**PHYS460, Test 2, Fall 2014**

**You must show work to get credit!!!!!!**

(1) (5 pts) Sketch the $P_\ell^m(x)$ function in the relevant range of $x$ for the case of $\ell = 7$, $m = 4$. Point out all of the relevant features for that $\ell, m$.

(2) (5 pts) In T. Li, *et al*, Phys. Rev. Lett. **109**, 163001 (2012), they proposed an experiment that would consist of a ring of 100 trapped $^9Be^+$ ions. As a first step, they will trap one ion. You can approximate the ion motion as confined to a ring of radius 50 nm in the $xy$-plane. What are the lowest 3 energy levels in Joules and in Kelvins?

(3) (5 pts) You have a 1D potential with the form $V(x) = 0$ for $|x| > a$ and $V(x) = -(1/10)\hbar^2\pi^2/(2M[2a]^2)$ for $|x| < a$. There is one bound state. Give the bound state energy in the form $E = -f\hbar^2\pi^2/(2M[2a]^2)$ with your value of $f$ good to 2 significant digits. Make sure to clearly write down your algorithm.

(4) (5 pts) Laser cooling and trapping techniques have progressed to the point where a quantum hamster with mass $M_h$ is in the ground state of an infinite square well potential, $V(x) = 0$ for $0 < x < a$ and $V(x) = \infty$ elsewhere. (a) What is the probability to measure the hamster's momentum between $p$ and $p + dp$? (b) Is it ethical to expose an innocent hamster to laser cooling and trapping techniques?

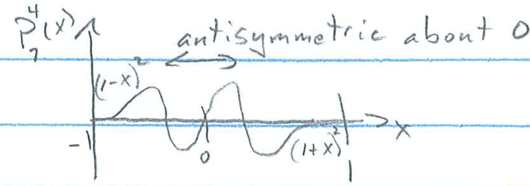
(5) (10 pts) The 3D potential energy for a quark can be (crudely) approximated as linearly increasing with distance from the origin. For a specified energy $E > 0$, give the first 4 nonzero terms in the power series expansion (in $r$) of the radial part of the wave function for $\ell = 2, m = -1$. Do not worry about normalization or whether $E$ is an eigenenergy.

(6) (10 pts) For classical particles, the equations for the angular momenta are $d\vec{L}/dt = \vec{N}$ where the torque $\vec{N} = \vec{r} \times \vec{F}(\vec{r})$. (a) For a quantum particle, find $d\langle\vec{L}\rangle(t)/dt = \langle ???\rangle$. (b) Evaluate the right hand side when the potential energy is spherically symmetric.

(7) (10 pts) You have a $2 \times 2$ Hamiltonian with elements $H_{11} = 3V$, $H_{22} = -3V$, and $H_{12} = 4V$. (1 pt) (a) What is the matrix element $H_{21}$? Give the reason for your answer. (3 pt) (b) Determine the two eigenenergies. (3 pt) (c) Determine the two eigenstates. (3 pt) (d) At time $t = 0$, the state is $|\Psi(0) >= |1 >$. Determine $|\Psi(t) >$.

4.27 and/or E8 Table 4.2

1)




3 nodes $= l - m$

2D Notes Bohr idea

2) The Hamiltonian is $H = \frac{L_z^2}{2MR^2} = -\frac{\hbar^2}{2MR^2}\frac{\partial^2}{\partial\varphi^2}$

The eigenstates are $\frac{1}{\sqrt{2\pi}} e^{im\phi}$ with eigenvalues $\frac{\hbar^2 m^2}{2MR^2}$

The lowest 3 energy levels are $0,\ \hbar^2/2MR^2,\ 4\,\hbar^2/2MR^2$

$M = 9 \cdot 1.66\times10^{-27}\,kg = 1.49\times10^{-26}\,kg$

$\hbar^2/2MR^2 = \frac{(1.055\times10^{-34}\,Js)^2}{2\ 1.49\times10^{-26}\,kg\ (50\times10^{-9}\,m)^2} = 1.49\times10^{-28}\,J = 1.08\times10^{-5}\,K$

$E = 0,\ 1.49\times10^{-28}\,J,\ 5.96\times10^{-28}\,J$

$0,\ 1.08\times10^{-5}\,K,\ 4.32\times10^{-5}\,K$

HWK 6 HWK 11

3) Need to satisfy Eq. 2.154 $\quad K = l \tan la$

$K = \sqrt{-2ME/\hbar^2} = \sqrt{2\cancel{M} f \cancel{\hbar^2}\pi^2/2\cancel{M}(2a)^2\cancel{\hbar^2}} = \sqrt{f}\ \frac{\pi}{2a}$

$l = \sqrt{2M(E+V_0)/\hbar^2} = \sqrt{2\cancel{M}(\frac{1}{10}-f)\cancel{\hbar^2}\pi^2/2\cancel{M}(2a)^2\cancel{\hbar^2}} = \sqrt{\frac{1}{10}-f}\ \frac{\pi}{2a}$

So the Eq. 2.154 becomes $\quad \sqrt{f} = \sqrt{\frac{1}{10}-f}\ \tan\left(\sqrt{\frac{1}{10}-f}\ \frac{\pi}{2}\right)$

Algorithm guess f, put in right hand side, use to find new f

$f_{new} = \left(\frac{1}{10} - f_{old}\right)\tan^2\left(\sqrt{\frac{1}{10}-f_{old}}\ \frac{\pi}{2}\right)$

| $f_{old}$ | $f_{new}$ | $f_{old}$ | $f_{new}$ |
|---|---|---|---|
| 0 | 0.0294 | 0.0194 | 0.0184 |
| 0.0294 | 0.0139 | 0.0184 | 0.01889 |
| 0.0139 | 0.0212 | 0.01889 | 0.01867 |
| 0.0212 | 0.0175 | 0.01867 | 0.01878 |
| 0.0175 | 0.0194 | 0.01878 | 0.01873 |

could stop here

Knowing $0 < f < 1/10$, you can find $\sqrt{f} - \sqrt{\frac{1}{10}-f}\ \tan\left(\sqrt{\frac{1}{10}-f}\ \frac{\pi}{2}\right) = 0$

to 1% with ~ 5 guesses at f

HWK 6 Prob 3.11 Simplified HWK8 3.28

4) a) The probability is $P(p)dp$ where $P(p) = |C(p)|^2$

From Eq. 3.53

$$C(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^a e^{-ipx/\hbar} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{2i} \sqrt{\frac{2}{a}} \int_0^a e^{i\left(\frac{\pi}{a} - P/\hbar\right)x} - e^{-i\left(\frac{\pi}{a} + P/\hbar\right)x} dx$$

ok to stop here

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{2i} \sqrt{\frac{2}{a}} \left\{ \frac{1}{i\left(\frac{\pi}{a} - \frac{P}{\hbar}\right)} \left[ e^{i\left(\frac{\pi}{a} - P/\hbar\right)a} - 1 \right] - \frac{1}{-i\left(\frac{\pi}{a} + \frac{P}{\hbar}\right)} \left[ e^{-i\left(\pi/a + P/\hbar\right)a} - 1 \right] \right\}$$

$$= \frac{1}{2\sqrt{\pi\hbar a}} \left[ \frac{e^{-iPa/\hbar} + 1}{\pi/a - P/\hbar} + \frac{e^{-iPa/\hbar} + 1}{\pi/a + P/\hbar} \right] \qquad \text{use } e^{i\frac{\pi}{a}a} = -1$$

$$= \frac{1}{\sqrt{\pi\hbar a}} e^{-ipa/2\hbar} \cos\left(\frac{pa}{2\hbar}\right) \frac{2\pi/a}{(\pi/a)^2 - (P/\hbar)^2}$$

$$|C(p)|^2 dp = \frac{\cos^2\left(\frac{pa}{2\hbar}\right)(4\pi^2/a^2)}{\pi\hbar a \left[(\pi^2/a^2) - (P/\hbar)^2\right]^2} dp$$

HWK 9 Prob 7 HWK 11 Prob 4.29 Many notes

5) Is any hamster truly innocent?

5) The solution $\psi = \frac{u(r)}{r} Y_2^{-1}(\theta, \varphi)$

From Eq 4.37

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\alpha \cdot r + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

For $l = 2$ $\qquad u = a_0 r^3 + a_1 r^4 + a_2 r^5 + a_3 r^6 + \dots$

$$-\frac{\hbar^2}{2m}\left[3\cdot 2 a_0 r + 4\cdot 3 a_1 r^2 + 5\cdot 4 a_2 r^3 + 6\cdot 5 a_3 r^4\right] + \alpha a_0 r^4 + \alpha a_1 r^5 + \alpha a_2 r^6 + \alpha a_3 r^7 \dots$$

$$+\frac{\hbar^2}{2m}\left[3\cdot 2 a_0 r + 3\cdot 2 a_1 r^2 + 3.2 a_2 r^3 + 3\cdot 2 a_3 r^4\right] = E a_0 r^3 + E a_1 r^4 + E a_2 r^5 + E a_3 r^6 \dots$$

$O(r^1)$ $\quad -\frac{\hbar^2}{2m} 3\cdot 2 a_0 + \frac{\hbar^2}{2m} 3\cdot 2 a_0 = 0 \Rightarrow a_0 = 1$

$O(r^2)$ $\quad -\frac{\hbar^2}{2m} 4\cdot 3 a_1 + \frac{\hbar^2}{2m} 3\cdot 2 a_1 = 0 \Rightarrow a_1 = 0$

$O(r^3)$ $\quad -\frac{\hbar^2}{2m} 5\cdot 4 a_2 + \frac{\hbar^2}{2m} 3\cdot 2 a_2 = E a_0 \Rightarrow a_2 = -E a_0 / (14\hbar^2/2m) = -E/(14\hbar^2/2m)$

ok to stop here

$O(r^4)$ $\quad -\frac{\hbar^2}{2m} 6\cdot 5 a_3 + \frac{\hbar^2}{2m} 3\cdot 2 a_3 + \alpha a_0 = E a_1 \Rightarrow a_3 = +\alpha/(24\hbar^2/2m)$

$O(r^5)$ $\quad -\frac{\hbar^2}{2m} 7\cdot 6 a_4 + \frac{\hbar^2}{2m} 3\cdot 2 a_4 + \alpha a_1 = E a_2 \Rightarrow a_4 = -E a_2/(42\hbar^2/m) = E^2 / \left[\frac{14\hbar^2}{2m} \frac{42\hbar^2}{2m}\right]$

HWK 8 3.17

6) $\vec{L} = \vec{r} \times \vec{p}$

From Eq. 3.71, $\frac{d\langle \vec{L} \rangle}{dt} = \frac{i}{\hbar} \langle [H, \vec{L}] \rangle$

$[H, \vec{L}] = [H, \vec{r} \times \vec{p}] = [H, \vec{r}] \times \vec{p} + \vec{r} \times [H, \vec{p}]$

$[H, \vec{L}] = [\frac{p^2}{2m}, \vec{r}] \times \vec{p} + \vec{r} \times [V, \vec{p}]$

$= \frac{\hbar}{im} \vec{p} \times \vec{p} + \vec{r} \times (i\hbar(\vec{\nabla} V)) = -i\hbar\, \vec{r} \times \vec{F}(\vec{r})$

$\Rightarrow \frac{d\langle \vec{L} \rangle}{dt} = \frac{i}{\hbar}(-i\hbar) \langle \vec{r} \times \vec{F}(\vec{r}) \rangle = \langle \vec{r} \times \vec{F}(\vec{r}) \rangle$

If $V$ is spherically symmetric $\vec{F} = -\hat{r} \frac{\partial V}{\partial r}$

This gives $\vec{r} \times \hat{r} = 0$

$\frac{d\langle \vec{L} \rangle}{dt} = 0$

HWK 8 Prob 7 Example 3.8

7) $H_{21} = H_{12}^* = 4V^* = 4V$ ($V$ must be real because $H_{ii}^* = H_{ii}$)

To find eigenvalues use $\det \begin{pmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{pmatrix} = 0$

$\det \begin{pmatrix} 3V - E & 4V \\ 4V & -3V - E \end{pmatrix} = (E - 3V)(E + 3V) - (4V)^2 = E^2 - 9V^2 - 16V^2 = E^2 - 25V^2 = 0$

$E_\pm = \pm 5V$ are eigen values

Eigen state for $E_+$ $\quad 3V C_1 + 4V C_2 = 5V C_1 \Rightarrow 2C_2 = C_1$

Use $C_1^2 + C_2^2 = 4C_2^2 + C_2^2 = 5C_2^2 = 1 \Rightarrow C_2 = 1/\sqrt{5} \quad C_1 = 2/\sqrt{5}$

Eigenstate for $E_-$ $\quad 3V C_1 + 4V C_2 = -5V C_1 \Rightarrow C_2 = -2C_1$

Use $C_1^2 + C_2^2 = C_1^2 + 4C_1^2 = 5C_1^2 = 1 \Rightarrow C_1 = 1/\sqrt{5} \quad C_2 = -2/\sqrt{5}$

$E_+ = 5V \quad \psi_+ = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \qquad E_- = -5V \quad \psi_- = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

$|\psi(0)\rangle = \frac{2}{\sqrt{5}} \psi_+ + \frac{1}{\sqrt{5}} \psi_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \psi(t) = \frac{2}{\sqrt{5}} \psi_+ e^{-i5Vt/\hbar} + \frac{1}{\sqrt{5}} e^{i5Vt/\hbar}$ OK to stop here

$$|\psi(t)\rangle = \begin{pmatrix} \frac{4}{5}e^{-i5vt/\hbar} + \frac{1}{5}e^{i5vt/\hbar} \\ \frac{2}{5}e^{-i5vt/\hbar} - \frac{2}{5}e^{i5vt/\hbar} \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{-i5vt/\hbar} + \frac{1}{5}e^{i5vt/\hbar} \\ -\frac{4i}{5}\sin\left(\frac{5vt}{\hbar}\right) \end{pmatrix}$$