Taylor series expansion

If a function $f(x)$ and all of its derivatives exist at $x=0$

$$f(x) = f_{TS}(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{1\cdot 2\cdot 3} f'''(0) + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} f^{IV}(0) + \dots$$

To show this evaluate $f_{TS}(0)$, $f'_{TS}(0)$, $f''_{TS}(0)$, ... and show $f(0) = f_{TS}(0)$, $f'(0) = f'_{TS}(0)$, $f''(0) = f''_{TS}(0)$, ...

Notation $O(x^4)$ means order x^4 is lowest order term left out

For example, $f(x) = f(0) + x f'(0) + O(x^2)$

Note $O(x^3) + O(x^4) = O(x^3)$

You can expand functions around points other than 0

$$f(x+\delta x) = f(x) + \delta x\, f'(x) + \frac{\delta x^2}{1\cdot 2} f''(x) + \frac{\delta x^3}{1\cdot 2\cdot 3} f'''(x) + O(\delta x^4)$$

Examples

① Compute all derivatives of $f(x) = (x+2)^3$ at $x=0$.
Use this to find the Taylor series expansion of $f(x)$. Compare to the exact function by expanding.

② Compute the first 4 derivatives of $f(x) = \frac{1}{1+x^2}$ at $x=0$ and show $f(x) = 1 - x^2 + x^4 + O(x^6)$

③ Compute all derivatives of $\cos(x)$ to show

$$\cos(x) = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \dots$$