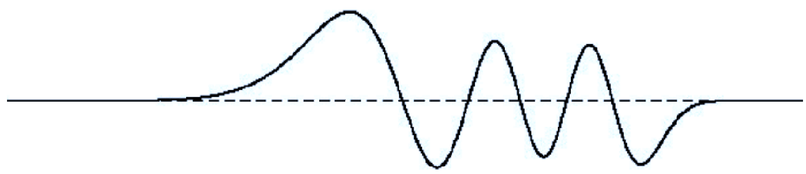
PHYS460, Test 1, Fall 2014

You must show work to get credit.

(1) (5 pts) Write as much information as you can about the eigenstate below and the potential that goes with it.



(2) (5 pts) I've acquired sock data for the class: 2 students have 0 socks, 5 students have 6 socks, 3 students have 7 socks, 4 students have 8 socks and 1 student has 9 socks. (a) Compute the average number of socks owned. (b) Compute the variance in the number of socks owned. (c) Is it OK to not own socks? Justify your answer.

(3) (5 pts) A plastic sphere with a density of 200 kg/m^3 and radius of 0.5 nm is held between two hard walls with separation L. You measure the $< v^2 >= 0.01$ m^2/s^2. (a) Is there a maximum or is there a minimum L? Explain. (b) If there is a minimum L, give that value. (c) If there is a maximum L, give that value.

(4) (5 pts) You have an infinite square well where $V(x) = 0$ for $0 < x < a$ and ∞ everywhere else. At $t = 0$, $\Psi(x, 0) = Ax(a - x)e^{ikx}$. Compute $< \hat{x} > (0)$ and $< \hat{p} > (0)$.

(5) (10 pts) You have an infinite square well where $V(x) = 0$ for $0 < x < a$ and ∞ everywhere else. (a) For $E = 0$, find the two linearly independent solutions of Schrodinger's equation in the region $0 < x < a$. (Note: $\psi(x) = 0$ is *not* one of the two solutions.) (b) Show that they can or show that they can't satisfy the boundary conditions when they are superposed. (c) Is there a solution if the boundary conditions are changed to $d\psi/dx = 0$ at $x = 0$ and at $x = a$? If yes, give the normalized eigenstate.

(6) (10 pts) The internal vibration of a CO molecule can be treated as a harmonic oscillator. At $t = 0$, the wave function is $\Psi(x, 0) = A[3\psi_n(x) + e^{i\phi}4\psi_{n+1}(x)]$. (a) Compute $< a_+ > (t)$, $< a_- > (t)$, $< a_+^2 > (t)$ and $< a_-^2 > (t)$. (b) Use this information to compute $< x > (t)$. (c) Without doing more calculations argue why $< x > (t) = 0$ for $\Psi(x, 0) = A[3\psi_n(x) + e^{i\phi}4\psi_{n+2}(x)]$.

(7) (10 pts) An electron starts at large **positive** x with **negative** momentum $-p$. It interacts with a potential $V(x) = -V_0$ for $x < 0$ and $V(x) = 0$ for $x > 0$. (a) From the wave function compute the reflection and transmission probabilities as a function of p. (b) Do they add up to 1? Explain why you should get the answer you found for the sum.

Test 1, Fall 2014

Based on Oral HWK

(1) The 6^{th} eigenstate (from the 5 nodes), (2) potential is not symmetric (from $\psi(x)$ not symmetric), (3) there are no ∞ walls (from $\psi(x)$ smoothly to 0 at ends), (4) potential increases more quickly on right then left (from how $\psi(x) \to 0$ on ends), and (5) the potential minimum near 2^{nd} minimum of $\psi(x)$ (from smallest wavelength).

Based on Prob 1.1

(2) $\langle j \rangle = \frac{2}{15} \cdot 0 + \frac{5}{15} 6 + \frac{3}{15} 7 + \frac{4}{15} 8 + \frac{1}{15} 9 = \boxed{6.1\bar{3}}$ (a)

$\langle j^2 \rangle = \frac{2}{15} \cdot 0^2 + \frac{5}{15} 6^2 + \frac{3}{15} 7^2 + \frac{4}{15} 8^2 + \frac{1}{15} 9^2 = 44.2\bar{6}$

(b) $\sigma_j^2 = \langle j^2 \rangle - \langle j \rangle^2 = \boxed{6.65}$

(c) <u>No</u>, need to wear shoes in the winter

Sec 1.6

(3) There is a minimum L. Use either Heisenberg uncertainty relation or ground state of particle in a box.

Need $m = \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi (\frac{1}{2} 10^{-9} m)^3 \, 200 \, kg/m^3 = 1.05 \times 10^{-25}$ kg

since $\langle v \rangle = 0$, $\sigma_p = 0.1 \, m/s \; m = 1.05 \times 10^{-26} \, kg \, m/s$

$\Rightarrow L \geq \frac{\hbar}{2\sigma_p} = \frac{1.05 \times 10^{-34} \, Js}{2 \cdot 1.05 \times 10^{-26} \, kg \, m/s} = \boxed{5 \times 10^{-9} \, m}$

OR

For ∞ square well $\langle p^2 \rangle = \frac{\hbar^2 \pi^2}{L^2} \Rightarrow L = \frac{\hbar \pi}{m\sqrt{\langle v^2 \rangle}} = \frac{h}{2\sigma_p}$

$\boxed{= 3 \times 10^{-8} m}$

Based on Prob 1.17

(4) Fast way: $|\Psi(x,0)|^2$ is symmetric about $x = a/2 \Rightarrow \boxed{\langle x \rangle = a/2}$

$\langle p \rangle = \int_0^a \Psi^*(x,0) \frac{\hbar}{i} \Psi'(x,0)\, dx = \int_0^a \Psi^*(x,0) \left[\hbar k \Psi(x,0) + \frac{\hbar}{i} A (a-2x) e^{ikx} \right] dx$

$= \hbar k \int \Psi^* \Psi \, dx + -i\hbar A^2 \underbrace{\cancel{\int_0^a x(a-x)(a-2x)\,dx}}_{\to 0} = \boxed{\hbar k}$

Slower way (next page)

$$\int_0^a |\Psi|^2 dx = A^2 \int_0^a x^2(a-x)^2 dx = A^2 \int_0^a x^2a^2 - 2ax^3 + x^4 dx$$
$$= A^2 a^5 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = A^2 a^5 \quad \frac{10-15+6}{30} = A^2 \frac{a^5}{30} = 1 \Rightarrow A^2 = \frac{30}{a^5}$$

$$\langle x \rangle = \int_0^a x |\psi|^2 dx = A^2 \int_0^a (x^3a^2 - 2ax^4 + x^5) dx = A^2 a^6 \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right)$$
$$= \frac{30}{a^5} a^6 \left(\frac{30-48+20}{120}\right) = \boxed{a/2}$$

same steps as previous page

$$\langle p \rangle = \hbar k \int \Psi^* \Psi \, dx - i\hbar A^2 \int_0^a (ax - x^2)(a-2x) dx$$
$$= \hbar k + -i\hbar A^2 \int_0^a a^2x - 2ax^2 - ax^2 + 2x^3 dx$$
$$= \hbar k - i\hbar A^2 a^4 \left[\frac{1}{2} - \frac{3}{3} + \frac{2}{4}\right] = \boxed{\hbar k}$$

Based on Prob 2.3 (5) Schrodinger's Eq $-\frac{\hbar^2}{2m}\psi''(x) = 0$

Two linearly indep. solutions are constant and x

a) $\boxed{\psi(x) = A + Bx}$

b) $\psi(0) = 0 \Rightarrow A = 0$ and $\psi(a) = 0 \Rightarrow B = -A/a = 0$

$\boxed{\text{No solution}}$

c) if $\psi'(0) = 0 \Rightarrow B = 0$, if $\psi'(a) = 0 \Rightarrow B = 0$

$\psi(x) = A$ is the solution

$$\int_0^a |\psi(x)|^2 dx = a|A|^2 \Rightarrow A = 1/\sqrt{a} \Rightarrow \boxed{\psi(x) = 1/\sqrt{a}}$$

Based on Prob 2.13 (6) First find A $\int \Psi^*(x,0)\Psi(x,0) dx = A^2 \int 9\psi_n^2 + 16\psi_{n+1}^2 dx$

(The terms with $\psi_n \psi_{n+1}$ integrate to 0)

$A^2(9+16) = A^2 \cdot 25 = 1 \Rightarrow A = 1/5$

$$\Psi(x,t) = \frac{3}{5}\psi_n(x) e^{-iE_nt/\hbar} + \frac{4}{5}e^{i\phi}\psi_{n+1} e^{-iE_{n+1}t/\hbar}$$

Strategy: Do not write out all of the terms

Only do the ones that are nonzero

$$a_+ \Psi(x,t) = \frac{3}{5}\sqrt{n+1}\,\psi_{n+1}\, e^{-iE_n t/\hbar} + (\;)$$

$$\langle a_+ \rangle = \frac{4}{5} e^{-i\varphi} e^{iE_{n+1}t/\hbar} \frac{3}{5}\sqrt{n+1}\, e^{-iE_n t/\hbar} = \boxed{\frac{12\sqrt{n+1}}{25} e^{-i\varphi} e^{i\omega t}}$$

(I used $E_{n+1} - E_n = \hbar\omega$)

$$a_- \Psi(x,t) = (\;) + \frac{4}{5} e^{i\varphi}\sqrt{n+1}\,\psi_n\, e^{-iE_{n+1}t/\hbar}$$

$$\langle a_- \rangle = \frac{3}{5} e^{iE_n t/\hbar} \frac{4}{5} e^{i\varphi}\sqrt{n+1}\, e^{-iE_{n+1}t/\hbar} = \boxed{\frac{12\sqrt{n+1}}{25} e^{i\varphi} e^{-i\omega t}}$$

$a_+^2 \Psi(x,t)$ gives $n+2$ and $n+3$, this means

$\boxed{\langle a_+^2 \rangle = 0}$

Similar argument for $\langle a_-^2 \rangle = 0$

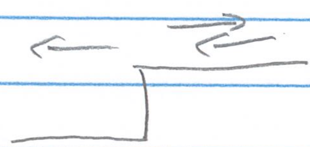
$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a}_+ + \hat{a}_-) \Rightarrow \langle \hat{x} \rangle = \left(\frac{\hbar(n+1)}{2m\omega}\right)^{1/2}\frac{12}{25}\left(e^{i(\omega t-\varphi)} + e^{-i(\omega t-\varphi)}\right)$$

$$= \left(\frac{\hbar(n+1)}{2m\omega}\right)^{1/2}\frac{24}{25}\cos(\omega t - \varphi)$$

(c) Must be 0 because the a_+ can't change ψ_n to ψ_{n+2} (and vice versa).

Based on Prob 2.34

7)



For $x<0$ $\quad -\frac{\hbar^2}{2m}\psi'' - V_0\psi = E\psi \Rightarrow \psi'' = -l^2\psi$

where $l = \sqrt{2m(E+V_0)}/\hbar = \sqrt{p^2 + 2mV_0}/\hbar$

For $x>0$ $\quad -\frac{\hbar^2}{2m}\psi'' = E\psi \Rightarrow \psi'' = -k^2\psi$

where $k = p/\hbar = \sqrt{2mE}/\hbar$

For $x<0$ $\quad \psi(x) = C\, e^{-ilx}$

For $x>0$ $\quad \psi(x) = A\, e^{-ikx} + B\, e^{ikx}$

Continuity $\quad \psi(0) = \psi(0) \Rightarrow C = A + B$

Continuity of $\psi'(x)$ $\quad \psi'(0) = \psi'(0) \Rightarrow -ilC = -ik(A-B)$

$$kA - kB = lA + lB \Rightarrow (k-l)A = (k+l)B$$

$$B = \frac{k-l}{k+l} A \qquad C = A + B = \left(1 + \frac{k-l}{k+l}\right) A = \frac{2k}{k+l} A$$

$$\text{Reflection} = \frac{\hbar k}{m}|B|^2 \Big/ \frac{\hbar k}{m}|A|^2 = \left(\frac{k-l}{k+l}\right)^2 = \left(\frac{p - \sqrt{p^2+2mV_0}}{p + \sqrt{p^2+2mV_0}}\right)^2$$

$$\text{Transmission} = \frac{\hbar l}{m}|C|^2 \Big/ \frac{\hbar k}{m}|A|^2 = \frac{l}{k}\left(\frac{2k}{k+l}\right)^2 = \frac{4kl}{(k+l)^2} = \frac{4p\sqrt{p^2+2mV_0}}{(p+\sqrt{p^2+2mV_0})^2}$$

$$R + T = \frac{(k-l)^2}{(k+l)^2} + \frac{4kl}{(k+l)^2} = \frac{(k^2 - 2kl + l^2 + 4kl)}{(k+l)^2} = \frac{k^2+2kl+l^2}{k^2+2kl+l^2}$$

$$= 1$$

Conservation of probability