PHYS460, Test 2, Fall 2016

You must show work to get credit!!!!!! Possible integrals you might need are in the back cover of the book.

(1) (5 pts) The potential $V(x, y, z) = -V_0 \exp(-(x^2 + y^2 + z^2)/a^2)$ has 12 bound states with $\ell = 2$. Sketch the radial part of the 5th eigenstate that has $\ell = 2$. Quantitatively describe as many independent features of this radial function as you can. (More features = more points)

(2) (5 pts) We've all experienced the strange propagation of odors. This is because odor molecules obey a bizarre Hamiltonian. For two spatial dimensions, it can be simplified to: $\hat{H} = \hat{p}_x\hat{p}_y/M$. The initial wave function gives $<\hat{x}>= x_0$, $<\hat{p}_x>= p_{x,0}$ and $<\hat{y}>= y_0$, $<\hat{p}_y>= p_{y,0}$. (a) Determine the expectation values of the position and momentum operators for all later times. (b) From your observation, order the *group average* personal hygiene of math, physics, chemistry, and biology majors from good to poor. *Please don't explain.*

(3) (5 pts) Buckminsterfullerene, C_{60}, has an average diameter of 7.1 Å. Treat this molecule as an infinitely stiff, spherical mass shell and ignore the trivial center of mass motion. Under these limitations, what kind of motion is left? Under these limitations, what is the Hamiltonian for this molecule and what are the eigenergies? Give your answer in J and K.

(4) (5 pts) Two operators are represented by the matrices

$$A = \xi \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad B = \eta \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5i \\ 0 & -5i & 0 \end{pmatrix} \tag{1}$$

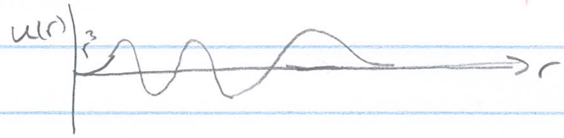
where ξ and η are real. (a) For each operator, specify whether it is Hermitian, anti-Hermitian, or neither. (b) Do the operators commute? (c) Is it possible to find a complete set of states that are simultaneously eigenstates of these two operators? Why?

(5) (10 pts) Fred, a well meaning quantum tardigrade of mass M_F, can only move in 2D. Fred is captured in the bound state of the potential $V(x) = -\alpha_x\delta(x) - \alpha_y\delta(y)$ where α_x and α_y are positive. (a) Determine the units of the α. (b) Determine Fred's normalized wave function. (c) What is the probability of finding Fred between x and $x + dx$? (d) What is the probability of finding Fred between x and $x + dx$ *and* between y and $y + dy$? (e) What is the probability that Fred's momentum is between p_x and $p_x + dp_x$? (f) What is the probability that Fred's momentum is between p_x and $p_x + dp_x$ *and* between p_y and $p_y + dp_y$?

(6) (10 pts) Fred's situation has deteriorated so that it can only move in 1D. Fred is traveling to the left from large positive x. The potential energy is $V(x) = \beta\delta(x)$ and $V(x) = 0$ for $x > 0$ and $V(x) = V_0 > 0$ for $x < 0$. (a) Set up the wave function that represents this situation when $E > V_0$. (b) Obtain all of the equations between the different parameters in your wave function. (c) Solve for the parameters to obtain the reflection and transmission probability.

(7) (10 pts) The operator $\hat{Q} = \hat{b}^\dagger\hat{b}$ where $\hat{b}$ is an operator that can be non-hermitian. (a) Show that $\hat{Q}$ must be hermitian, must be non-hermitian, or could be either. Make sure to show intermediate steps so I know that you're not just bluffing. (b) If you write $\hat{b} = \sum_{n,m} |n> b_{nm} < m|$ with $|n>$ orthonormal states, give the expression for Q_{nm}. (c) *Show* that the real part of $<\hat{Q}>$ is always greater than or equal to 0. (d) *Show* that the real part of every eigenvalue of $\hat{Q}$ is greater than or equal to 0.

1) The function $u(r)$ should behave like r^3 at small r, should have 4 nodes, smoothly go to 0 as $r \to \infty$, oscillate somewhat faster at small r



2) a) Use Eq 3.71 $\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[H,\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

$$[P_x, H] = 0 \qquad \frac{\partial P_x}{\partial t} = 0$$

$$[P_y, H] = 0 \qquad \frac{\partial P_y}{\partial t} = 0$$

This means $\frac{d\langle P_x\rangle}{dt} = 0 \Rightarrow \langle\hat{P}_x\rangle = P_{x,0}$

and similar for $\langle P_y\rangle = P_{y,0}$

$$[H, x] = P_y/m\,[P_x, x] + \cancel{[P_y, x]}^{\to 0} P_x/m = -i\hbar\,P_y/m$$

and $[H, y] = P_y/m\,\cancel{[P_x, y]}^{\to 0} + [P_y, y]\,P_x/m = -i\hbar\,P_x/m$

This gives

$$\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\left(-i\hbar\frac{\langle P_y\rangle}{m}\right) = \frac{\langle P_y\rangle}{m} = \frac{P_{y,0}}{m} \Rightarrow \langle x\rangle = x_0 + \frac{P_{y0}}{m}t$$

$$\frac{d\langle y\rangle}{dt} = \frac{i}{\hbar}\left(-i\hbar\frac{\langle P_x\rangle}{m}\right) = \frac{\langle P_x\rangle}{m} = \frac{P_{x,0}}{m} \Rightarrow \langle y\rangle = y_0 + \frac{P_{x,0}}{m}t$$

b) ???

3) The only motion left is rotation. You can either appeal to classical mechanics $H = L^2/2I$ or use Eq 4.14 with all derivatives with respect to r set to 0.

$$I = MR^2 = 60 \cdot 12 \cdot 1.66\times10^{-27}\,kg \cdot \left(\frac{7.1\times10^{-10}\,m}{2}\right)^2 = 1.51\times10^{-43}\,kg\,m^2$$

Using Eq. 4.18 the eigen energies are $\frac{\hbar^2 l(l+1)}{2I} = E_l$

$$E_l = \frac{(1.0545\times10^{-34}\,Js)^2}{2\cdot 1.51\times10^{-43}\,kg\,m^2}\,l(l+1) = 3.69\times10^{-26}\,J \cdot l(l+1)$$

$$= 2.67\times10^{-3}\,K\;l(l+1)$$

4) a) They are both Hermitian because the conjugate transpose is the same $A^{T*} = A$ and $B^{T*} = B$

b) $AB - BA = \xi\eta\left[\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5i \\ 0 & -5i & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5i \\ 0 & -5i & 0 \end{pmatrix}\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right]$

$= \xi\eta\left[\begin{pmatrix} 21 & 0 & 0 \\ 0 & 0 & 10i \\ 0 & -10i & 0 \end{pmatrix} - \begin{pmatrix} 21 & 0 & 0 \\ 0 & 0 & 10i \\ 0 & -10i & 0 \end{pmatrix}\right] = 0$ commute

c) Yes: Because they commute. The eigenstates are

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$

5) Because the potential is separable, the wave function can be written as $\psi(x,y) = \psi^{(1)}(x)\,\psi^{(2)}(y)$

a) The $\delta(x)$ has units of $1/m \Rightarrow$ units of $\alpha = J \cdot m$

b) Use Eq. 2.129 for the bound state of a delta function

$\psi(x,y) = \frac{\sqrt{m\alpha_x}}{\hbar} e^{-m\alpha_x|x|/\hbar^2} \frac{\sqrt{m\alpha_y}}{\hbar} e^{-m\alpha_y|y|/\hbar^2}$

c) Fred can have any y $\quad P(x)\,dx = |\psi^{(1)}(x)|^2 dx = \frac{m\alpha_x}{\hbar^2} e^{-2m\alpha_x|x|/\hbar^2} dx$

d) $P(x,y)\,dx\,dy = |\psi(x,y)|^2 dx\,dy = \frac{m^2\alpha_x\alpha_y}{\hbar^4} e^{-2m(\alpha_x|x| + \alpha_y|y|)/\hbar^2} dx\,dy$

e) Again, Fred can have any P_y. Look at Example 3.4

$\Phi(P_x,t) = \sqrt{\frac{2}{\pi}} \left(\frac{m\alpha_x}{\hbar}\right)^{3/2} e^{-iEt/\hbar} / \left(P_x^2 + \left(\frac{m\alpha_x}{\hbar}\right)^2\right)$

$P(P_x)\,dP_x = |\Phi(P_x,t)|^2 = \left[\frac{2}{\pi}\left(\frac{m\alpha_x}{\hbar}\right)^3 / \left(P_x^2 + \left(\frac{m\alpha_x}{\hbar}\right)^2\right)^2\right] dP_x$

f) $P(P_x,P_y)\,dP_x\,dP_y = \left[\frac{2}{\pi}\left(\frac{m\alpha_x}{\hbar}\right)^3 / \left[P_x^2 + \left(\frac{m\alpha_x}{\hbar}\right)^2\right]^2\right]\left[\frac{2}{\pi}\left(\frac{m\alpha_y}{\hbar}\right)^3 / \left(P_y^2 + \left(\frac{m\alpha_y}{\hbar}\right)^2\right)^2\right] dP_x\,dP_y$

6) a) Separate the wave function into two regions

$\psi_I(x) = T e^{-ik_I x}$ $\qquad \frac{\hbar^2 k_I^2}{2m} + V_0 = E$

$\psi_{II}(x) = e^{-ik_{II} x} + R e^{ik_{II} x}$ $\qquad \frac{\hbar^2 k_{II}^2}{2m} = E$

↑reflection

Continuity of $\psi \Rightarrow \psi_I(0) = \psi_{II}(0) \Rightarrow T = 1 + R$

The derivative is not continuous. Eq 2.124

$$\frac{d\psi_{II}(0)}{dx} - \frac{d\psi_I(0)}{dx} = -\frac{2m\beta}{\hbar^2}\psi_I(0)$$

$ik_{II}R - ik_{II} + ik_I T = -\frac{2m\beta}{\hbar^2}T$ $\qquad$ Put $T = 1 + R$

$ik_{II}R - ik_{II} + ik_I + ik_I R = -\frac{2m\beta}{\hbar^2} - \frac{2m\beta}{\hbar^2}R$ $\Big]$ algebra

$\left(ik_{II} + ik_I + \frac{2m\beta}{\hbar^2}\right)R = \left(ik_{II} - ik_I - \frac{2m\beta}{\hbar^2}\right)$

$$R = \left(k_{II} - k_I + \frac{i2m\beta}{\hbar^2}\right) \Big/ \left(k_{II} + k_I - \frac{i2m\beta}{\hbar^2}\right)$$

$$T = 1 + R = 2k_{II} \Big/ \left(k_{II} + k_I - \frac{i2m\beta}{\hbar^2}\right)$$

The reflection probability is $|R|^2$. The transmission probability is $\frac{k_I}{k_{II}}|T|^2 = 1 - |R|^2$

7) a) $\hat{Q}^\dagger = (\hat{b}^\dagger \hat{b})^\dagger = \hat{b}^\dagger \hat{b}^{\dagger\dagger} = \hat{b}^\dagger \hat{b} = \hat{Q}$ $\Rightarrow$ Hermitian

b) $b = \sum_{mn} |m\rangle b_{mn} \langle n|$ $\qquad \hat{b}^\dagger = \sum_{mn} |n\rangle b^*_{mn} \langle m| = \sum_{mn} |m\rangle b^*_{nm} \langle n|$

$\hat{b}^\dagger \hat{b} = \left(\sum_{mn'} |m\rangle b^*_{n'm} \langle n'|\right)\left(\sum_{nm'} |m'\rangle b_{m'n} \langle n|\right)$

$= \sum_{mn} |m\rangle \left(\sum_{n'} b^*_{n'm} b_{n'n}\right) \langle n| \Rightarrow Q_{mn} = \sum_{n'} b^*_{n'm} b_{n'n}$

c) Since $\hat{Q}$ is Hermitian $\langle \hat{Q} \rangle$ = real

$\langle \hat{Q} \rangle \underset{\text{definition}}{=} \langle \Psi | \hat{b}^\dagger \hat{b} | \Psi \rangle \underset{\text{definition}}{=} \langle \hat{b}\Psi | \hat{b}\Psi \rangle = \langle g | g \rangle$

with $|g\rangle = \hat{b}|\Psi\rangle$

$\langle g | g \rangle = \int g^*(x) g(x)\,dx = \int |g(x)|^2 dx \geq 0$ since $|g(x)|^2 \geq 0$

d) In terms of the eigenstates $\hat{Q}|f_n\rangle = q_n |f_n\rangle$

$|\Psi\rangle = \sum_n c_n |f_n\rangle$ and $\langle \hat{Q} \rangle = \sum_n |c_n|^2 q_n \geq 0$ This has to be true for any $|c_n|^2$. If any $q_n < 0$ make $c_n = 1$ and all others 0. This would give $\langle \hat{Q} \rangle < 0$ which can't be true.