PHYS460, Test 1, Fall 2015

You must show work to get credit. Possible integrals you might need are in the back cover of the book.

(1) (5 pts) A particle is in the potential well that has the form $V(x) = Cx^4$ where $C > 0$. Describe the features of the eigenstates and/or eigenvalues. You will get 1 pt for each *independent* feature that is correct, but you will be deducted 1 pt for each feature that is wrong.

(2) (5 pts) For this problem, you are scientific advisor for the science fiction movie *Shelby's Conundrum* where the premise is $\hbar = 1$ J s. The hero, Shelby McShelby III, measures the position and velocity of a pack of killer gerbils ($M_{gerbil} = 1000$ kg) that are all in the same (angry) quantum state (don't ask why; I didn't say it was a good movie). He measured: 8.1 m 4 times, 8.3 m 10 times, 8.4 m 4 times, and 8.5 m 2 times for the position and -0.50 m/s 4 times, -0.51 m/s 12 times, and -0.52 m/s 4 times for the velocity. (a) Are these measurements consistent with the quantum mechanics of Shelby's world? (b) Would you use your real name in the movie credits?

(3) (5 pts) A particle experiences a constant force in the $-x$ direction. There is an infinite wall at $x = 0$ so that the particle is only measurable at $x > 0$. Give the wave function at some random energy $E > 0$ as a power series in x through the term proportional to x^5.

(4) (5 pts) The wave function $\Psi(x, 0) = C \exp(-\alpha|x| + ibx)$ where α, b, and C are positive real constants and $-\infty < x < \infty$. Compute the expectation value of $\hat{x}$ and the expectation value of $\hat{p}$.

(5) (10 pts) You have a potential $V(x) = -\alpha[\delta(x - a) + \delta(x + a)]$. Determine the ground state wave function and the transendental equation for the ground state energy.

(6) (10 pts) You have an electron in an infinite square well with $0 < x < a$. The wave function $\Psi(x, 0) = Bx^2(a - x)$. Compute the average energy that you would measure. Compare your result to the ground state energy.

(7) (10 pts) You have a mass that experiences the potential $V(x) = (1/2)M\omega^2x^2$. The wave function at $t = 0$ is $\Psi(x, 0) = Dx(1 - 4\sqrt{M\omega/\hbar x}) \exp(-M\omega x^2/[2\hbar])$. (a) Compute the average energy you would measure. (b) Compute $\Psi(x, t)$.

PHYS 460 Test 1

oral 1st class

1) There are many possible features. My list is probably missing a few: 1) $\psi_n(x)$ is continuous for all x, 2) $\psi'(x)$ is continuous for all x, 3) $\psi_n(x) = (-1)^{n+1}\psi_n(-x)$ if the ground state is $n=1$, 4) the $\psi_n(x)$ oscillates fastest near $x=0$, 5) the number of nodes is $n-1$, 6) for $|x| > (E_n/c)^{1/4}$, the $\psi_n(x)$ rapidly but smoothly goes to 0 as $|x|\to\infty$, 7) the energies are discrete (not continuous) for all n, 8) $E_{n+1}-E_n$ is an increasing function of n, 9) the amplitude of the oscillation in x is smallest $x\sim 0$, 10) etc

HWK Prob 1.1 and 1.9d

2) a) You need to check whether $\sigma_x \sigma_p \geq \hbar/2$. For this problem $\hbar/2 = \frac{1}{2}$ Js

$$\langle x\rangle = (4 \times 8.1\text{m} + 10\times 8.3\text{m} + 4\times 8.4\text{m} + 2\times 8.5\text{m})/20 = 8.3\text{m}$$

$$\sigma_x^2 = [4\times(0.2\text{m})^2 + 10\times(0\text{m})^2 + 4\times(0.1\text{m})^2 + 2\times(0.2\text{m})^2]/20 = 0.014\text{m}^2$$

$$\sigma_x = 0.12\text{ m}$$

$$\langle v\rangle = [4\times(-0.50\,\text{m/s}) + 12\times(-0.51\,\text{m/s}) + 4\times(-0.52\,\text{m/s})]/20 = -0.51\,\text{m/s}$$

$$\sigma_v^2 = [4\times(0.01\,\text{m/s})^2 + 12\times(0\,\text{m/s})^2 + 4\times(0.01\,\text{m/s})^2]/20 = 4\times10^{-5}\,\text{m}^2/\text{s}^2$$

$$\sigma_v = 6.3\times10^{-3}\,\text{m/s}$$

$$\sigma_x\sigma_p = 0.12\text{m}\ 1000\text{kg}\ 6.3\times10^{-3}\,\text{m/s} = 0.76\text{ Js}$$

This is larger than $\hbar/2$. So this is OK.

b) No. Everyone needs a stage name.

Similar to Eqs. 2.78-2.81

3) The time independent Schrodinger Eq. is $V(x) = F\cdot x$

$$-\frac{\hbar^2}{2m}\psi(x) + Fx\,\psi(x) = E\psi(x)$$

The wave function must be 0 at $x=0$

$$\psi(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

Substitute into the Sch. Eq. Can set $\underline{a_1 = 1}$

$$-\frac{\hbar^2}{2m}\left(2\cdot 1\, a_2 + 3\cdot 2\, a_3 x + 4\cdot 3\, a_4 x^2 + 5\cdot 4\, a_5 x^3 + \dots\right.$$

$$+ \qquad F a_1 x^2 + F a_2 x^3 + \dots$$

$$= \qquad E a_1 x + E a_2 x^2 + E a_3 x^3 + \dots$$

There is only term with x^0

There is only one term with $x^0 \Rightarrow \underline{a_2 = 0}$

Term with $x^1 \Rightarrow -\frac{3\hbar^2}{m}a_3 = E a_1 \Rightarrow a_3 = \underline{-\frac{mE}{3\hbar^2}}$

Term with $x^2 \Rightarrow -\frac{6\hbar^2}{m}a_4 + F a_1 = E a_2 = 0 \Rightarrow a_4 = \underline{\frac{Fm}{6\hbar^2}}$

Term with $x^3 \Rightarrow -\frac{10\hbar^2}{m}a_5 + F a_2 = E a_3 \Rightarrow a_5 = \frac{-mE}{10\hbar^2}a_3 = \underline{\frac{m^2E^2}{30\hbar^4}}$

HWk Prob 1.53, 2.4 in class, and many others

4) $$\langle\hat{x}\rangle = \int_{-\infty}^{\infty}\Psi^*(x,0)\,x\,\Psi(x,0)\,dx = C^2\int_{-\infty}^{\infty} x\, e^{-2\alpha|x|}\,dx$$

The function $x e^{-2\alpha|x|}$ is odd $\Rightarrow \boxed{\langle\hat{x}\rangle = 0}$

$$\langle\hat{p}\rangle = \int_{-\infty}^{\infty}\Psi^*(x,0)\,\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,0)\,dx$$

$$= C^2\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-ibx}\left(\hbar b\, e^{-\alpha|x|}e^{ibx} + \frac{\hbar}{i}e^{ibx}\frac{d}{dx}e^{-\alpha|x|}\right)dx$$

$$= \hbar b\, C^2\int_{-\infty}^{\infty} e^{-2\alpha|x|}dx + C^2\frac{\hbar}{i}\int_{-\infty}^{\infty} e^{-\alpha|x|}\left(\frac{d}{dx}e^{-\alpha|x|}\right)dx$$

From normalization $C^2\int_{-\infty}^{\infty} e^{-2\alpha|x|}\,dx = 1$

For any function $f(x)$ $\int_{-\infty}^{\infty} f(x)\,f'(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{dx}\left(f^2(x)\right)dx$

$$= \frac{1}{2}f^2(x)\Big|_{-\infty}^{\infty}$$

$$\langle\hat{p}\rangle = \hbar b\cdot 1 + C^2\frac{\hbar}{i}\frac{1}{2}e^{-2\alpha|x|}\Big|_{-\infty}^{\infty} = \boxed{\hbar b}$$

prob 2.27 + oral

5) For the ground state $E < 0$. Write $E = -\frac{\hbar^2 k^2}{2m}$

Except at the δ functions $\psi'' = k^2 \psi$. In each region ψ can be superposition of e^{-kx} and e^{kx}

$$\psi_I(x) = A e^{kx} + \underbrace{\cancel{B e^{-kx}}}_{\to 0} \quad \text{because } \psi(x \to -\infty) \to 0$$

$$\psi_{II}(x) = B(e^{kx} + e^{-kx}) \quad \text{because } \psi(-x) = \psi(x) \text{ for ground state}$$

$$\psi_{III}(x) = A e^{-kx}$$

Continuity $\psi_{II}(a) = \psi_{III}(a) \Rightarrow B(e^{ka} + e^{-ka}) = A e^{-ka}$

Discontinuity of ψ' at $x = a$

$$-\frac{\hbar^2}{2m}\left(\psi'_{III}(a) - \psi'_{II}(a)\right) = \alpha \psi_{III}(a)$$

$$\frac{\hbar^2 k}{2m}\left(A e^{-ka} - B(e^{-ka} - e^{ka})\right) = \alpha A e^{-ka}$$

From continuity $B = A e^{-ka} \frac{1}{e^{ka} + e^{-ka}}$ substitute

$$\frac{\hbar^2 k}{2m} \cancel{A e^{-ka}} \left(1 - \frac{e^{-ka} - e^{ka}}{e^{-ka} + e^{ka}}\right) = \alpha \cancel{A e^{-ka}}$$

$$\frac{\hbar^2 k}{\cancel{2m}} \frac{2 e^{ka}}{e^{ka} + e^{-ka}} = \alpha \Rightarrow \boxed{k = \frac{m\alpha}{\hbar^2}\left(1 + e^{-2ka}\right)}$$

Hwk Prob 2.9 and oral

6) $\langle \hat{H} \rangle = \int_0^a \Psi^*(x,0) \frac{-\hbar^2}{2m} \Psi''(x,0)\, dx$

$= \frac{-\hbar^2}{2m} B^2 \int_0^a x^2 (a-x)(2a-6x)\, dx$

$= \frac{-\hbar^2}{2m} B^2\, 2 \int_0^a x^2 (a^2 - 4ax + 3x^2)\, dx = \frac{-\hbar^2}{2m} B^2\, 2 \left(a^2 \frac{x^3}{3} - a x^4 + \frac{3}{5} x^5 \right)\Big|_0^a$

$= -\frac{\hbar^2}{m} B^2 a^5 \left(\frac{1}{3} - 1 + \frac{3}{5} \right) = \frac{\hbar^2}{15m} a^5 B^2$

To find B^2 normalize

$1 = B^2 \int_0^a x^4 (a-x)^2\, dx = B^2 \int_0^a x^4 (a^2 - 2ax + x^2)\, dx$

$= B^2 \left(a^2 \frac{x^5}{5} - a \frac{x^6}{3} + \frac{x^7}{7} \right)\Big|_0^a = B^2 a^7 \left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7} \right) = B^2 a^7 \frac{21 - 35 + 15}{3 \cdot 5 \cdot 7}$

$= \frac{B^2 a^7}{15 \cdot 7} \Rightarrow \quad B^2 = \frac{15 \cdot 7}{a^7}$

$\langle \hat{H} \rangle = \frac{\hbar^2\, 7}{a^2 m} = \frac{\hbar^2\, 14}{2 a^2 m}$

prob 2.41 done in class

7) You can do this with either the ladder operators or by using the recursion relation for Hermite pols. I will do it with ladder operators.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \quad \text{and} \quad e^{-m\omega x^2/2\hbar} = \left(\frac{\pi\hbar}{m\omega}\right)^{1/4}\psi_0$$

$$\Psi(x,0) = D\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)(1 - 2\sqrt{2}a_+ - 2\sqrt{2}a_-)\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}\psi_0$$

$$= D\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{\pi\hbar}{m\omega}\right)^{1/4}(\hat{a}_+ - 2\sqrt{2}\hat{a}_+^2 - 2\sqrt{2}\hat{a}_+\hat{a}_- + \hat{a}_- - 2\sqrt{2}\hat{a}_-\hat{a}_+ - 2\sqrt{2}\hat{a}_-^2)\psi_0$$

$$= D\frac{\pi^{1/4}}{\sqrt{2}}\left(\frac{\hbar}{m\omega}\right)^{3/4}(\sqrt{1}\psi_1 - 2\sqrt{2}\sqrt{2\cdot 1}\psi_2 - 0 + 0 - 2\sqrt{2}\sqrt{1}^2\psi_0 - 0)$$

$$= D\frac{\pi^{1/4}}{\sqrt{2}}\left(\frac{\hbar}{m\omega}\right)^{3/4}(\psi_1 - 4\psi_2 - 2\sqrt{2}\psi_0)$$

To find D normalize

$$\int_{-\infty}^{\infty}\Psi^*(x,0)\Psi(x,0)dx = D^2\frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{3/2}(1 + 16 + 8) = 1$$

$$\Rightarrow D = \frac{\sqrt{2}}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{3/4}\frac{1}{\sqrt{25}}$$

$$\Psi(x,0) = \frac{\sqrt{2}}{\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{3/4}\frac{\pi^{1/4}}{\sqrt{2}}\left(\frac{\hbar}{m\omega}\right)^{3/4}\frac{1}{5}(\psi_1 - 4\psi_2 - 2\sqrt{2}\psi_0)$$
$$= -\frac{2\sqrt{2}}{5}\psi_0 + \frac{1}{5}\psi_1 - \frac{4}{5}\psi_2$$

$$\Psi(x,t) = -\frac{2\sqrt{2}}{5}\psi_0 e^{-i\omega t/2} + \frac{1}{5}\psi_1 e^{-3i\omega t/2} - \frac{4}{5}\psi_2 e^{-5i\omega t/2}$$

where the ψ_0, ψ_1, ψ_2 are from Eq. 2.85

$$\langle H\rangle = \frac{8}{25}\frac{\hbar\omega}{2} + \frac{1}{25}\frac{3\hbar\omega}{2} + \frac{16}{25}\frac{5\hbar\omega}{2} = \frac{91}{50}\hbar\omega$$