Physics - Connection between force and potential energy

Newton's Equation: $m\dot{v} = F(x(t))$

The energy is defined as $E(t) = \frac{1}{2} m v^2(t) + U(x(t))$

① Use the chain rule to show $\frac{dE}{dt} = 0$ if $F(x) = -\frac{dU(x)}{dx}$

$U(x)$ is the potential energy

② For a Harmonic Oscillator, $F(x) = -k \cdot x$ with k a constant.

(a) Find $U(x)$.

(b) Show $x(t) = x_0 \cos(\omega \cdot t) + \frac{v_0}{\omega} \sin(\omega t)$ is the solution that goes with $x(0) = x_0$ and $v(0) = v_0$

(c) Show $E(t) = \text{constant} = \frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2$

③ For Euler's method, show

$$E(t+\delta t) = E(t) + \frac{1}{2}\left[\frac{F^2(x(t))}{m} + v^2(t)\, U''(x(t))\right]\delta t^2$$

④ For a Harmonic oscillator and Euler's method, show that the energy increases with every time step like

$$E(t+\delta t) = E(t) + \frac{k}{m} E(t)\,\delta t^2 = \left(1 + \frac{k\,\delta t^2}{m}\right) E(t)$$

This means that after n time steps $E(n \cdot \delta t) \cong e^{\frac{n k \delta t^2}{m}} E(0)$
(the energy increases exponentially with rate $k\delta t/m$)