E_{n\ell}u_{n\ell}(r)$

Harmonic oscillator eigenstates are

$$u_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/(2\hbar)}$$

Prob 1 As in Prob 7-12 $H = L_z^2/2MR^2$ with $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$

The eigenstates are $\frac{e^{im\varphi}}{\sqrt{2\pi}}$ with the condition $e^{im\varphi} = e^{im(\varphi + \frac{2\pi}{100})}$

Allowed $m = 0, \pm 100, \pm 200, \ldots$ and the $M = 100\, M_{Be}$

The lowest 3 energies are $0, \frac{10^4 \hbar^2}{2MR^2}, \frac{4\times 10^4 \hbar^2}{2MR^2}$

$$\frac{\hbar^2}{2MR^2} = \frac{(1.05\times 10^{-34})^2}{2\cdot 900\cdot 1.7\times 10^{-27}\cdot (50\times 10^{-9})^2} = 1.44\times 10^{-30}\ \text{J} = 1.04\times 10^{-7}\ \text{K}\ k_B$$

$$\mathcal{E} = 0,\ 1.4\times 10^{-26}\ \text{J},\ 5.8\times 10^{-26}\ \text{J}$$
$$= 0,\ 1.0\times 10^{-3}\ \text{K},\ 4.1\times 10^{-3}\ \text{K}$$

Prob 2 $\hat{M} = e^{ia\hat{p}/\hbar} = e^{a\frac{\partial}{\partial x}}$ a must have units of meter

Use $e^b = 1 + b + \frac{b^2}{2} + \frac{b^3}{3!} + \ldots$

$$\hat{M} f(x) = f(x) + a f'(x) + \frac{a^2}{2} f''(x) + \frac{a^3}{3!} f'''(x) + \ldots$$

$$\Rightarrow g(x) = f(x+a)$$

Prob 3 $\frac{\partial}{\partial t}\langle \psi_1(t) | \psi_2(t)\rangle = \frac{\partial\langle \psi_1(t)|}{\partial t} |\psi_2(t)\rangle + \langle \psi_1(t)| \frac{\partial |\psi_2(t)\rangle}{\partial t}$

Schrod. Eq $\frac{\partial|\psi(t)\rangle}{\partial t} = -\frac{i}{\hbar} H(t)|\psi(t)\rangle$ and

$$\frac{\partial\langle\psi(t)|}{\partial t} = \left(\frac{\partial|\psi(t)\rangle}{\partial t}\right)^+ = \left(-\frac{i}{\hbar} H(t)|\psi(t)\rangle\right)^+ = \frac{i}{\hbar}\langle\psi(t)|H^+(t) = \frac{i}{\hbar}\langle\psi(t)|H(t)$$

$$\frac{\partial}{\partial t}\langle\psi_1(t)|\psi_2(t)\rangle = \frac{i}{\hbar}\langle\psi_1(t)|H(t)|\psi_2(t)\rangle - \frac{i}{\hbar}\langle\psi_1(t)|H(t)|\psi_2(t)\rangle = 0$$

Solution $\langle\psi_1(t)|\psi_2(t)\rangle = \langle\psi_1(0)|\psi_2(0)\rangle$

Prob 4 $\quad H = KE + V = \frac{p^2}{2m} + \frac{(p^2)^2}{8m^3c^2} + V$

(a) $H\psi = E\psi \rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^4}{8m^3c^2}\nabla^2(\nabla^2\psi) + V\psi = E\psi$

(b) Can show by either using Eq 8-12 or showing $[L^2, H] = 0$

$$[L^2, \frac{p^2}{2m} + \frac{(p^2)^2}{8m^3c^2} + V(r)] = \cancel{[L^2, \frac{p^2}{2m}]}\to 0 + [L^2, \frac{(p^2)^2}{8m^3c^2}] + \cancel{[L^2, V(r)]}\to 0$$

$$[L^2, p^2p^2] = p^2\cancel{[L^2, p^2]}\to 0 + \cancel{[L^2, p^2]}\to 0\, p^2 = 0$$

These commutators are zero because $\vec{L}$ commutes with any scalar. For example

$$[L_x, x^2+y^2+z^2] = [yP_z, z^2] - [zP_y, y^2] = 2\frac{\hbar}{i}yz - 2\frac{\hbar}{i}zy = 0$$

Prob 5 $\quad H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - \frac{p^2p^2}{8m^3c^2}$

To find matrix elements use raising and lowering operators

$$A - A^+ = i\sqrt{\frac{2}{m\omega\hbar}}\,P \Rightarrow P = \sqrt{\frac{m\omega\hbar}{2}}\, i\,(A^+ - A)$$

$$H = \hbar\omega(A^+A + \tfrac{1}{2}) - \frac{m^2\omega^2\hbar^2}{4\cdot 8m^3c^2}(A^+ - A)^4$$

Since $H_{on} = H_{no}$ only do H_{no}

$$\langle n|(A^+-A)^4|0\rangle = \langle n|(A^+-A)^3|1\rangle = \langle n|(A^+-A)^2[\sqrt{2}|2\rangle - |0\rangle]$$
$$= \langle n|(A^+-A)[\sqrt{6}|3\rangle - 2|1\rangle - |1\rangle] = \langle n|[\sqrt{24}|4\rangle - 3\sqrt{2}|2\rangle - 3\sqrt{2}|2\rangle + 3|0\rangle$$

$$\Rightarrow H_{n,0} = \frac{\hbar\omega}{2}\delta_{n,0} - \frac{\omega^2\hbar^2}{32mc^2}\left(\sqrt{24}\,\delta_{n,4} - 6\sqrt{2}\,\delta_{n,2} + 3\,\delta_{n,0}\right)$$

(b) $\text{Parity}(H\psi) = H(\text{Parity}\,\psi) \Rightarrow [H, \text{Parity}] = 0 \Rightarrow$ <u>yes</u>

(c) Smaller. We know the lowest energy is below any $\langle H\rangle$

$\langle 0|H|0\rangle = E_0 - \frac{1}{8m^3c^2}\langle 0|p^2p^2|0\rangle = E_0 - \frac{1}{8m^3c^2}\langle p^2 0|p^2 0\rangle$

We know $\langle\psi|\psi\rangle \geq 0 \Rightarrow \langle p^2 0|p^2 0\rangle \geq 0$

This means $E_0^{corr} < E_0$

Prob 6 (a) Normalize

$$A^2 \int_0^\infty \int_0^{2\pi} \rho^2 e^{-2\beta\rho^2} \rho \, d\rho \, d\phi \qquad s = 2\beta\rho^2 \quad ds = 4\beta\rho \, d\rho$$

$$= A^2 2\pi \frac{1}{8\beta^2} \int_0^\infty s e^{-s} ds = A^2 \frac{\pi}{4\beta^2} = 1 \Rightarrow \quad A = \frac{2\beta}{\sqrt{\pi}}$$

(b) $\psi = A \, \rho(\cos\varphi + i \sin\varphi) \, e^{-\beta(x^2+y^2)} = A(x+iy) e^{-\beta(x^2+y^2)}$

(c) $$J_x = \frac{\hbar}{m} \mathrm{Im}\left(\psi^* \frac{\partial \psi}{\partial x}\right) = \frac{\hbar}{m} A^2 \, \mathrm{Im}\left[(x-iy) e^{-\beta\rho^2} (-2\beta x(x+iy) + 1) e^{-\beta\rho^2} \right]$$
$$= \frac{\hbar}{m} A^2 (-y) e^{-2\beta\rho^2}$$

$$J_y = \frac{\hbar}{m} \mathrm{Im}\left(\psi^* \frac{\partial \psi}{\partial y}\right) = \frac{\hbar}{m} A^2 \, \mathrm{Im}\left[(x-iy) e^{-\beta\rho^2} (-2\beta y(x+iy) + i) e^{-\beta\rho^2} \right]$$
$$= \frac{\hbar}{m} A^2 x e^{-2\beta\rho^2}$$

(d) J_x is odd in $y \Rightarrow \int J_x \, dx\,dy = 0$

J_y " " " $x \Rightarrow \int J_y \, dx\,dy = 0$

(e) $$M \vec{\rho} \times \vec{j} = \hat{z} \, m(x J_y - y J_x) = \hat{z} \, \hbar A^2 (x^2+y^2) e^{-2\beta\rho^2}$$

$$\int M \vec{\rho} \times \vec{j} \, d^2 r = \hat{z} \hbar A^2 \int_0^\infty \int_0^{2\pi} \rho^2 e^{-2\beta\rho^2} d\phi \, \rho \, d\rho = \hat{z} \hbar$$

Coincidence this is $\hbar \hat{z}$? No!

Prob 7 $\quad H = \frac{p^2}{2\mu} + Fr \quad$ with $\mu = \frac{M}{2}$ for reduced mass

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + Fr \, u(r) = E \, u(r)$$

Do change of variables $r = ax + b$ to get $u'' = x u$

$$-\frac{\hbar^2}{2ma^2} u'' + (Fax + Fb) u = E u \Rightarrow u'' = \frac{2ma^2}{\hbar^2}(Fax + Fb - E)$$

$$\Rightarrow b = E/F \qquad \frac{2ma^3 F}{\hbar^2} = 1 \Rightarrow a = \left(\hbar^2/2mF\right)^{1/3}$$

$$\underline{u(r) = C \, Ai\left(\tfrac{r-b}{a}\right)} + \cancel{D \, Bi\left(\tfrac{r-b}{a}\right)}^{\,0} \qquad D = 0 \text{ because } Bi \text{ diverges}$$

Boundary condition $u(r=0) = 0 \Rightarrow -\frac{b}{a} = \alpha_n \Rightarrow \underline{E_n = -Fa\alpha_n = -F\left(\frac{\hbar^2}{2mF}\right)^{1/3} \alpha_n}$