PHYS550, Test 1, Fall 2018

You must show work to get credit. You should read the important math info at the bottom of the page before working any problem; it might save you steps.

(1) (5 pts) A ^{85}Rb atom is initially stationary. A photon with wave length 795 nm is traveling in the $+y$-direction. The photon is absorbed by the Rb atom and then emitted in the $+x$-direction. (a) Determine the initial energy of the photon (J or eV are OK). (b) Determine the velocity vector in m/s of the ^{85}Rb after this process. (c) How much energy does Rb gain/lose in the same units as (a)? (d) How much energy does the photon gain/lose?

(2) (5 pts) An electron is in an infinite square well of length 10 nm with a wave function $\psi(x,0) = Ax(a-x)$. What is the probability for finding the electron in the central 1 nm of the well, that is, between $x = 4.5$ and 5.5 nm? (Give a number.)

(3) (5 pts) A neutron is in a well where it can move over a distance of 2×10^{-15} m. (a) Roughly, what is the smallest kinetic energy it can have (in MeV)? (b) Roughly, what is the largest kinetic energy it can have? For this problem, the mass of a neutron and a proton are the same.

(4) (5 pts) A proton is in a harmonic oscillator well. Its wave function at $t = 0$ is $\psi(x,0) = A[3e^{i\alpha_1}u_1(x) + 5e^{i\alpha_2}u_2(x) + 2e^{i\alpha_3}u_3(x)]$ where $\alpha_n = n\pi/10$. Give the expression for its average energy vs. t.

(5) (10 pts) An electron starts at large *positive* x with a negative momentum $-p$. It interacts with a potential $V(x) = \hbar^2\lambda\delta(x)/(2Ma)$. (a) Write down the form of the wave function for all x. (b) Write down all boundary conditions. (c) Compute the reflection probability and the transmission probability.

(6) (10 pts) An electron in an infinite square well of length a has the wave function $\psi(x,0) = A[u_5(x) + 2u_6(x)]$ at t=0. (a) Determine A. (b) Compute the probability density at time $t \geq 0$. (c) Compute the flux at time $t \geq 0$.

(7) (10 pts) For a harmonic oscillator eigenstate, determine the function $xu_n(x)$ in terms of a superposition of other harmonic oscillator states: $C_0u_0(x) + C_1u_1(x) + ...$

Math info:

If a function doesn't vary much between b and c then $\int_b^c f(x)dx \simeq (c-b)f([b+c]/2)$

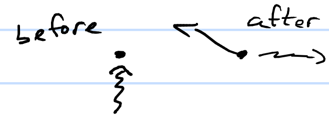
Possibly useful integral $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{b^2/[4a]}$

Harmonic oscillator eigenstates are

$$u_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/(2\hbar)}$$

1) a) $E = hf = hc/\lambda = \dfrac{6.626\times10^{-34}\,\mathrm{Js}\;3\times10^{8}\,\mathrm{m/s}}{795\,\mathrm{nm}} = \underline{2.5\times10^{-19}\,\mathrm{J}} = \underline{1.56\,\mathrm{eV}}$ Eg 1-13

b) Cons of mom.   Eg 1-10



$M = 85\cdot1.66\times10^{-27}\,\mathrm{kg} = 1.41\times10^{-25}\,\mathrm{kg}$

$V_x = -\dfrac{h}{\lambda m} = -\dfrac{6.626\times10^{-34}\,\mathrm{Js}}{795\,\mathrm{nm}\;\;1.41\times10^{-25}} = \underline{-5.91\times10^{-3}\,\mathrm{m/s}}$

$V_y = h/\lambda m = \underline{5.91\times10^{-3}\,\mathrm{m/s}}$

c) $E = \frac{1}{2}m(V_x^2+V_y^2)$

$= 1.41\times10^{-25}\,\mathrm{kg}\,(5.91\times10^{-3}\,\mathrm{m/s})^2 = \underline{4.93\times10^{-30}\,\mathrm{J}} = \underline{3.08\times10^{-11}\,\mathrm{eV}}$

d) Photon lost the amount of energy gained by atom.

2) $P = \int_{x_0}^{x_f} |\Psi(x)|^2\,dx$ Eg 2-14

First normalize   Eg 2-29 $1 = A^2\int_0^a x^2(a^2-2ax+x^2)\,dx = A^2\left(\frac{a^5}{3} - \frac{2a^5}{4} + \frac{a^5}{5}\right)$

$= A^2 a^5\left(-\frac{1}{6}+\frac{1}{5}\right) = A^2\frac{a^5}{30} \Rightarrow A = \sqrt{\frac{30}{a^5}}$

Use approx at bottom of test

$P = (x_f - x_0)\left|\Psi\left(\frac{x_f+x_0}{2}\right)\right|^2 = 1\,\mathrm{nm}\,\frac{30}{(10\,\mathrm{nm})^5}\,(5\,\mathrm{nm})^4 = 0.1875 \simeq 0.19$

3) a) Can solve using particle in an infinite square well or Heisenberg unc. relation

$KE_{min} \sim \dfrac{\hbar^2\pi^2}{2ma^2} \sim \dfrac{(10^{-34}\,\mathrm{Js}\cdot\pi)^2}{2\cdot1.67\times10^{-27}\,\mathrm{kg}\,(2\times10^{-15}\,\mathrm{m})^2} \sim 7\times10^{-12}\,\mathrm{J} \sim \underline{46\,\mathrm{MeV}}$

b) There is no upper limit.

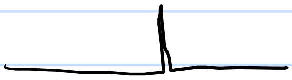
4) For a wave function expanded in eigenstates $\Psi(x) = C_1u_1(x) + \ldots$

$\langle E\rangle = |C_1|^2E_1 + |C_2|^2E_2 + \ldots$ Eg 3-33

Get A from normalization   Eg 3-35 $A^2(9+25+4) = 1 \Rightarrow A = 1/\sqrt{38}$

$\langle E\rangle = \frac{9}{38}\,\frac{3}{2}\hbar\omega + \frac{25}{38}\,\frac{5}{2}\hbar\omega + \frac{4}{38}\,\frac{7}{2}\hbar\omega = \frac{180}{38\cdot2}\hbar\omega = \underline{\frac{45}{19}\hbar\omega}$

5)



a) $u_I(x) = T e^{-ikx}$ $\qquad u_{II}(x) = e^{-ikx} + R e^{ikx}$ $\qquad E = \frac{\hbar^2 k^2}{2m}$

b) continuity $\quad u_I(0) = u_{II}(0) \Rightarrow \quad T = 1 + R$

derivative $\quad u'_{II}(0) - u'_I(0) = \frac{\lambda}{a} u_I(0)$ $\qquad$ Eq 4-68 $\quad \lambda \to -\lambda$

$ik(-1 + R) - ik(-T) = \frac{\lambda}{a} T$

c) $T = 1 + R$ and $-1 + R + T = -\frac{i\lambda}{ka} T \Rightarrow 2R = -\frac{i\lambda}{ka} - \frac{i\lambda}{ka} R$

$\Rightarrow R = -\frac{i\lambda}{2ka + i\lambda}$ $\qquad T = \frac{2ka}{2ka + i\lambda}$

Prob refl $= |R|^2 = \frac{\lambda^2}{4k^2a^2 + \lambda^2}$ $\qquad$ Prob trans $= |T|^2 = \frac{4k^2a^2}{4k^2a^2 + \lambda^2}$

6) a) Normalize $\quad A^2(1 + 2^2) = 1 \Rightarrow A = 1/\sqrt{5}$

b) $\Psi(x,t) = \frac{1}{\sqrt{5}}\left(u_5(x) e^{-iE_5 t/\hbar} + 2\, u_6(x)\, e^{-iE_6 t/\hbar}\right)$

$P(x,t) = |\Psi(x,t)|^2 = \frac{1}{5}\left[u_5^2(x) + 4u_6^2(x) + 2u_5(x)u_6(x)\left\{e^{i(E_5-E_6)t/\hbar} + e^{-i(E_5-E_6)t/\hbar}\right\}\right.$

could stop here $= \frac{1}{5}\left\{u_5^2(x) + 4u_6^2(x) + 4u_5(x)u_6(x)\cos[(E_6 - E_5)t/\hbar]\right\}$

or

here $= \frac{2}{5a}\left\{\sin^2\left(\frac{5\pi x}{a}\right) + 4\sin^2\left(\frac{6\pi x}{a}\right) + 4\sin\left(\frac{5\pi x}{a}\right)\sin\left(\frac{6\pi x}{a}\right)\cos\left[\frac{(E_6 - E_5)t}{\hbar}\right]\right\}$

c) $j(x) = \frac{\hbar}{2im}\left[\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi(x)\right] = \frac{\hbar}{m}\mathrm{Im}\left[\Psi^* \frac{\partial \Psi}{\partial x}\right]$

$= \frac{\hbar}{m}\frac{1}{5}\mathrm{Im}\left[\left(u_5(x)e^{iE_5t/\hbar} + 2u_6(x)e^{iE_6t/\hbar}\right)\left(u_5'(x)e^{-iE_5t/\hbar} + u_6'(x)e^{-iE_6t/\hbar}\right)\right]$

$= \frac{\hbar}{m}\frac{1}{5}\mathrm{Im}\left[\cancel{u_5 u_5'} + \cancel{4u_6 u_6'} + 2u_6(x)u_5'(x)e^{i(E_6-E_5)t/\hbar} + 2u_5(x)u_6'(x)e^{-i(E_6-E_5)t/\hbar}\right]$

could stop here $= \frac{\hbar}{m}\frac{2}{5}\left[u_6(x)u_5'(x) - u_5(x)u_6'(x)\right]\sin\left[(E_6 - E_5)t/\hbar\right]$

or

here $= \frac{4\hbar\pi}{5ma^2}\left[5\sin\left(\frac{6\pi x}{a}\right)\cos\left(\frac{5\pi x}{a}\right) - 6\sin\left(\frac{5\pi x}{a}\right)\cos\left(\frac{6\pi x}{a}\right)\right]\sin\left[\frac{(E_6 - E_5)t}{\hbar}\right]$

7) From bottom of test $\quad U_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}$

From Eq 4-108 $\quad y H_n(y) = n\, H_{n-1}(y) + \frac{1}{2} H_{n+1}(y)$

$$x U_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-\frac{m\omega x^2}{2\hbar}} \underbrace{x H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right)}$$

$$\downarrow$$

$$\underbrace{\sqrt{\frac{\hbar}{m\omega}} \sqrt{\frac{m\omega}{\hbar}} x H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right)}$$

$$\downarrow$$

$$\sqrt{\frac{\hbar}{mm}} \left[n H_{n-1} + \frac{1}{2} H_{n+1} \right]$$

$$\frac{n}{\sqrt{2^n n!}} = \frac{1}{\sqrt{2}} \sqrt{\frac{n \cdot n}{2^{n-1} n!}} = \sqrt{\frac{n}{2}} \frac{1}{\sqrt{2^{n-1}(n-1)!}}$$

$$\frac{1}{2} \sqrt{\frac{1}{2^n n!}} = \frac{1}{\sqrt{2}} \sqrt{\frac{n+1}{2^{n+1}(n+1)!}} = \sqrt{\frac{n+1}{2}} \frac{1}{\sqrt{2^{n+1}(n+1)!}}$$

$$x U_n(x) = \sqrt{\frac{n\hbar}{2m\omega}} U_{n-1}(x) + \sqrt{\frac{(n+1)}{2} \frac{\hbar}{m\omega}} U_{n+1}(x)$$