PHYS460, Test 2, Fall 2015

You must show work to get credit!!!!!! Possible integrals you might need are in the back cover of the book.

(1) (5 pts) The evil genius Flexinor has created quantum tardigrades (aka water bears or moss piglets). As part of his evil plot, he's put one in a *spherically symmetric* potential. The eigenstate energies are labeled by the number of radial nodes, n_r, in the wave function and on ℓ and m: $E_{n_r,\ell,m}$. For each of the following pairs, put them in order from lower to higher energy. (Hint: two of the pairs are *trick questions*! For those pairs say what the trick is.) (a) $E_{5,4,3}, E_{4,4,3}$ (b) $E_{5,4,3}, E_{5,5,3}$, (c) $E_{5,4,3}, E_{5,4,4}$, (d) $E_{5,4,5}, E_{5,3,5}$, (e) $E_{1,4,3}, E_{0,4,3}$

(2) (5 pts) A quantum tardigrade is confined so it can only move in the x-direction. The potential energy is 0 for $|x| > a$ and is $-V_0$ for $|x| < a$. Derive the transcendental equation for the allowed energies for the odd bound states.

(3) (5 pts) (a) When is the sum of two hermitian operators hermitian? (b) When is the product of two hermitian operators hermitian? (c) Is the operator $\hat{\vec{r}} \times \hat{\vec{p}}$ hermitian? (d) Is the operator $\hat{\vec{r}} \cdot \hat{\vec{p}}$ hermitian?

(4) (5 pts) The tardigrade is now confined in a 2D infinite circular well: $V = 0$ for $\rho < a$ and is infinite otherwise. The radial part of the Schrödinger equation for $\rho < a$ is

$$-\frac{\hbar^2}{2M}\left(\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} - \frac{m^2}{\rho^2}F\right) = EF \tag{1}$$

where m is the azimuthal quantum number. (a) Transform the equation in terms of $s = \alpha\rho$. (b) Determine what α is in terms of E, $\hbar$, M, etc. (c) For the case $m = 0$, write F as a power series in s and give the recursion relation that lets you get c_n from c_{n-1} and c_{n-2}.

(5) (10 pts) Flexinor constructs a Hilbert space of three orthonormal states to blow the tardigrade's tiny mind. The Hermitian operator $\hat{Q} = \alpha|\psi_1 >< \psi_3| + \beta|\psi_3 >< \psi_1|$ where α and β are complex numbers. (a) Is there a relation between α and β? If yes, give the relationship and the explanation for it. (b) Give the expression for $\hat{Q}^2$ in terms of bra's and ket's of ψ's. (c) For the case where α and β are real and positive, find all of the eigenvalues and eigenvectors of $\hat{Q}$.

(6) (10 pts) The mad scientist Grblnxyklmr gets into the tardigrade business by confining one in a spherical potential which is 0 for $r > a$ and is $-V_0$ for $r < a$. (a) For the case $\ell = 0$, give the equation that determines the bound state energies. (b) Is there always a bound state? If no, give the minimum value of V_0 that allows at least one bound state. (c) Would you rather be an evil genius or a mad scientist? Briefly explain.

(7) (10 pts) Compute the matrix elements $< Y_{\ell'}^{m'}|\cos(\theta)|Y_\ell^m >$. The recursion relation

$$(\ell + 1 - m)P_{\ell+1}^m(x) = x(2\ell + 1)P_\ell^m(x) - (\ell + m)P_{\ell-1}^m(x) \tag{2}$$

may be useful.

Long discussions in class

1) a) $E_{4,4,3} < E_{5,4,3}$ because states with same l, m have E increase w/ n_r

b) $E_{5,4,3} < E_{5,5,3}$ " " " " n_r, m " " " l

c) $E_{5,4,3} = E_{5,4,4}$ " " " " n_r, l have same E

d) neither $E_{5,4,5}$ or $E_{5,3,5}$ exist because $l \geq |m|$

e) $E_{0,4,3} < E_{1,4,3}$ see a)

See HWK6 2.29

2) See Eq. 2.151 except $\psi(x) = \begin{cases} Fe^{-\kappa x} & x > a \\ D \sin lx & |x| < a \\ -Fe^{\kappa x} & x < -a \end{cases}$

Continuity $\quad Fe^{-\kappa a} = D \sin(la)$

Continuity of ψ' $\quad -\kappa F e^{-\kappa a} = l D \cos(la)$

Divide $\quad \frac{1}{l} \tan(la) = -\frac{1}{\kappa} \Rightarrow \boxed{\tan(la) = \frac{-l}{\kappa}}$

where l is from Eq 2.148 and κ is from 2.146

Prob 3.4

3) Define $\hat{A}$ and $\hat{B}$ with $\hat{A}^\dagger = \hat{A}$ and $\hat{B}^\dagger = \hat{B}$

a) $\hat{Q} = \hat{A} + \hat{B} \Rightarrow \hat{Q}^\dagger = \hat{A}^\dagger + \hat{B}^\dagger = \hat{A} + \hat{B} = \hat{Q}$ always

b) $\hat{Q} = \hat{A}\hat{B} \Rightarrow \hat{Q}^\dagger = \hat{B}^\dagger \hat{A}^\dagger = \hat{B}\hat{A} = \hat{A}\hat{B} + [\hat{B}, \hat{A}]$

$= \hat{Q} + [\hat{B}, \hat{A}]$ only when $\hat{A}$ and $\hat{B}$ commute

c) $\hat{L} = \hat{r} \times \hat{p}$ for example $L_x = yP_z - zP_y$

This means $L_x^\dagger \stackrel{\checkmark}{=} L_x$ since $[y, P_z] = 0$ and $[z, P_y] = 0$

d) $\hat{r} \cdot \hat{p} = \hat{x}\hat{P}_x + \hat{y}\hat{P}_y + \hat{z}\hat{P}_z \neq (\hat{r} \cdot \hat{p})^\dagger$ since $[\hat{x}, \hat{P}_x]$ etc don't commute see Hwk 9

4) a) $\rho = s/\alpha \Rightarrow -\frac{\hbar^2\alpha^2}{2m}\left(\frac{d^2F}{ds^2} + \frac{1}{s}\frac{dF}{ds} - \frac{m}{s^2}F\right) = EF$

b) so set $E = \hbar^2\alpha^2/2m$

c) $F'' + \frac{1}{s}F' + F = 0$

$F = C_0 + C_1 s + C_2 s^2 + C_3 s^3 + \dots$

$F'' \quad 2\cdot1\, C_2 s^0 + 3\cdot2\cdot C_3 s^1 + 4\cdot3\, C_4 s^2 + \dots$

$+\frac{1}{s}F' \quad \frac{C_1}{s} + 2C_2 s^0 + 3C_3 s^1 + 4C_4 s^2 + \dots$

$+F \quad C_0 + C_1 + C_2 + \dots$

$=0 \quad \frac{C_1}{s} + (C_0 + 2^2 C_2)s^0 + (C_1 + 3^2 C_3)s^1 + (C_2 + 4^2 C_4)s^2 + \dots$

$\Rightarrow \boxed{C_1 = 0}$ and $\boxed{C_0 = \text{anything} \neq 0}$

$C_{j+2} = -C_j/(j+2)^2$ or $\boxed{C_n = -C_{n-2}/n^2}$ but only even n, all odd n are 0

from orals

5) $\hat{Q} = \alpha|\psi_1\rangle\langle\psi_3| + \beta|\psi_3\rangle\langle\psi_1| = \hat{Q}^+$

$\hat{Q}^+ = \alpha^*|\psi_3\rangle\langle\psi_1| + \beta^*|\psi_1\rangle\langle\psi_3|$

a) yes $\beta = \alpha^*$

b) $\hat{Q}\hat{Q} = (\alpha|\psi_1\rangle\langle\psi_3| + \alpha^*|\psi_3\rangle\langle\psi_1|)(\alpha|\psi_1\rangle\langle\psi_3| + \alpha^*|\psi_3\rangle\langle\psi_1|)$

$= \alpha^2|\psi_1\rangle\cancel{\langle\psi_3|\psi_1\rangle}^{0}\langle\psi_3| + \alpha\alpha^*|\psi_1\rangle\cancel{\langle\psi_3|\psi_3\rangle}^{1}\langle\psi_1|$

$+ \alpha^{*2}|\psi_3\rangle\cancel{\langle\psi_1|\psi_3\rangle}^{0}\langle\psi_1| + \alpha\alpha^*|\psi_3\rangle\cancel{\langle\psi_1|\psi_1\rangle}^{1}\langle\psi_3|$

$= |\alpha|^2(|\psi_1\rangle\langle\psi_1| + |\psi_3\rangle\langle\psi_3|)$

c) $\hat{Q} = \alpha(|\psi_1\rangle\langle\psi_3| + |\psi_3\rangle\langle\psi_1|)$ $\quad \hat{Q}|\psi_q\rangle = q|\psi_q\rangle$

$\hat{Q}|\psi_2\rangle = 0 \Rightarrow |\psi_2\rangle$ is eigenstate with eigenval 0

$\hat{Q}(a_1|\psi_1\rangle + a_3|\psi_3\rangle) = \alpha(a_1|\psi_3\rangle + a_3|\psi_1\rangle) = q(a_1|\psi_1\rangle + a_3|\psi_3\rangle)$

$\Rightarrow \alpha a_3 = q a_1$ and $\alpha a_1 = q a_3$

$a_3 = \frac{\xi}{\alpha} a_1$ from 1st plug into 2nd $\alpha a_1 = \frac{\xi^2}{\alpha} a_1 \Rightarrow \xi = \pm\alpha$

for $\xi = \alpha$ $\quad a_1 = a_3 = 1/\sqrt{2}$ $\quad \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_3\rangle$

for $\xi = -\alpha$ $\quad a_1 = -a_3 = 1/\sqrt{2}$ $\quad \frac{1}{\sqrt{2}}|\psi_1\rangle - \frac{1}{\sqrt{2}}|\psi_3\rangle$

See HWk 6 2.29

6) For $l=0$ $\quad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_0 u = Eu \quad r<a$

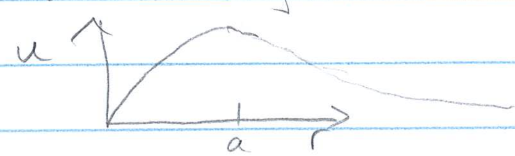
$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu \quad r>a$$

For bound states $E<0$

These are the $\underline{\text{same}}$ equations as Sec 2.6 except you need $u(0)=0$. This boundary condition is the same as problem 2 on this test

a) $\Rightarrow \boxed{\tan(la) = -\frac{l}{\kappa}}$ where l is from Eq 2.148 and κ is 2.146

To see where bound states, the wave function needs to be decreasing as $r \to a$ from below



The $\sin(lr)$ is decreasing if $la \geq \frac{\pi}{2}$ as $E \to 0$

b) $\Rightarrow \frac{\sqrt{2mV_0}}{\hbar} a \geq \frac{\pi}{2} \Rightarrow \boxed{V_0 \geq \left(\frac{\hbar\pi}{2a}\right)^2 \frac{1}{2m}}$

c) Vote gave 7 mad scientists (fun, charismatic, moral issues...) 5 evil geniuses (death ray in back yard, genius!, power), and 2 abstain computationally hardest question

7) I'm going to do $m \geq 0$ so I don't need $|m|$

From Eq 4.32 $Y_l^m(\theta,\phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, e^{im\phi} P_l^m(\cos\theta)$

From test $x P_l^m(x) = \frac{l+1-m}{2l+1} P_{l+1}^m(x) + \frac{l+m}{2l+1} P_{l-1}^m(x)$

This gives

$$\cos\theta\, Y_l^m(\theta,\phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \left[\frac{l+1-m}{2l+1} P_{l+1}^m(\cos\theta) e^{im\phi} + \frac{l+m}{2l+1} P_{l-1}^m(\cos\theta) e^{im\phi}\right]$$

Now use $e^{im\phi} P_{l+1}^m(\cos\theta) = (-1)^m \sqrt{\frac{4\pi}{2l+3}\frac{(l+1+m)!}{(l+1-m)!}}\, Y_{l+1}^m(\theta,\phi)$

and $e^{im\phi} P_{l-1}^m(\cos\theta) = (-1)^m \sqrt{\frac{4\pi}{2l-1}\frac{(l-1+m)!}{(l-1-m)!}}\, Y_{l-1}^m(\theta,\phi)$

$$\cos\theta\, Y_l^m(\theta,\phi) = C_l^m\, Y_{l+1}^m(\theta,\phi) + D_l^m\, Y_{l-1}^m(\theta,\phi)$$

$$C_l^m = \cancel{(-1)^m} \sqrt{\frac{2l+1}{\cancel{4\pi}}\frac{(l-m)!}{(l+m)!}}\, \frac{l+1-m}{2l+1}\, \cancel{(-1)^m} \sqrt{\frac{\cancel{4\pi}}{2l+3}\frac{(l+1+m)!}{(l+1-m)!}} = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}$$

where I used $(l+1+m)!/(l+m)! = l+1+m$ and $(l-m)!/(l+1-m)! = \frac{1}{l+1-m}$

$$D_l^m = \cancel{(-1)^m} \sqrt{\frac{(2l+1)}{\cancel{4\pi}}\frac{(l-m)!}{(l+m)!}}\, \frac{l+m}{2l+1}\, (-1)^m \sqrt{\frac{\cancel{4\pi}}{2l-1}\frac{(l-1+m)!}{(l-1-m)!}} = \sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}}$$

where I used $(l-m)!/(l-1-m)! = l-m$ and $(l-1+m)!/(l+m)! = \frac{1}{l+m}$

Since the answer only depends on m^2 will work for all m

$$\langle Y_{l'}^{m'} | \cos\theta | Y_l^m \rangle = \left(C_l^m \delta_{l',l+1} + D_l^m \delta_{l',l-1}\right)\delta_{mm'}$$

$$= \left(\sqrt{\frac{l'^2 - m^2}{4l'^2 - 1}}\, \delta_{l',l+1} + \sqrt{\frac{l^2 - m^2}{4l^2 - 1}}\, \delta_{l'+1,l}\right)\delta_{mm'}$$